Lecture 44: Continuous functions on metric spaces

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Companion to Advanced Calculus

Continuity on metric spaces: Definition

Let $X \subseteq \mathbb{R}$, let $f : X \to \mathbb{R}$ and let $x_0 \in X$. Recall that f is continuous at x_0 if, and only if,

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \ \ (|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon).$$

If d denotes the usual metric on $\mathbb{R}$, then this can be written as

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \ \ (d(x, x_0) < \delta \Longrightarrow d(f(x), f(x_0)) < \varepsilon).$$

This definition can be generalized to functions on arbitrary metric spaces as follows:

Definition 2.1.1: (Continuous functions) Let (X, d_X) be a metric space, let (Y, d_Y) be another metric space, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is continuous at x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. In symbols:

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \ \ (d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon).$$

We say that f is continuous iff it is continuous at every point $x \in X$.

Example 1

Notice that it follows immediately from this definition that if $f : X \to Y$ is continuous at $x_0 \in X$, then the restriction $f \restriction K$ to any $K \subseteq X$ with x_0 is also continuous at x_0.

Example 1: Let X be any set, let (Y, d_y) be a metric space, let $x_0 \in X$, and let $f : X \to Y$ be any function. If we consider X with the discrete metric d_{disc}, then f will always be continuous at x_0.

Question L44.1: Let $\varepsilon > 0$. How should we choose $\delta > 0$ so that $\forall x \in X \ \ (d_{disc}(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon)$?

Here any positive $\delta \leq 1$ will work, since $d_{disc}(x, x_0) < 1$ implies that $x = x_0$.

Continuity and convergent sequences

Recall that if $x_0 \in X \subseteq \mathbb{R}$ and if $f : X \to \mathbb{R}$, then f is continuous at x_0 (with respect to the usual metric on $\mathbb{R}$) if and only if for every sequence $(x_n)_{n=1}^{\infty}$ that converges to x_0 the sequence of function values $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$.

This result generalizes as follows:

Theorem 2.1.4: (Continuity preserves convergence) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X.
Then the following three statements are logically equivalent:

(a) f is continuous at x_0.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X that converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

(c) For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subseteq V$.

About the proof of Theorem 2.1.4

The equivalence between parts (a) and (b) of this theorem can be proved in the exact same way as for the special case of functions from $X \subseteq \mathbb{R}$ into $\mathbb{R}$ that we have already seen.

For the implication $(c) \Longrightarrow (a)$, let f, x_0 be as in the assumption, and consider $\varepsilon > 0$. Let $V := B(f(x_0), \varepsilon)$.

Then V is open in Y, so that by (c) there exists an open $U \subseteq X$ such that $x_0 \in U$ and $f(U) \subseteq V$, which means that $\forall x \in U \; f(x) \in V$.

By our choice of V, then $\forall x \in U \; d_Y(f(x), f(x_0)) < \varepsilon$.

Question L44.2: Why does there exist $\delta > 0$ such that $\forall x \in X \;\; (d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon)$?

Because if $x_0 \in U$ and U is open, then there exists an open ball $B(x_0, r) \subseteq U$ with radius $r > 0$. We can then take $\delta := r > 0$.

We will prove the implication $(a) \Longrightarrow (c)$ in Module 49.

A characterization of continuous functions

We now obtain the following characterization of continuous functions between metric spaces, that is, functions that are continuous at every point in their domains:

Theorem 2.1.5: Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following four statements are equivalent:

(a) f is continuous.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y .

(c) Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : \ f(x) \in V\}$ is an open set in X.

(d) Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : \ f(x) \in F\}$ is a closed set in X.

About the proof of Theorem 2.1.5

The equivalence between parts (a) and (b) follows immediately from Theorem 2.1.4(a)(b) and the definition of continuous function as functions that are continuous at each x in their domains.

For the implication $(a) \Longrightarrow (c)$, assume $f : X \to V$ is continuous let $V \subseteq Y$, and $x \in f^{-1}(V)$. Then $f(x) \in V$, and f is continuous at x.

