Lecture 38: Riemann integrability of continuous and of monotone functions

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Companion to Advanced Calculus

Uniformly continuous functions are Riemann integrable

Theorem 11.5.1: Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is Riemann integrable.

For the proof of this theorem, it suffices to find, for every $\varepsilon > 0$, a piecewise constant function $\ell_\varepsilon : I \to \mathbb{R}$ that minorizes f and a piecewise constant function $u_\varepsilon : I \to \mathbb{R}$ that majorizes f such that $p.c. \int_I u_\varepsilon - p.c. \int_I \ell_\varepsilon \leq \varepsilon$.

Let $\varepsilon > 0$ be arbitrary, and let $\gamma > 0$ be a number that we will choose in a moment.

By uniform continuity of f, there exists $\delta > 0$ such that $\forall x, y \in I \; (|x - y| < \delta \to |f(x) - f(y)| \leq \gamma$.

Pick such δ. We can find a partition $\mathbf{P}$ of I into pairwise disjoint nonempty subintervals J of length $|J| < \delta$ each.

For each $J \in \mathbf{P}$, pick $x_J \in J$, and define $\ell_\varepsilon(x) := f(x_J) - \gamma$ and $u_\varepsilon(x) := f(x_J) + \gamma$ for all $J \in \mathbf{P}$ and all $x \in J$.

Question L38.1: How to choose γ for this argument to work?

The proof of Theorem 11.5.1 completed

We distinguish two cases:

Case 1: $|I| = 0$.

Then f is piecewise constant and hence Riemann integrable.

Case 2: $|I| > 0$.

Then we choose $\gamma := \frac{\varepsilon}{2|I|}$.

By construction, ℓ_ε minorizes f, u_ε majorizes f, and both $\ell_\varepsilon, u_\varepsilon$ are piecewise constant.

Moreover, $u_\varepsilon(x) - \ell_\varepsilon(x) = 2\gamma = \frac{\varepsilon}{|I|}$ for all $x \in I$, so that

$p.c. \int_I u_\varepsilon - p.c. \int_I \ell_\varepsilon = p.c. \int_I (u_\varepsilon - \ell_\varepsilon) = \frac{\varepsilon}{|I|}|I| = \varepsilon.$ □

Combining Theorem 11.5.1 with Theorem 9.9.16, we obtain:

Corollary 11.5.2: Let $[a, b]$ be a closed interval, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Does it suffice to assume continuity of f?

Conjecture L38.1: Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be continuous on I. Then f is Riemann integrable.

Question L38.2: Is this conjecture true?

No. For example, the function $f(x) := \frac{1}{x}$ is continuous on $(0, 1]$, but unbounded. So it cannot be Riemann integrable.

It turns out that a slight modification of Conjecture 38.1 is true:

Proposition 11.5.3: Let I be a bounded interval, and let
$f : I \to \mathbb{R}$ be both continuous and bounded.
Then f is Riemann integrable.

Note that by Corollary 11.5.2 and our earlier observation about functions on degenerate intervals, Proposition 11.5.3 is of interest only if $I = (a, b)$ or $I = (a, b]$ or $I = [a, b)$ for some $a < b$.

Then for every a^+, b^- with $a < a^+ < b^- < b$ the restriction $f \restriction [a^+, b^-]$ is Riemann integrable by Corollary 11.5.2.

Proof of Proposition 11.5.3

We prove the result for $I = (a, b)$ with $a < b$; the proof in the other cases is similar.

Let f be as in the assumptions, and let $M > 0$ be such that $-M \leq f(x) \leq M$ for all $x \in I$.

Let $\varepsilon > 0$, and let $a < a^+ < b^- < b$.

Pick piecewise constant function $\ell'_\varepsilon, u'_\varepsilon : [a^+, b^-] \to \mathbb{R}$ with respect to some partition $\mathbf{P}^-$ of $[a^+, b^-]$ such that ℓ'_ε minorizes $f \restriction [a^+, b^-]$, u'_ε majorizes $f \restriction [a^+, b^-]$, and $p.c. \int_{[a^+,b^-]} u'_\varepsilon - p.c. \int_I \ell'_\varepsilon \leq \frac{\varepsilon}{3}$.

