Lecture 4: Recursive definitions

Winfried Just
Department of Mathematics, Ohio University

Companion to Advanced Calculus

An familiar example

We probably all have seen the following definition:

$$
\begin{aligned}
a_0 &:= 1, \\
a_{n+1} &:= (n+1)a_n \qquad \text{for all natural numbers} \quad n.
\end{aligned}
\tag{1}
$$

Then $a_1 = a_{0+1} = (1)(1) = 1$, $a_2 = a_{1+1} = (2)(1) = 2$, $a_3 = a_{2+1} = (3)(2) = 6$, $a_4 = a_{3+1} = (4)(6) = 24$, and so on.

We can see that (1) rigorously defines $a_n = 1 \cdot 2 \cdot 3 \cdots n$ for all $n \geq 1$. The number a_n is called *n factorial* and usually denoted by $n!$ This notation is useful, because there is no formula that would express $n!$ in terms of arithmetic operations without using potentially ambiguous $\ldots$ (called ellipsis).

It is important for us to note that (1) does define $a_n = n!$ rigorously and unambiguously, without the need for an explicit formula. It is an example of a *recursive definition* or, more generally, *recursive construction.*

Recursion *vs.* induction

People often say that numbers a_n like the ones on the previous slide are "defined by induction." This is not correct. *Induction* is a method for proving theorems, *recursion* is a method for constructing or defining mathematical objects.

However, in a sense recursion and induction are two sides of the same coin: When a sequence of numbers a_n is defined by recursion, then mathematical induction is usually the method of choice for proving that this sequence has certain properties.

For example, we can prove $P(n): \ n! > 0$ as follows:

Basic step: $0! = 1 > 0$

Inductive step: $P(k) \Longrightarrow P(k+1)$ for all natural numbers k:
Assume by induction that $k! > 0$. Then $(k+1)! = (k+1)k!$ is the product of two positive numbers, hence positive.

It now follows from the PMI that $n! > 0$ for all natural numbers n.

Recursive definitions in Peano arithmetic

We will use recursion quite a lot in this course. In particular, recursive definitions are permissible in Peano arithmetic in view of the following result:

Proposition 2.1.16: (Recursive definitions) Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.

Question L4.1: Find the number c in our definition of n!

Here $c = 1$.

Question L4.2: Find the function f_n in our definition of n!

Here we could use the function defined by $f_n(x) = (n + 1)x$.

Recursive definition of addition

We can now define addition in Peano arithmetic as follows:

Definition 2.2.1: (Addition of natural numbers) Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n++$ to m by defining $(n++) + m := (n + m)++$.

This definition fits into the framework of Proposition 2.1.16 if we think about it as recursively defining, for every natural number m separately, a sequence of numbers a_n^m as follows:

$a_0^m = m$ and $a_{n++}^m = a_n^m ++$.

We then can use our familiar notation $n + m$ for a_n^m.

However, it is not immediately clear that the axioms of Peano arithmetic imply that the operation that we have just defined satisfies the familiar properties of addition, like commutativity and associativity. This need to be rigorously proved. We will return to this issue in our next lecture.