By Theorem 2.1.4(c), there exists an open $U_x \subset X$ with $x \in U_x$ such that $f(U_x) \subseteq V$. But then $\{x\} \subseteq U_x \subseteq f^{-1}(U_x) \subseteq f^{-1}(V)$. Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{x\} = \bigcup_{x \in f^{-1}(V)} U_x \subseteq f^{-1}(V)$. Since the union of any collection of open sets is open, we conclude that $f^{-1}(V)$ is an open subset of X, which proves part (c).

Now assume part(c) is true, and let F be a closed subset of Y. Then $V := Y \backslash F$ is open and $F = Y \backslash V$, so that $f^{-1}(F) = X \backslash f^{-1}(V)$. By part (c), $f^{-1}(V)$ is open, so that its complement $f^{-1}(F)$ is closed in Y, and part (d) follows.

We will conclude the proof of Theorem 2.1.5 in Module 49.

Continuity of compositions

As in the special case of continuous functions on subsets of $\mathbb{R}$, continuity is preserved under compositions:

Corollary 2.1.7: (Continuity is preserved by composition) Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.

(a) If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $g \circ f(x) := g(f(x))$, is continuous at x_0.

(b) If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.

Note that part (b) immediately follows from part (a).

Proof of Corollary 2.1.7(a)

For the proof of part (a), let f be as in the assumption, and let $x_0 \in X$.

We will prove the property of Theorem 2.1.4(c).

Specifically we need to show:

For every open set $V \subseteq Z$ such that $g(f(x_0)) \in V$ there exists an open set $U \subseteq X$ such that $x_0 \in U$ and $g \circ f(U) \subseteq V$.

By Theorem 2.1.4(c) applied to the function g at $f(x_0)$, there exists an open set $W \subseteq Y$ such that $f(x_0) \in W$ and $g(W) \subseteq V$.

By Theorem 2.1.4(c) again, now applied to the function f at x_0, there exists an open set $U \subseteq X$ such that $x_0 \in U$ and $f(U) \subseteq W$.
Then $g \circ f(U) = g(f(U)) \subseteq g(W) \subseteq V$, as required. $\square$

Examples 2 and 3

The analogues of Theorem 2.1.5(c),(d) for images fail:

Example 2: Let (X, d) be any metric space, and let $f : X \to \mathbb{R}$ be the constant function defined by $f(x) := 2020$ for all $x \in X$.

Then f is continuous, since for every sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X with $\lim_{n\to\infty} x^{(n)} = x_0$ the sequence $\lim_{n\to\infty} f(x^{(n)}) = f(x_0) = 2020$.

But X is open X, while $f(x) = \{2020\}$ is not an open subset of $\mathbb{R}$.

Example 3: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \arctan x$ for all $x \in \mathbb{R}$.

Then f is continuous, $F := \mathbb{R}$ is closed in $\mathbb{R}$,

but $f(F) = (-\frac{\pi}{2}, \frac{\pi}{2})$ is not closed in $\mathbb{R}$.

Continuity for different metrics

Remark L44.1: Although we usually don't specify the metrics in our terminology, it is important to realize that if d_x and d'_x are two different metrics on X or d_y and d'_y are two different metrics on Y, then continuity of a given function f depends on which metric we are talking about.

Question L44.3: Let $f : \mathbb{R} \to \mathbb{R}$. If we consider the discrete metric d_{disc} on the domain of f and the usual metric d on the range of f, what can we deduce about continuity of f?

It will always be continuous; see Example 1.

Question L44.4: If $f(x) := x$ for all x in $\mathbb{R}$ and we consider the usual metric d on the domain and the discrete metric d_{disc} on the range, what can we deduce about the continuity of f?

It will be discontinuous at every x_0 in $\mathbb{R}$ since there are lots of convergent sequences $(x^{(n)})_{n=1}^{\infty}$ in $(\mathbb{R}, d)$ with limit x_0 that are not eventually constant, but $(f(x^{(n)}))_{n=1}^{\infty}$ converge to $f(x_0)$ in $(\mathbb{R}, d_{disc})$ only if $(x^{(n)})_{n=1}^{\infty} = (f(x^{(n)}))_{n=1}^{\infty}$ is eventually constant.