Let $\mathbf{P} = \mathbf{P}' \cup \{(a, a^+), (b^-, b)\}$, and let $\ell_\varepsilon(x) := -M$, $u_\varepsilon(x) := M$ for $x \in (a, b) \backslash [a^+, b^-]$,
while $\ell_\varepsilon(x) := \ell'_\varepsilon(x)$ and $u_\varepsilon(x) = u'_\varepsilon(x)$ for $x \in [a^+, b^-]$.

Then $\ell_\varepsilon, u_\varepsilon$ are piecewise constant, ℓ_ε minorizes f, and u_ε majorizes f.

Question L38.3: How should we choose a^+, b^- so that $p.c. \int_I u_\varepsilon - p.c. \int_I \ell_\varepsilon \leq \varepsilon$?

Proof of Proposition 11.5.3

We prove the result for $I = (a, b)$ with $a < b$; the proof in the other cases is similar. Let f be as in the assumptions, and let $M > 0$ be such that $-M \leq f(x) \leq M$ for all $x \in I$.

Let $\varepsilon > 0$, and let $a < a^+ < b^- < b$.

Pick piecewise constant function $\ell'_\varepsilon, u'_\varepsilon : [a^+, b^-] \to \mathbb{R}$ with respect to some partition $\mathbf{P}^-$ of $[a^+, b^-]$ such that ℓ'_ε minorizes $f \restriction [a^+, b^-]$, u'_ε majorizes $f \restriction [a^+, b^-]$, and $p.c. \int_{[a^+,b^-]} u'_\varepsilon - p.c. \int_I \ell'_\varepsilon \leq \frac{\varepsilon}{3}$.

Let $\mathbf{P} = \mathbf{P}' \cup \{(a, a^+), (b^-, b)\}$, and let $\ell_\varepsilon(x) := -M$, $u_\varepsilon(x) := M$ for $x \in (a, b) \backslash [a^+, b^-]$,
while $\ell_\varepsilon(x) := \ell'_\varepsilon(x)$ and $u_\varepsilon(x) = u'_\varepsilon(x)$ for $x \in [a^+, b^-]$.

Then $\ell_\varepsilon, u_\varepsilon$ are piecewise constant, ℓ_ε minorizes f, and u_ε majorizes f.

It is sufficient to choose these numbers so that $a^+ - a \leq \frac{\varepsilon}{6M}$ and $b - b^- \leq \frac{\varepsilon}{6M}$. Then $p.c. \int_I u_\varepsilon - \ell_\varepsilon$
$= p.c. \int_{(a,a^+)} u_\varepsilon - \ell_\varepsilon + p.c. \int_{[a^+,b^-]} u_\varepsilon - \ell_\varepsilon + p.c. \int_{(b^-,b)} u_\varepsilon - \ell_\varepsilon$
$= 2M(a^+ - a) + p.c. \int_{[a^+,b^-]} u_\varepsilon - \ell_\varepsilon + 2M(b - b^-) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$.

Riemann integrability of bounded monotone functions

Proposition 11.6.1: Let $[a, b]$ be a closed and bounded interval and let $f : [a, b] \to \mathbb{R}$ be a monotone function.
Then f is Riemann integrable.

Again, the analogue for open or half-open intervals is false, as the example of $f(x) := \frac{1}{x}$ on $(0, 1]$ shows.

Note that the assumptions of Proposition 11.6.1 imply that the function values of f must all be in the interval $[f(a), f(b)]$ (if f is increasing) or in the interval $[f(b), f(a)]$ (if f is decreasing).

So it would seem more natural to call Proposition 11.6.1 a corollary of the following result:

Corollary 11.6.3: Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be both monotone and bounded. Then f is Riemann integrable.

We will prove Corollary 11.6.3 without using Proposition 11.6.1 as a stepping-stone.

Proof of Corollary 11.6.3

Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be monotone and bounded. We focus on the case when $|I| > 0$ and f is increasing.

Let $M > 0$ be such that $f(I) \subseteq [-M, M)$, and let $\varepsilon > 0$.

