Lecture 41: Some point-set topology of metric spaces

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Companion to Advanced Calculus

Adherent points in metric spaces

Let (X, d) be a metric space, let $E \subseteq X$, and let $x \in X$.

We can then call x an adherent point of E if, and only if,
$\forall \varepsilon > 0 \, \exists y \in E \;\, d(x, y) \leq \varepsilon$.

This definition is a straightforward generalization of the definition for the special case when $X = \mathbb{R}$ with the usual metric.

We can see from this definition that if there exists a sequence $(x^{(n)})_{n=m}^{\infty}$ of elements of E that converges to x,
then x must be an adherent point of E.

Conversely, if x_0 is an adherent point of E, then for every $n \geq 1$ we can pick $x^{(n)} \in E$ such that $\forall n \geq 1 \;\; d(x^{(n)}, x) \leq \frac{1}{n}$, so that the sequence $(x^{(n)})_{n=1}^{\infty}$ converges to x.

As in the special case of subsets of the real line, the adherent points of E are the limits of convergent sequences of points in E.

It will, however, help our geometric intuition if we rephrase this and other definitions in this section in terms of (open) balls.

Definition 1.1.2: (Balls) Let (X, d) be a metric space, let $x_0 \in X$, and let $r > 0$. We define the (open) ball $B_{(X,d)}(x_0, r)$ in X, centered at x_0, and with radius r, in the metric d, to be the set

$B_{(X,d)}(x_0, r) := \{x \in X : \ d(x, x_0) < r\}.$

When it is clear what the metric space (X, d) is, we shall abbreviate $B_{(X,d)}(x_0, r)$ as just $B(x_0, r)$.

- When $(X, d) = (\mathbb{R}^3, d_{\ell^2})$, then $B_{(X,d)}(x_0, r)$ is indeed what we would normally think of as the ball centered at x_0 and with radius r.
- When $(X, d) = (\mathbb{R}, d_{\ell^2})$, then $B_{(X,d)}(x_0, r) = (x_0 - r, x_0 + r)$. This is an open interval around x_0, and we can see why the notion of an open ball generalizes the notion of an open interval. The textbook does not write "open ball," but most authors do.

Open balls in $\mathbb{R}^2$ with metrics induced by norms

Question L41.1: What are the open balls centered around the origin with radius r in $(\mathbb{R}^2, d_{\ell 1}), (\mathbb{R}^2, d_{\ell 2})$, and $(\mathbb{R}^2, d_{\ell \infty})$?

$B_{(\mathbb{R}^2, d_{\ell 1})}((0,0), r)$ is the open square with vertices $(-r, 0), (0, r), (r, 0), (0, -r)$.

$B_{(\mathbb{R}^2, d_{\ell 2})}((0,0), r)$ is the open circular disc with radius r that is centered at the origin.

$B_{(\mathbb{R}^2, d_{\ell \infty})}((0,0), r)$ is the open square with vertices $(-r, -r), (-r, r), (r, r), (r, -r)$.

Open balls in the discrete metric

Let $X \neq \emptyset$, and let d_{disc} be the discrete metric on X.

Question L41.2: Which sets are open balls $B(x_0, r)$ for $x_0 \in X$ and $r > 0$ in (X, d_{disc})?

Each open ball $B(x_0, r)$ in any metric space is nonempty, as we require $r > 0$ and thus always have $x_0 \in B(x_0, r)$.

When $r \leq 1$, then $B_{(X,d_{disc})} = \{x_0\}$.

When $r > 1$, then $B_{(X,d_{disc})} = X$.

Exterior, interior, and boundary points

Definition 1.2.5: Let (X, d) be a metric space, let $E \subseteq X$, and let $x_0 \in X$. We say that

- x_0 is an interior point of E if there exists a radius $r > 0$ such that $B(x_0, r) \subseteq E$,
- x_0 is an exterior point of E if there exists a radius $r > 0$ such that $B(x_0, r) \cap E = \emptyset$,
- x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.

The interior $int(E)$ of E is the set of all interior points of E.
The exterior $ext(E)$ of E is the set of all exterior points of E.
The boundary $\partial(E)$ of E is the set of all boundary points of E.

Proposition L41.1: Let (X, d) be a metric space and $E \subseteq X$. Then:

(a) $int(E) \subseteq E$. Moreover, $int(E) = ext(X \backslash E)$.

(b) $ext(E) \cap E = \emptyset$. Moreover, $ext(E) = int(X \backslash E)$.

(c) $\partial E = \{x_0 \in X : \ \forall r > 0 \ \ (B(x_0, r) \cap E \neq \emptyset \ \wedge \ B(x_0, r) \backslash E \neq \emptyset)\}$.

(d) X is the union of the pairwise disjoint sets $int(E), \partial(E), ext(E)$.

Exterior, interior, and boundary points: Examples

Example 1: Let $(X, d) = (\mathbb{R}^2, d_{\ell^2})$, and let $E = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \leq 25\}$. Then:

- $int(E) = B_{(\mathbb{R}^2, d_{\ell^2})}((0,0), 5)\}$ is the open circular disc with radius 5 that is centered at the origin.
- $\partial(E) = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 = 25\}$ is the circle with radius 5 that is centered at the origin.
- $ext(E) = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 > 25\}$ is the set of all points in $\mathbb{R}^2$ whose Euclidean distance from the origin is larger than 5.

