Lecture 35: The definition of the derivative of a function

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Companion to Advanced Calculus

The textbook definition of the derivative

Definition 10.1.1 (Differentiability at a point) Let $X \subseteq \mathbb{R}$, and let $x_0 \in X$ be an element of X that is also a limit point of X. Let $f : X \to \mathbb{R}$ be a function. If

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} := \lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = L, \quad (1)$$

i.e., if the limit exists and is equal to the real number L, then we say that f is differentiable at x_0 on X with derivative L, and write $f'(x_0) := L$ (or sometimes $\frac{df}{dx}(x_0) := L$).

If the limit does not exist, or if x_0 is not an element of X or not a limit point of X, we leave $f'(x_0)$ undefined, and say that f is not differentiable at x_0 on X.

Note that we do not require X to contain an interval.

Our version introduces a simplified notation on the left-hand side of (1). It can only be used if X is implied by the context.

Example: Derivative of x^2

Example 10.1.3: Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) := x^2$, and let x_0 be any real number. Then:

$$\lim_{x \to x_0; x \in \mathbb{R} \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0; x \in \mathbb{R} \setminus \{x_0\}} \frac{x^2 - x_0^2}{x - x_0}$$

$$= \lim_{x \to x_0; x \in \mathbb{R} \setminus \{x_0\}} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0; x \in \mathbb{R} \setminus \{x_0\}} x + x_0$$

$$= x_0 + x_0 = 2x_0.$$

Thus the function $f(x)$ is differentiable at x_0 and its derivative there is $2x_0$.

We see why it is important that we use $\lim_{x \to x_0; x \in X \setminus \{x_0\}}$ instead of $\lim_{x \to x_0; x \in X}$ in the definition of the derivative.

The quotient $\frac{f(x) - f(x_0)}{x - x_0}$ is always undefined for $x = x_0$, but in the above example we could use cancellation for all $x \neq x_0$.

Proposition 10.1.10: (Differentiability implies continuity) Let $X \subseteq \mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X \to \mathbb{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.

Proof: Let X, f, x_0 be as above, and let $L \in \mathbb{R}$ be the limit $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = L$. Let $\varepsilon > 0$. Pick $\delta > 0$ such that

$\forall x \in X \cap (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$ $\; L - \varepsilon \leq \frac{f(x)-f(x_0)}{x-x_0} \leq L + \varepsilon.$

Thus
$\forall x \in X \cap (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$ $\; |L| - \varepsilon \leq \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq |L| + \varepsilon,$
which implies that for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$ we have
$(|L| - \varepsilon)|x - x_0| \leq |f(x) - f(x_0)| \leq (|L| + \varepsilon)|x - x_0|.$

Now it follows from the squeeze theorem that
$\lim_{x \to x_0} |f(x) - f(x_0)| = 0 = \lim_{x \to x_0} f(x) - f(x_0),$
so that $\lim_{x \to x_0} f(x) = f(x_0).$

Thus f is continuous at x_0. $\square$

Example: Derivative of $|x|$

Example 10.1.6: (modified from textbook) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) := |x|$, let $g := f \restriction (0, \infty)$, let $h := f \restriction (-\infty, 0]$, and let $x_0 = 0$.

Question L35.1: What is $g'(0)$?

Question L35.2: What is $h'(0)$?

$g'(0)$ is undefined, since 0 is not in the domain of g.

$h'(0) = \lim_{x \to x_0; x \in (-\infty, 0)} \frac{|x|}{x} = \lim_{x \to x_0; x \in (-\infty, 0)} \frac{-x}{x} = -1.$

Thus h is differentiable at 0. Since 0 is an endpoint of its domain, the derivative here is what in calculus would be called a one-sided derivative.

Similarly, if $j := f \restriction [0, \infty)$, then

$j'(0) = \lim_{x \to x_0; x \in (0, -\infty)} \frac{|x|}{x} = \lim_{x \to x_0; x \in (0, \infty)} \frac{x}{x} = 1.$

It follows that the entire function $|x|$ is not differentiable at 0.

Differentiability and linear approximations

Proposition 10.1.7: (Newton's approximation) Let $X \subseteq \mathbb{R}$, let $x_0 \in X$ be a limit point of X, let $f : X \to \mathbb{R}$ be a function, and let L be a real number. Then the following statements are logically equivalent:

(a) f is differentiable at x_0 on X with derivative L.

(b) $\forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in X \cap [x_0 - \delta, x_0 + \delta] \;\; |f(x) - (f(x_0) + L(x - x_0))| \leq \varepsilon |x - x_0|$.

This result essentially says that $f'(x_0) = L$ if, and only if, the linear function $D(x) := f(x_0) + L(x - x_0)$ is an arbitrarily good approximation of $f(x)$ for points $x \in X$ that are sufficiently close to x_0.

We recognize the graph of D as the tangent line to the graph of f at x_0 and $L = f'(x_0)$ as its slope.

For $|x| \upharpoonright (-\infty, 0]$, the slope of the tangent line at 0 is -1; for $|x| \upharpoonright [0, \infty)$, the slope of the tangent line at 0 is 1; and $|x|$ itself does not have a tangent line at 0.