We partition $[-M, M)$ into pairwise disjoint intervals $[y_{i-1}, y_i)$ such that $-M = y_0 < y_1 < \cdots < y_n = M$ and $0 < y_i - y_{i-1} \leq \frac{\varepsilon}{|I|}$ for all $i \in \{1, \dots, n\}$.

Let $x, y, z \in I$ with $x < z < y$. Then if $f(x), f(y) \in [y_{i-1}, y_i)$ for some i, we also have $f(z) \in [y_{i-1}, y_i)$, since f was assumed monotone. Thus $f^{-1}([y_{i-1}, y_i))$ is an interval.

Thus $\mathbf{P} := \{f^{-1}([y_{i-1}, y_i)) : \; 1 \leq i \leq n\}$ is a partition of I.

Question L38.4: How should we choose $\ell_\varepsilon, u_\varepsilon$ for this proof?

Let $\ell_\varepsilon(x) := y_{i-1}$ and $u_\varepsilon(x) := y_i$ for $x \in f^{-1}([y_{i-1}, y_i))$ and $1 \leq i \leq n$.

Then $\ell_\varepsilon, u_\varepsilon$ are piecewise constant with respect to $\mathbf{P}$, ℓ_ε minorizes f, u_ε majorizes f, and $p.c \int_I u_\varepsilon - p.c. \int_I \ell_\varepsilon \leq \varepsilon$. □

The integral test

Proposition 11.6.4: (Integral test) Let $f : [0, \infty) \to \mathbb{R}$ be a monotone decreasing function which is non-negative (i.e., $f(x) \geq 0$ for all $x \geq 0$). Then the sum $\sum_{n=0}^{\infty} f(n)$ is convergent if, and only if, $\sup_{N>0} \int_{[0,N]} f$ is finite.

This test gives the following result, which does not follow from any of the tests for convergence of series that we had seen previously:

Corollary 11.6.5: Let p be a real number. Then $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges absolutely when $p > 1$ and diverges when $p \leq 1$.

Question L38.5: How to choose f for deriving Corollary 11.6.5?

We fix $p \in \mathbb{R}$ and let $f(x) = (x+1)^{-p}$.

Anticipating a bit, from the Fundamental Theorem of Calculus:
$\int_{[0,N]} f = \frac{1}{1-p}(N+1)^{-p+1} - \frac{1}{1-p}$ if $p \neq 1$ and
$\int_{[0,N]} f = \ln(N+1) - \ln 1 = \ln(N+1)$ if $p = 1$.
The result follows by examining these expressions as $N \to \infty$.

The proof of the integral test

Proposition 11.6.4: (Integral test) Let $f : [0, \infty) \to \mathbb{R}$ be a monotone decreasing function which is non-negative (i.e., $f(x) \geq 0$ for all $x \geq 0$). Then the sum $\sum_{n=0}^{\infty} f(n)$ is convergent if, and only if, $\sup_{N>0} \int_{[0,N]} f$ is finite.

Let f be as in the assumption, and fix $N > 0$. Then:

- $f \restriction [0, N]$ is Riemann integrable by Proposition 11.6.1.
- The piecewise constant function $u(x) := f(\lfloor x \rfloor)$ majorizes f on $[0, N]$.
- The piecewise constant function $\ell(x) := f(\lfloor x \rfloor + 1)$ minorizes f on $[0, N]$.
- $\int_{[0,N]} \ell = \sum_{n=1}^{N+1} f(n) \leq \int_{[0,N]} f \leq \sum_{n=0}^{N} f(n) = \int_{[0,N]} f(n)$.
- Now the result follows from the observation that the partial sums of the series $\sum_{n=0}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} f(n)$ either both converge or both increase without bound. $\square$

Take-home message

In this lecture we saw that:

- Bounded, continuous functions on bounded intervals are Riemann integrable.
- Bounded, monotonous functions on bounded intervals are Riemann integrable.

As an application, we derived the integral test for absolute convergence of series.

The integral test allowed us to determine that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, while for any $p > 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is absolutely convergent.