**Definition 1.4.6:** (Cauchy sequences) Let \((x^{(n)})_{n=m}^\infty\) be a sequence of points in a metric space \((X, d)\). We say that this sequence is a **Cauchy sequence** if, and only if,
\[
\forall \varepsilon > 0 \exists N \geq m \forall j, k \geq N \; d(x^{(j)}, x^{(k)}) < \varepsilon.
\]

The above definition generalizes the one that is already familiar to us for the special case of sequences of real numbers, except that the textbook phrases it in terms of “\(< \varepsilon\)” rather than “\(\leq \varepsilon\).” We have already seen that we get equivalent definitions when we make this substitution.

**Question L43.1:** Which sequences are Cauchy sequences in a discrete metric space \((X, d_{disc})\)?
Example 43.1: In any discrete metric space \((X, d_{\text{disc}})\), a sequence \((x^{(n)})_{n=m}^{\infty}\) is a Cauchy sequence if, and only if,

\[\exists N \geq m \forall j, k \geq N \ d(x^{(j)}, x^{(k)}) < 1,\]

which in view of the definition of \(d_{\text{disc}}\) means means that

\[\exists N \geq m \forall j, k \geq N \ x^{(j)} = x^{(k)},\]

so that the sequence is eventually constant.

Recall that this is the case if, and only if, \((x^{(n)})_{n=m}^{\infty}\) is convergent in \((X, d_{\text{disc}})\).

The following lemma also generalizes the “if”-direction of the above observation to arbitrary metric spaces:

Lemma 1.4.7: (Convergent sequences are Cauchy sequences) Let \((x^{(n)})_{n=m}^{\infty}\) be a sequence in \((X, d)\) which converges to some limit \(x_0\). Then \((x^{(n)})_{n=m}^{\infty}\) is also a Cauchy sequence.
Lemma 1.4.7: (Convergent sequences are Cauchy sequences) Let 
\((x^{(n)})_{n=m}^{\infty}\) be a sequence in \((X, d)\) which converges to some limit \(x_0\). Then \((x^{(n)})_{n=m}^{\infty}\) is also a Cauchy sequence.

Question L43.2: How would you prove Lemma 1.4.7?

Proof: Let \((x^{(n)})_{n=m}^{\infty}\) and \(x_0\) be as in the assumptions, and let \(\varepsilon > 0\).

We need to find \(N \geq m\) such that \(\forall j, k \geq N\) \(d(x^{(j)}, x^{(k)}) < \varepsilon\).

By assumption, \(\forall \delta > 0 \exists N \geq m \forall n \geq N\) \(d(x^{(n)}, x_0) \leq \delta\).

In particular, let \(N \geq m\) be such that \(\forall j, k \geq N\) \(d(x^{(n)}, x_0) \leq \frac{\varepsilon}{3}\).

By the triangle inequality and symmetry, for all \(j, k \geq N\) we then have

\[
d(x^{(j)}, x^{(k)}) \leq d(x^{(j)}, x_0) + d(x_0, x^{(k)}) = d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,
\]

as required. □
The converse of Lemma 1.4.7 is not true in every metric space; as we have seen in Chapter 4, there are Cauchy sequences on \( \mathbb{Q} \) with the usual metric that do not converge (to any \( x_0 \in \mathbb{Q} \)). But we have also seen that the converse of Lemma 1.4.7 is true in \( \mathbb{R} \) with the usual metric, as well as in all discrete metric spaces. This motivates the following definition:

**Definition 1.4.10:** (Complete metric spaces) A metric space \( (X, d) \) is said to be **complete** if, and only if, every Cauchy sequence in \( (X, d) \) is in fact convergent in \( (X, d) \).

Thus \( \mathbb{Q} \) with the usual metric is not complete, \( \mathbb{R} \) with the usual metric is, and any discrete metric space \( (X, d_{\text{disc}}) \) is also complete.
Let us recall that in MATH4/5301 we constructed the set $\mathbb{R}$ of real numbers in such a way that for the usual metric $d$, every $x \in \mathbb{R}$ is the limit of a Cauchy sequence in the subspace $(\mathbb{Q}, d)$.

This construction can be generalized: Whenever $(Y, d_Y)$ is a metric space, then it is a subspace of a complete metric space $(X, d_X)$ such that every $x \in X$ is the limit of a Cauchy sequence of elements of $Y$. The space $(X, d_X)$ is called the \textit{completion} of $(Y, d_Y)$.

Exercise 1.4.8 in the textbook gives details of this construction, but we will omit them here.
Banach spaces

Other important examples of complete metric spaces include the spaces \((\mathbb{R}^n, d_{\ell^1}), (\mathbb{R}^n, d_{\ell^2}), (\mathbb{R}^n, d_{\ell^\infty})\) for \(n \geq 2\), and \((\ell^1, d_{\ell^1}), (\ell^2, d_{\ell^2}), (\ell^\infty, d_{\ell^\infty})\).