Continuity for equivalent metrics

Recall that we consider two metrics d, d' on the same space X equivalent if every sequence of elements of X that is convergent with respect to one of these metrics is also convergent, to the same limit with respect to the other metric.

It follows from Theorem 2.1.4 that if d_x and d'_x are two equivalent metrics on X and d_y and d'_y are two equivalent metrics on Y, any given function $f : X \rightarrow Y$ is continuous at any given $x_0 \in X$ with respect to d_x and d_y if and only if, it is continuous at x_0 with respect to d'_x and d'_y.

In particular, when we study continuity of functions $f : X \rightarrow \mathbb{R}^m$, where $X \subseteq \mathbb{R}^n$, then it does not matter whether on the domain and range we consider the Euclidean metric d_{ℓ^2}, the taxi-cab metric d_{ℓ^1}, or the sup-norm metric d_{ℓ^∞}, since they are all equivalent.

Example 4: Let $n \geq 1$, and let $i \in \{1, \dots, n\}$. Then the projection $\pi_i : \mathbb{R}^n \to \mathbb{R}$ on the i^{th} coordinate defined by $\pi_i\,(x_1, \dots, x_n) = x_i$ is continuous.

It now follows from continuity of compositions that if $f : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ then the projection $\pi_i \circ f$ is also continuous at x_0.

Example 5: Given two functions $f : X \to Y$ and $g : X \to Z$, one can define their direct sum $f \oplus g : X \to Y \times Z$ by $f \oplus g(x) := (f(x), g(x))$. This is the function taking values in the Cartesian product $Y \times Z$ whose first co-ordinate is $f(x)$ and whose second co-ordinate is $g(x)$.

For instance, if $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) := x^2 + 3$, and $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = 4x$, then $f \oplus g : \mathbb{R} \to \mathbb{R}^2$ is the function $f \oplus g(x) := (x^2 + 3, 4x)$.

Lemma 2.2.1: Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $f \oplus g : X \to \mathbb{R}^2$ be their direct sum. We give $\mathbb{R}^2$ the Euclidean metric.

(a) If $x_0 \in X$, then f and g are both continuous at x_0 iff $f \oplus g$ is continuous at x_0.

(b) f and g are both continuous iff $f \oplus g$ is continuous.

Lemma 2.2.2: The addition function $(x, y) \mapsto x + y$,
the subtraction function $(x, y) \mapsto x - y$,
the multiplication function $(x, y) \mapsto xy$,
the maximum function $(x, y) \mapsto max(x, y)$,
and the minimum function $(x, y) \mapsto min(x, y)$
are all continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$.
The division function $(x, y) \mapsto x/y$ is a continuous function from $\mathbb{R} \times (\mathbb{R}\backslash 0) = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ to $\mathbb{R}$.

Continuity of arithmetic operations

Corollary 2.2.3(a): Let (X, d) be a metric space, let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Let c be a real number. If $x_0 \in X$ and f and g are continuous at x_0, then the functions:
$f + g : X \to \mathbb{R}$ and $f - g : X \to R$,
$fg : X \to R$ and $cf : X \to R$,
$max(f, g) : X \to \mathbb{R}$ and $min(f, g) : X \to \mathbb{R}$
are also continuous at x_0.
If $\forall x \in X \;\; g(x) \neq 0$, then $f/g : X \to \mathbb{R}$ is also continuous at x_0.

Sketch of the proof: Let us prove here that if $f, g : X \to \mathbb{R}$ are continuous at x_0, then the function $f + g$ is also continuous.

Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) = x + y$.

Note that $f + g = h \circ (f \oplus g)$.

Thus the composition $h(f \oplus g)$ is also continuous at x_0.

The proofs of the other parts of Corollary 2.2.3(a) are similar. $\square$