Lecture 47: Real-analytic functions

Winfried Just
Department of Mathematics, Ohio University

Companion to Advanced Calculus

Real-analytic functions

Definition 4.2.1: (Real-analytic functions) Let $E \subseteq \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a function.

If a is an interior point of E, we say that *f is real-analytic at a* if there exists an open interval $(a - r, a + r) \subseteq E$ for some $r > 0$ such that there exists a power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a - r, a + r)$.

If E is an open set, and f is real analytic at every point a of E, we say that *f is real-analytic on E.*

We will see later where the name “real-analytic” comes from.

Example 1: The function $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(x - a)^n$ is real-analytic at $a = 0$. However, this function cannot be real-analytic at $a = 1$, since $f(1)$ is undefined for $f(x) = \frac{1}{1-x}$.
It can be shown that $f(x)$ of this example is real-analytic for every $a \neq 1$, that is, real analytic on $E := (-\infty, 1) \cup (1, \infty)$.

k-times differentiability and smoothness

Now suppose that f is given and real-analytic at a.
The question is how we can find the coefficients c_n of the series $\sum_{n=0}^{\infty} c_n(x-a)^n = f(x)$.
We make several observations.

Definition 4.2.4: (k-times differentiability) Let $E \subseteq \mathbb{R}$. We say a function $f : E \to \mathbb{R}$ is once differentiable on E iff it is differentiable. More generally, for any $k \geq 2$ we say that $f : E \to \mathbb{R}$ is k times differentiable on E, or just k times differentiable, iff f is differentiable, and f' is $k-1$ times differentiable.

If f is k times differentiable, we define the k^{th} derivative $f^{(k)} : E \to \mathbb{R}$ by the recursive rule $f^{(1)} := f'$, and $f^{(k)} = (f^{(k-1)})'$ for all $k \geq 2$.

We also define $f^{(0)} := f$ (this is f differentiated 0 times), and we consider every function to be zero times differentiable.

A function is said to be infinitely differentiable (or smooth) iff it is k times differentiable for every $k \geq 0$.

The k^{th} derivatives of real-analytic functions

You will recall from calculus that most "nice" functions, like polynomial functions, exponential functions, and trig functions, are k times differentiable for all $k \in \mathbb{N}$, that is, are infinitely differentiable. This property is shared by all real-analytic functions:

Proposition 4.2.6: (Real-analytic functions are k-times differentiable). Let $E \subseteq \mathbb{R}$, let a be an interior point of E, and let f be a function which is real-analytic at a and has the power series expansion $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (a-r, a+r)$ for some $r > 0$.

Then for every $k \geq 0$, the function f is k-times differentiable on $(a-r, a+r)$, and for each $k \geq 0$ and all $x \in (a-r, a+r)$:

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=0}^{\infty} c_{n+k}(n+1)(n+2)\dots(n+k)(x-a)^n \\ &= \sum_{n=0}^{\infty} c_{n+k}\frac{(n+k)!}{n!}(x-a)^n. \end{aligned}$$

Real-analytic functions are infinitely differentiable

Sketch of the proof: For $k = 1$, Proposition 4.2.6 follows immediately from Theorem 4.1.6. In this case we have $c_{n+1}\frac{(n+1)!}{n!} = c_{n+1}\frac{1 \cdots n\,(n+1)}{1 \cdots n} = c_{n+1}(n+1)$,
so that we get $f^{(1)}(x) = f'(x) = \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$.

For $k > 1$ the result can be proved by induction over k; we will do this in Module 58.

Corollary 4.2.7: (Real-analytic functions are infinitely differentiable) Let E be an open subset of $\mathbb{R}$,
and let $f : E \rightarrow \mathbb{R}$ be a real-analytic function on E.

Then f is infinitely differentiable on E.

Also, all derivatives of f are also real-analytic on E.

Not every infinitely differentiable real function is real-analytic though; we will explore an example in Module 58.