All of these examples are Banach spaces, that is, vector spaces where the distance is induced by a norm and gives the metric space the structure of a complete space.

A proof that a given metric space \((X, d)\) is complete usually proceeds in three steps:

**Step 1:** We need to pick, for a given Cauchy sequence in \((X, d)\) a candidate \(x_0\) for its limit.

**Step 2:** We need to show that \(x_0\) is actually an element of \(X\).

**Step 3:** We need to show that \(x_0\) is actually the limit of the Cauchy sequence.

Sometimes step 2 is trivial, and sometimes it is more convenient to switch the order of steps 2 and 3.
Let us illustrate this procedure for the Banach spaces described above. So let \((X, d)\) be one of these spaces.

Consider a Cauchy sequence \((x^{(n)})_{n=m}^\infty\) in this space.

For every \(n \geq m\), let \(x_i^{(n)}\) denote the \(i^{th}\) coordinate of \(x^{(n)}\), where \(i\) ranges from 1 to \(n\) if \(X = \mathbb{R}^n\) and \(i \in \mathbb{N}\) if \(X\) is one of the spaces \(\ell^1, \ell^2, \ell^\infty\).

For Step 1, notice that for all metrics \(d\) under consideration and for all \(i, j, k\) the inequality \(d(x^{(j)}, x^{(k)}) \geq |x_i^{(j)} - x_i^{(k)}|\) holds. Thus for each relevant \(i\), the sequence \((x_i^{(n)})_{n=m}^\infty\) is a Cauchy sequence of real numbers, and is therefore convergent.

We let \(x_i := \lim_{n \to \infty} x_i^{(n)}\) for each relevant \(i\), and let \(x\) denote the vector with coordinates \(x_i\), where \(i\) ranges from 1 to \(n\) if \(X = \mathbb{R}^n\) and \(i \in \mathbb{N}\) if \(X\) is one of the spaces \(\ell^1, \ell^2, \ell^\infty\). This is our candidate for the limit of our Cauchy sequence.
Now consider the case when $X = \mathbb{R}^n$ for some positive natural number $n$. Then it follows immediately form our choice of $x$ that $x \in X$, so that Step 2 becomes trivial.

We have seen in Module 44 that analogue of Proposition 1.1.18 fails in the spaces $\ell^1, \ell^2, \ell^\infty$, and we will need to work a little harder in these cases to complete Step 2. We will return to this issue in Module 46.

For Step 3, let us again focus on the case when $X = \mathbb{R}^n$ for some positive natural number $n$. Then it follows from our construction that $(x^{(n)})_{n=m}^\infty$ and $x$ are as in Proposition 1.1.18(d), and parts (a)–(c) of the same proposition tell us that $x = \lim_{n \to \infty} x^{(n)}$ with respect to each of the metrics $d_{\ell^1}, d_{\ell^2}, d_{\ell^\infty}$.

We will discuss the proofs for the other three spaces in Module 46.
Subspaces of complete spaces

**Question L43.3:** Suppose $Y$ is a subspace of a complete metric space. Is then $Y$ always complete?

No. Consider, for example, $\mathbb{Q} \subseteq \mathbb{R}$ with the usual metric.

However, closed subspaces of complete metric spaces are always complete:

**Proposition 1.4.12:** Let $(X, d)$ be a metric space, and let $(Y, d \restriction Y \times Y)$ be a subspace of $(X, d)$.

(a) If $(Y, d \restriction Y \times Y)$ is complete, then $Y$ must be closed in $X$.

(b) Conversely, suppose that $(X, d)$ is a complete metric space, and $Y \subseteq X$ is closed.

Then the subspace $(Y, d \restriction Y \times Y)$ is also complete.

**Proof:** For Part (b), we observe that every Cauchy sequence in $(Y, d \restriction Y \times Y)$ is also a Cauchy sequence in $X$, and therefore must have a limit in $X$. But if $Y$ is closed, then this limit must be an element of $Y$. 
Proposition 1.4.12: Let \((X, d)\) be a metric space, and let \((Y, d \restriction Y \times Y)\) be a subspace of \((X, d)\).

(a) If \((Y, d \restriction Y \times Y)\) is complete, then \(Y\) must be closed in \(X\).

(b) Conversely, suppose that \((X, d)\) is a complete metric space, and \(Y \subseteq X\) is closed. Then the subspace \((Y, d \restriction Y \times Y)\) is also complete.

Question L43.4: How would you prove Part (a)?

Proof: For Part (a), assume that \((x^{(n)})_{n=m}^{\infty}\) is a sequence of elements of \(Y\) that converges to some \(x_0 \in X\). By Lemma 1.4.7, \((x^{(n)})_{n=m}^{\infty}\) is then a Cauchy sequence of elements of \(Y\), and converges to some \(y_0 \in Y\). Thus \(x_0 = y_0 \in Y\), and it follows that \(Y\) is a closed subset of \(X\). □