Example: Derivative of $\sqrt{x}$

Example L35.1: Let $f : [0, \infty) \to \mathbb{R}$ be the function $f(x) := \sqrt{x}$, and let $x_0 = 0$.

$\lim_{x \to x_0; x \in (0,\infty)} \frac{\sqrt{x}}{x} = \lim_{x \to x_0; x \in (0,\infty)} \frac{1}{\sqrt{x}} = +\infty.$

Recall that $+\infty$ is not a real number. We have found that the quotient increases without bound as x approaches 0 in the domain of the function f. Thus $f'(0)$ does not exist; the function $f(x) := \sqrt{x}$ is not differentiable at 0.

Notice that if we were to draw a tangent line to the graph of $\sqrt{x}$ at 0, it would be vertical, that is, would have infinite slope. Such a line is is not considered a linear approximation as it doesn't give us any information about the approximate values of $\sqrt{x}$ for x near 0.

Definition 10.1.11: (Differentiability on a domain) Let $X \subseteq \mathbb{R}$, and let $f : X \to \mathbb{R}$. We say that f is differentiable (on X) if, and only if, for every limit point $x_0 \in X$, the function f is differentiable at x_0 on X.

Question L35.3: Let $X = \mathbb{Z}$, and let $f : X \to \mathbb{R}$.

What can we say about the differentiability of f?

According to the above definition, every such function would be considered differentiable.

Since each point in the domain of f is isolated, there are are no limit points x_0 of $X = \mathbb{Z}$ where f would not be differentiable.

This certainly seems a bit counterintuitive; other authors define f to be differentiable whenever f is differentiable at every point x_0 in its domain X. But the variant chosen by the textbook author has some advantages; for example, all constant functions on any domain will then be considered differentiable.

Theorem 10.1.13: (Differential calculus) Let $X \subseteq \mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $f, g : X \to \mathbb{R}$.

(a) If $f(x) := c$ for some $c \in \mathbb{R}$ is a constant function, then f is differentiable at x_0 and $f'(x_0) = 0$.

(b) If $f(x) := x$ for all $x \in X$ is the identity function, then f is differentiable at x_0 and $f'(x_0) = 1$.

(c) (Sum rule) If f and g are differentiable at x_0, then $f + g$ is also differentiable at x_0, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

(d) (Product rule) If f and g are differentiable at x_0, then fg is also differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(e) If f is differentiable at x_0 and $c \in \mathbb{R}$, then cf is also differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.

Theorem 10.1.13, completed

Theorem 10.1.13: (Differential calculus) Let $X \subseteq \mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $f, g : X \to \mathbb{R}$.

(f) (Difference rule) If f and g are differentiable at x_0, then $f - g$ is also differentiable at x_0, and $(f-g)'(x_0) = f'(x_0) - g'(x_0)$.

(g) If g is differentiable at x_0, and g is non-zero on X (i.e., if $g(x) \neq 0$ for all $x \in X$), then $\frac{1}{g}$ is also differentiable at x_0, and $\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}$.

(h) (Quotient rule) If f and g are differentiable at x_0, and g is non-zero on X, then $\frac{f}{g}$ is also differentiable at x_0, and $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$.

No part of this theorem comes as a big surprise to us. We are already familiar with it, and presumably with its proof, from calculus.

Theorem 10.1.15: (Chain rule) Let $X, Y \subseteq \mathbb{R}$, let $x_0 \in X$ be a limit point of X, let $y_0 \in Y$ be a limit point of Y, let $f : X \to Y$ be differentiable at x_0 and such that $f(x_0) = y_0$. Suppose that $g : Y \to \mathbb{R}$ is differentiable at y_0.
Then the function $g \circ f : X \to \mathbb{R}$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0). \tag{2}$$

When we use variables y for $f(x)$ and z for $g(y)$, we can write the chain rule as $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$.
This expression in Leibniz notation is very intuitive and often convenient, especially in the study of differential equations. Your textbook author has some concerns about it though. They seem somewhat overblown, but it needs to be kept in mind that the derivatives of f and g in (2) are calculated at different points, so that the chain rule should really be written in Leibniz notation as $\frac{dz}{dx}(x_0) = \frac{dz}{dy}(f(x_0))\frac{dy}{dx}(x_0)$.

Take-home message

This lecture formally introduces the definition of a derivative of a function $f : X \to \mathbb{R}$ at a point x_0 in its domain.

This definition coincides with the one from calculus when X is an interval and x_0 is not one of the endpoints of this interval. However, it is more general and only requires x_0 to be both an element and a limit point of X. In particular, when X is a nondegenerate interval and $x_0 \in X$ is one of its endpoints, then $f'(x)$ as defined here coincides with what was called a one-sided derivative in calculus.

The remainder of the lecture essentially confirms standard results from calculus in this slightly more general setting.

One difference between the textbook approach and what you may have seen in calculus is that here we consider a function differentiable if it is differentiable at all limit points in its domain, even when the domain contains isolated points where the derivative is undefined.