Lecture 42: Relative topology

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Companion to Advanced Calculus

Restricting the metric to a subset of X

Let (X, d) be any metric space, and let $Y \subseteq X$. Then we can restrict the metric function $d : X \times X \to [0, +\infty)$ to the subset $Y \times Y$ of $X \times X$ to create a restricted metric function $d \restriction Y \times Y : Y \times Y \to [0, +\infty)$ of Y.

This is called the metric on Y *induced* by the metric d on X.

Question L42.1: How do we know $d \restriction Y \times Y$ is a metric?

In order to see that $d \restriction Y \times Y$ is indeed a metric, it suffices to notice that all conditions that we need to verify depend only on values that d takes on pairs $(x, y) \in Y \times Y$. They must be satisfied since d is a metric on X.

Subspaces of a metric space (X, d)

The pair $(Y, d \upharpoonright Y \times Y)$ is a metric space and is known *the subspace of (X, d) induced by Y.*

When the meaning clear from the context, we often write simply (Y, d) instead of $(Y, d \upharpoonright Y \times Y)$.

Note that while $d \upharpoonright Y \times Y$ is uniquely determined by Y and d, we cannot in general infer d from its restriction $d \upharpoonright Y \times Y$:

Example L42.1: If we treat the real line as the x-axis, then $(\mathbb{R}, d_{\ell^2})$ becomes a subspace of $(\mathbb{R}^2, d_{\ell^1})$, $(\mathbb{R}^2, d_{\ell^2})$, and of $(\mathbb{R}^2, d_{\ell^\infty})$.

Topological properties of subspaces

Example L42.2: Let d be the usual metric on $\mathbb{R}$, let $X := \mathbb{R}$, and let $Y := \mathbb{Z}$. Then (Y, d) is a subspace of (X, d).

Question L42.2: Is $\mathbb{N}$ an open subset?

Not in the space (X, d). But in the space (Y, d), the set $\mathbb{N}$ is open.

We can see from the last example that a set $E \subseteq Y \subseteq X$ can have very different topological properties depending on whether E is considered in the metric space (X, d) or in the subspace $(Y, d \restriction Y \times Y)$.

To clarify this distinction, we make a definition.

Definition 1.3.3: (Relative topology). Let (X, d) be a metric space, and let $E \subseteq Y \subseteq X$. We say that E is *relatively open with respect to Y* if it is open in the metric subspace $(Y, d \restriction Y \times Y)$. Similarly, we say that E is *relatively closed with respect to Y* if it is closed in the metric space $(Y, d \restriction Y \times Y)$.

A characterization of relatively open/closed sets

In Example L42.2, the set $\mathbb{N}$ is both closed in $\mathbb{R}$ and relatively closed with respect to $\mathbb{Z}$, not open in $\mathbb{R}$, but relatively open with respect to $\mathbb{Z}$.

For a subspace Y of X, the phrases "open in Y" and "relatively open with respect to Y" actually mean the same thing; similarly "closed in Y" and "relatively closed with respect to Y" are synonymous. But we often add "relatively" for emphasis and clarity.

The relationship between open (or closed) sets in X, and relatively open (or relatively closed) sets in Y, is characterized as follows.

Proposition 1.3.4: Let (X, d) be a metric space, and let $E \subseteq Y \subseteq X$. Then:

(a) E is relatively open with respect to Y if, and only if, $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.

(b) E is relatively closed with respect to Y if, and only if, $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.

A characterization of relatively open/closed sets: Proof

Proposition 1.3.4: Let (X, d) be a metric space, and let $E \subseteq Y \subseteq X$. Then:

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(b) E is relatively closed with respect to Y if, and only if, $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.

The proof of part (a), given in the textbook, is based on the observation that E as above is relatively open with respect to Y if, and only if, for every $x \in E$ there exists an $r_x > 0$ such that the open ball $B_{(Y,d)}(x, r_x) \subseteq E$.

By definition, $B_{(Y,d)}(x, r_x) = B_{(X,d)}(x, r_x) \cap Y$.

The set V can then be taken as the union $V := \bigcup_{x \in E} B_{(X,d)}(x, r_x)$ of the corresponding open balls in X.