Finding the coefficients of the power series

If f is real-analytic, then we can use Proposition 4.2.6 to find the sequence of coefficients c_n. Here is how this works:

Assume $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (-r+a, a+r)$.

Then $f(a) = \sum_{n=0}^{\infty} c_n(a-a)^n = c_0 + \sum_{n=1}^{\infty} c_n(0)^n = c_0$.

Similarly: $f'(a) = \sum_{n=0}^{\infty}(n+1)c_{n+1}(a-a)^n = (0+1)c_1 = c_1$.

$f''(a) = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}(a-a)^n = 2c_2$. Thus $c_2 = \frac{f''(a)}{2}$.

And so on. Now we can prove by induction over k the following:

Corollary 4.2.10 (Taylor's formula) Let $E \subseteq \mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real-analytic at a and has the power series expansion $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (a-r, a+r)$ and some $r > 0$.

Then for any integer $k \geq 0$ we have $f^{(k)}(a) = k!c_k$.

In particular, for all x in $(a-r, a+r)$ we have Taylor's formula

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Taylor and Maclaurin series: An example

For example, we have shown that the function $f(x) := \ln(x+1)$ is real-analytic at $a := 0$, with $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ on $(-1, 1)$.

When we take derivatives at $a := 0$ we get:

$f^{(0)}(0) = \ln(1+0) = 0$

$f'(0) = \left(\frac{1}{(1+x)}\right)(0) = 1 = \frac{(-1)^{1+1}}{1}$

Question L47.1: What is $f''(0)$?

$$f''(0) = \left(\frac{-1}{(1+x)^2}\right)(0) = -1 = \frac{(-1)^{2+1}}{2} 2!$$

Similarly, $f^{(3)}(0) = \left(\frac{2}{(1+x)^3}\right)(0) = 2 = \frac{(-1)^{3+1}}{3} 3!$

And so on.

The power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of f around a. In the special case when $a := 0$, this is also called the Maclaurin series of f.

Uniqueness of power series

We have found that the Maclaurin series of $f(x) := \ln(1+x)$ is given by the formula $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$.

This example is an instance of a more general observation:

When $f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n$, on some interval $(-r+a, a+r)$, then the coefficients c_n must be the ones given by the Taylor series at a:

Corollary 4.2.12: (Uniqueness of power series) Let $E \subseteq \mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real-analytic at a. Suppose that f has two power series expansions

$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $f(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$

centered at a, each with a non-zero radius of convergence.

Then $c_n = d_n$ for all $n \geq 0$.

The power series for the natural exponential function

Now let us consider the function $f(x) := e^x$ on $\mathbb{R}$.
This function is real-analytic.

Let us compute the coefficients of its Maclaurin series
$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(e^x)^{(n)}(0)}{n!}(x-a)^n$.

Question L47.2: Find a formula for c_n.

Since $(e^x)' = e^x$ and $e^0 = 1$, we get $f^{(n)}(0) = 1$ for all $n \in \mathbb{N}$.
It follows that $c_n = \frac{1}{n!}$ for each $n \in \mathbb{N}$.

As we have mentioned, e^x is real-analytic.

Therefore $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.
One can show that this equality holds for all x in the interval of convergence of this series, which is the entire real line, as we have seen in Module 57.

The function(s) e^x and $\exp(x)$

The textbook takes a slightly different approach and actually defines a function $\exp(x) := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

This definition works for all $x \in \mathbb{R}$, since the power series has a radius of convergence $R = \infty$ as we saw in Module 57.

A different definition of the function e^x was given in Volume I of the textbook.

If we know that the function e^x is real-analytic, then it follows from Corollary 4.2.12 that $\exp(x)$ and e^x must be the same function.

However, we shouldn't take on faith that our familiar function e^x is actually real-analytic. This can be rigorously proved, but the proof goes beyond the scope of our course.