Lecture 43: Cauchy sequences and complete metric spaces

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Companion to Advanced Calculus

Cauchy sequeces in metric spaces: Definition

Definition 1.4.6: (Cauchy sequences) Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). We say that this sequence is a *Cauchy sequence* if, and only if,
$\forall \varepsilon > 0 \, \exists N \geq m \, \forall j, k \geq N \;\; d(x^{(j)}, x^{(k)}) < \varepsilon$.

The above definition generalizes the one that is already familiar to us for the special case of sequences of real numbers, except that the textbook phrases it in terms of "$< \varepsilon$" rather than "$\leq \varepsilon$." We have already seen that we get equivalent definitions when we make this substitution.

Question L43.1: Which sequences are Cauchy sequences in a discrete metric space (X, d_{disc})?

Convergent sequences are Cauchy

Example 43.1: In any discrete metric space (X, d_{disc}), a sequence $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence if, and only if, $\exists N \geq m \, \forall j, k \geq N \;\; d(x^{(j)}, x^{(k)}) < 1$, which in view of the definition of d_{disc} means means that $\exists N \geq m \, \forall j, k \geq N \;\; x^{(j)} = x^{(k)}$, so that the sequence is eventually constant.

Recall that this is the case if, and only if, $(x^{(n)})_{n=m}^{\infty}$ is convergent in (X, d_{disc}).

The following lemma also generalizes the "if"-direction of the above observation to arbitrary metric spaces:

Lemma 1.4.7: (Convergent sequences are Cauchy sequences) Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence.

The proof of Lemma 1.4.7

Lemma 1.4.7: (Convergent sequences are Cauchy sequences) Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence.

Question L43.2: How would you prove Lemma 1.4.7?

Proof: Let $(x^{(n)})_{n=m}^{\infty}$ and x_0 be as in the assumptions, and let $\varepsilon > 0$.

We need to find $N \geq m$ such that $\forall j, k \geq N \;\; d(x^{(j)}, x^{(k)}) < \varepsilon$.

By assumption, $\forall \delta > 0 \, \exists N \geq m \, \forall n \geq N \;\; d(x^{(n)}, x_0) \leq \delta$.

In particular, let $N \geq m$ be such that $\forall j, k \geq N \;\; d(x^{(n)}, x_0) \leq \frac{\varepsilon}{3}$.

By the triangle inequality and symmetry, for all $j, k \geq N$ we then have

$d(x^{(j)}, x^{(k)}) \leq d(x^{(j)}, x_0) + d(x_0, x^{(k)}) =$
$d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$, as required. $\square$

The converse of Lemma 1.4.7 is not true in every metric space; as we have seen in Chapter 4, there are Cauchy sequences on $\mathbb{Q}$ with the usual metric that do not converge (to any $x_0 \in \mathbb{Q}$). But we have also seen that the converse of Lemma 1.4.7 is true in $\mathbb{R}$ with the usual metric, as well as in all discrete metric spaces. This motivates the following definition:

Definition 1.4.10: (Complete metric spaces) A metric space (X, d) is said to be *complete* if, and only if, every Cauchy sequence in (X, d) is in fact convergent in (X, d).

Thus $\mathbb{Q}$ with the usual metric is not complete, $\mathbb{R}$ with the usual metric is, and any discrete metric space (X, d_{disc}) is also complete.

Let us recall that in MATH4/5301 we constructed the set $\mathbb{R}$ of real numbers in such a way that for the usual metric d, every $x \in \mathbb{R}$ is the limit of a Cauchy sequence in the subspace $(\mathbb{Q}, d)$.

This construction can be generalized: Whenever (Y, d_Y) is a metric space, then it is a subspace of a complete metric space (X, d_X) such that every $x \in X$ is the limit of a Cauchy sequence of elements of Y. The space (X, d_X) is called the *completion* of (Y, d_Y).

Exercise 1.4.8 in the textbook gives details of this construction, but we will omit them here.

Other important examples of complete metric spaces include the spaces $(\mathbb{R}^n, d_{\ell^1}), (\mathbb{R}^n, d_{\ell^2}), (\mathbb{R}^n, d_{\ell^\infty})$ for $n \geq 2$, and $(\ell^1, d_{\ell^1}), (\ell^2, d_{\ell^2}), (\ell^\infty, d_{\ell^\infty})$.