Example 2: Let $(X, d) = (X, d_{disc})$ and let $E \subseteq X$. Then:

- $int(E) = E$.
- $\partial(E) = \emptyset$.
- $ext(E) = X \backslash E$.

Exterior, interior, and boundary points: More Examples

Example 3: Let $(X, d) = (\mathbb{R}, d_{\ell^2})$, and let $E = \mathbb{Q}$.

Question L41.3: Find $int(E)$, $\partial(E)$, and $ext(E)$.

Here every open ball $B(x_0, r)$ is a non-degenerate interval $(x_0 - r, x_0 + r)$ that contains both elements of E (rational numbers) and elements of $X \backslash E$ (irrational numbers).

Thus $int(E) = ext(E) = \emptyset$ and $\partial E = \mathbb{R}$.

Example 4: Let (X, d) be any metric space.

Question L41.4: Find $int(X)$, $int(\emptyset)$, $\partial(X)$, $\partial(\emptyset)$, $ext(X)$, and $ext(\emptyset)$.

- $int(X) = ext(\emptyset) = X$.
- $int(\emptyset) = ext(X) = \emptyset$.
- $\partial(X) = \partial(\emptyset) = \emptyset$.

Definition 1.2.9: (Closure) Let (X, d) be a metric space, let $E \subseteq X$, and let $x_0 \in X$. We say that x_0 is an adherent point of E if for every radius $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E. The set of all adherent points of E is called the closure of E and is denoted by $\overline{E}$.

This definition of an adherent point rephrases the one on the first slide in the terminology of open balls. It then follows that the closure of E is the set of all limits of convergent sequences of elements of E (Proposition 1.2.10).

We also get the following characterization immediately from Proposition L41.1:

Corollary 1.2.11: Let (X, d) be a metric space, and let $E \subseteq X$. Then $\overline{E} = int(E) \cup \partial(E) = X \backslash ext(E)$.

Open and closed sets

Definition 1.2.12: (Open and closed sets) Let (X, d) be a metric space, and let $E \subseteq X$. We say that E is closed if it contains all of its boundary points, i.e., $\partial(E) \subseteq E$. We say that E is open if it contains none of its boundary points, i.e., $\partial(E) \cap E = \emptyset$.
If E contains some of its boundary points but not others, then it is neither open nor closed.

- To make sense of this definition, notice that for $a, b \in \mathbb{R}$ with $a < b$, for the usual metric on $\mathbb{R}$ we have $\partial((a, b)) = \partial([a, b)) = \partial((a, b]) = \partial([a, b]) = \{a, b\}$.
- It follows from Corollary 1.2.11 that E is closed iff $\overline{E} = E$.
- It follows from Proposition L41.1 that E is open iff $int(E) = E$.
- Note that the empty set $\emptyset$ and the entire space X are always simultaneously closed and open (see Example 4).
- With respect to the discrete metric d_{disc} on X, every subset $E \subseteq X$ is simultaneously closed and open (see Example 2).

Basic properties of open and closed sets

Proposition 1.2.15: Let (X, d) be a metric space.

(a) Let $E \subseteq X$. Then E is open if, and only if, $E = int(E)$.
In other words, E is open iff $\forall x \in E \, \exists r > 0 \;\; B(x, r) \subseteq E$.

(b) Let $E \subseteq X$. Then E is closed if, and only if, E contains all its adherent points. In other words, E is closed if, and only if, for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n\to\infty} x_n$ lies in E.

(c) For any $x_0 \in X$ and $r > 0$, the ball $B(x_0, r)$ is an open set. The set $\{x \in X : \; d(x, x_0) \leq r\}$ is a closed set. (This set is sometimes called the closed ball of radius r centered at x_0.)

(d) Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.

(e) A set $E \subseteq X$ is open iff the complement $X \backslash E$ is closed.

We have already shown (a) and the first part of (b).

The second part of (b) follows from (Proposition 1.2.10).

Since every sequence in $\{x_0\}$ is constant, (d) follows from (b).

Part (e) follows from the observation that for every $E \subseteq X$ we have $int(E) = ext(X \backslash E)$ and $\partial(E) = \partial(X \backslash E)$.

More properties of open and closed sets

Proposition 1.2.15: Let (X, d) be a metric space.

(f) If $E_1, \dots, E_n$ are a finite collection of open sets in X, then $E_1 \cap \dots \cap E_n$ is also open. If $F_1, \dots, F_n$ are a finite collection of closed sets in X, then $F_1 \cup \dots \cup F_n$ is also closed.

(g) If $\{E_\alpha\}_{\alpha \in I}$ is a collection of open sets in X, then the union $\bigcup_{\alpha \in I} E_\alpha = \{x \in X : \ \exists \alpha \in I \ x \in E_\alpha\}$ is also open. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then the intersection $\bigcap_{\alpha \in I} F_\alpha = \{x \in X : \ \forall \alpha \in I \ x \in E_\alpha\}$ is also closed.

(h) If $E \subseteq X$, then $int(E)$ is the largest open subset of E; in other words, $int(E)$ is open, and given any open set $V \subseteq E$, we have $V \subseteq int(E)$. Similarly, $\overline{E}$ is the smallest closed superset of E; in other words, $\overline{E}$ is closed, and given any closed set $K \supseteq E$, we have $K \supseteq \overline{E}$.

Notice that in part (f) we consider finite collections of subsets of X, whereas in part (g) the index set I could be finite, countable, or uncountable.