All of these examples are *Banach spaces,* that is, vector spaces where the distance is induced by a norm and gives the metric space the structure of a complete space.

A proof that a given metric space (X, d) is complete usually proceeds in three steps:

Step 1: We need to pick, for a given Cauchy sequence in (X, d) a candidate x_0 for its limit.

Step 2: We need to show that x_0 is actually an element of X.

Step 3: We need to show that x_0 is actually the limit of the Cauchy sequence.

Sometimes step 2 is trivial, and sometimes it is more convenient to switch the order of steps 2 and 3.

Proving that (X, d) is a Banach space, Example

Let us illustrate this procedure for the Banach spaces described above. So let (X, d) be one of these spaces.

Consider a Cauchy sequence $(x^{(n)})_{n=m}^{\infty}$ in this space.

For every $n \geq m$, let $x_i^{(n)}$ denote the i^{th} coordinate of $x^{(n)}$, where i ranges from 1 to n if $X = \mathbb{R}^n$ and $i \in \mathbb{N}$ if X is one of the spaces $\ell^1, \ell^2, \ell^\infty$.

For Step 1, notice that for all metrics d under consideration and for all i, j, k the inequality $d(x^{(j)}, x^{(k)}) \geq |x_i^{(j)} - x_i^{(k)}|$ holds. Thus for each relevant i, the sequence $(x_i^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence of real numbers, and is therefore convergent.

We let $x_i := \lim_{n\to\infty} x_i^{(n)}$ for each relevant i, and let x denote the vector with coordinates x_i, where i ranges from 1 to n if $X = \mathbb{R}^n$ and $i \in \mathbb{N}$ if X is one of the spaces $\ell^1, \ell^2, \ell^\infty$. This is our candidate for the limit of our Cauchy sequence.

Proving that (X, d) is a Banach space, Example, continued

Now consider the case when $X = \mathbb{R}^n$ for some positive natural number n. Then it follows immediately form our choice of x that $x \in X$, so that Step 2 becomes trivial.

We have seen in Module 44 that analogue of Proposition 1.1.18 fails in the spaces $\ell^1, \ell^2, \ell^\infty$, and we will need to work a little harder in these cases to complete Step 2. We will return to this issue in Module 46.

For Step 3, let us again focus on the case when $X = \mathbb{R}^n$ for some positive natural number n. Then it follows from our construction that $(x^{(n)})_{n=m}^\infty$ and x are as in Proposition 1.1.18(d), and parts (a)–(c) of the same proposition tell us that $x = \lim_{n\to\infty} x^{(n)}$ with respect to each of the metrics $d_{\ell^1}, d_{\ell^2}, d_{\ell^\infty}$.

We will discuss the proofs for the other three spaces in Module 46.

Subspaces of complete spaces

Question L43.3: Suppose Y is a subspace of a complete metric space. Is then Y always complete?

No. Consider, for example, $\mathbb{Q} \subseteq \mathbb{R}$ with the usual metric.

However, closed subspaces of complete metric spaces are always complete:

Proposition 1.4.12: Let (X, d) be a metric space, and let $(Y, d \restriction Y \times Y)$ be a subspace of (X, d).

(a) If $(Y, d \restriction Y \times Y)$ is complete, then Y must be closed in X.

(b) Conversely, suppose that (X, d) is a complete metric space, and $Y \subseteq X$ is closed.
Then the subspace $(Y, d \restriction Y \times Y)$ is also complete.

Proof: For Part (b), we observe that every Cauchy sequence in $(Y, d \restriction Y \times Y)$ is also a Cauchy sequence in X, and therefore must have a limit in X. But if Y is closed, then this limit must be an element of Y.

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(b) Conversely, suppose that (X, d) is a complete metric space, and $Y \subseteq X$ is closed.
Then the subspace $(Y, d \restriction Y \times Y)$ is also complete.

Question L43.4: How would you prove Part (a)?

Proof: For Part (a), assume that $(x^{(n)})_{n=m}^{\infty}$ is a sequence of elements of Y that converges to some $x_0 \in X$. By Lemma 1.4.7, $(x^{(n)})_{n=m}^{\infty}$ is then a Cauchy sequence of elements of Y, and converges to some $y_0 \in Y$. Thus $x_0 = y_0 \in Y$, and it follows that Y is a closed subset of X. $\square$