

Lecture 6: Forming Sets

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Companion to Advanced Calculus

The goal of this lecture

While not the subject of this course, learning some set theory will be important for us for several reasons:

- Almost all of modern mathematics can be formalized in terms of set theory. In particular, all the mathematical concepts of analysis can be formalized as sets.
- Thus set theory provides an important *language* for our work in this course. We need to become fluent in this language if we want to study the concepts defined in this terminology.
- The study of set theory will give us an excellent opportunity to hone our skills in understanding and writing proofs while working with fairly simple concepts.

In this lecture we will begin our study of this language by introducing some elementary operations on sets and discuss some of their properties.

Axiomatic set theory

As its very name implies, set theory is also a mathematical theory.

Like Peano arithmetic, it can, and should, be developed as an axiomatic theory. In fact, since practically all of modern mathematics can be expressed in the language of set theory, such an axiomatic theory is powerful enough to serve as a foundation of modern mathematics as a whole.

Your textbook does develop set theory as an axiomatic theory. This has clearly the advantage of giving us some guidance about what we can take as given and what we need to prove. We will follow this approach only to a limited extent as familiarity with set theoretic language and some set theoretic tools is much more important for the study of advanced calculus than the details of its axiomatic development.

Naive set theory

Most axioms of set theory essentially boil down to assuring us that certain sets can be formed. Once the set of interest are formed though, the axioms are usually no longer needed to prove any properties of these sets.

So, for the most part we will work here in the framework of *naive set theory*. Here the mindset essentially is: “If a set can be defined or constructed from previously given sets, then it exists.”

We will see in Conversation 4 why this mindset of naive set theory is somewhat problematic and why axioms that explicitly permit us to form certain sets are needed.

However, the existence of the sets that are actually used in analysis is so uncontroversial that we can get away with the mindset of naive set theory and focus on the aspects of set theory that are crucial for studying this subject.

The undefined notions of axiomatic set theory

Definition 3.1.1: (Informal) We define a *set* A to be any unordered collection of objects, e.g., $\{3, 8, 5, 2\}$ is a set. If x is an *object*, we say that x is an *element of* A or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

This definition tells us right away what our **undefined notions** of axiomatic set theory will be:

- *Object*: Anything that we want to talk about.
- *Set*: Any collection of objects. The word “collection” here is just a device to help our intuition. Essentially, a sneaky way of “defining” a set without being *too obviously circular*. In set theory, “collection” and “set” are synonymous.
- *Membership relation*: \in .
The statement $\sim (x \in A)$ will be abbreviated as $x \notin A$.

In the beginning there were urelements

Intuitively, we can form a set by putting some objects into an abstract box that is denoted by $\{\dots\}$. For example, we can form a set $\{3, 4\}$ of the natural numbers 3 and 4.

We certainly want to talk about sets in set theory, so sets must be objects. We can put them into other boxes to form new sets.

Like this one: $\{3, 4, \{3, 4\}\}$.

In our version of set theory, natural numbers are not considered sets. Objects other than sets are sometimes called *urelements*. This is a funny concoction of the German “ur” (primordial, primeval) and the English “elements.” It gives you a good intuition of how sets are formed: Start with collecting some urelements, put them in a box called “set.” When you have several such boxes, you can put them again into a box, together with some urelements, perhaps. And then repeat as often as you want.

Like this: $\{3, 4, \{3, 4\}, \{3, 4, \{3, 4\}\}\}$.

When are two sets equal?

Definition 3.1.4: (Equality of sets) Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B , and every element y of B belongs also to A .

Question L6.1: Are the sets $A = \{1, 2, 3\}$ and $B = \{3, 2, 1, 3\}$ equal?

Yes. They have the same elements. They are listed in different order and one element of B is listed twice, but this does not matter in the sense of Definition 3.1.4.

Question L6.2: Are the sets $A = \{4, 3\}$ and $B = \{3, 4, \{3, 4\}\}$ equal?

No. The set $\{3, 4\}$ is an element of B , but not of A .

Sets are never equal to urelements

Note that the definition on the previous slide covers *only* equality of sets. For urelements a, b , objects that are not set, we need to assume that we know from external sources whether or not $a = b$.

Important fact: *A set is never equal to an urelement.*

Thus, for example, $\{7\} \neq 7$. This is because 7 is an urelement, while $\{7\}$ is a set whose only element is 7.

We will often orally refer to sets of the form $\{a\}$ as “*singleton a.*”

In the beginning there was nothing

Axiom 3.2: (Empty set) There exists a set \emptyset , known as *the empty set*, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

For example, if we let A be the set of all prime numbers that are divisible by 6 and if we let B be the set of all nondifferentiable polynomial functions, then both A and B are empty.

While A is by definition a set of natural numbers and B is by definition a set of functions, these sets contain the same elements (none). Thus $A = B$.

This is why does Axiom 3.2 talks about *the* empty set. In other words, all “empty sets” are the same set in the sense of our definition of equality of sets.

Think of the empty set as an empty box $\{\}$.

More precisely, as *the* empty box, as we distinguish between boxes only in terms of their contents, not in terms of their origins from particular mathematical constructions.

The union of two sets

Axiom 3.4: (Pairwise union) Given any two sets A, B , there exists a set $A \cup B$, called the *union* of A and B , whose elements consists of all the elements which belong to A or B or both.

In other words, for any object x ,

$$x \in A \cup B \iff (x \in A \vee x \in B).$$

Note how using the symbol “ \vee ” for the logical connective “or” immediately provides a visual cue for the interpretation of “ \cup .”

It should be kept in mind though that \vee connects propositions, while \cup forms a new set.

For example, the union of the set of all even integers and the set of all odd integers is the set of all integers.

Question L6.3: Let $A = \{1, 2, 3\}$ and let $B = \{1, 3, 5\}$. Find $A \cup B$.

$A \cup B = \{1, 2, 3, 5\}$. It would not be actually wrong to write $A \cup B = \{1, 2, 3, 1, 3, 5\}$, only wasteful.

However, $A \cup B \neq \{2, 5\}$, since the “or” in the definition of the union of two sets is the inclusive or, not the exclusive or that would give the set $\{2, 5\}$ of objects that belong to exactly one of the sets A, B .

The intersection of two sets

Definition 3.1.23: (Intersections) The *intersection* $A \cap B$ of two sets A, B is defined to be the set that consists of all the elements which belong to both A and B . Thus, for all objects x ,

$$x \in A \cap B \iff (x \in A \wedge x \in B).$$

Note how using the symbol “ \wedge ” for the logical connective “and” immediately provides a visual cue for the interpretation of “ \cap .” It should be kept in mind though that \wedge connects propositions, while \cap forms a new set.

For example, the intersection of the set of all even integers and the set of all odd integers is the empty set.

Question L6.4: Let A be the set of all even natural numbers and let B be the set of all prime numbers. Find $A \cap B$.

$$A \cap B = \{2\}.$$

It would be wrong here to write $A \cap B = 2$, since $A \cap B$ is by definition a set, while 2 is not a set.

Definition 3.1.15: (Subsets) Let A, B be sets. We say that A is a *subset of B* , denoted $A \subseteq B$, iff every element of A is also an element of B , i.e.

For any object x , $x \in A \implies x \in B$.

Question L6.5: Let $A = \{2\}$ and let B be the set of even prime numbers. Is then A a subset of B ?

Yes. The conditions of Definition 3.1.15 are satisfied, even though the sets are equal, which somewhat clashes with our intuitive notion of “subset” meaning a “smaller set.”

Definition 3.1.15: (Continued) We say that A is a *proper subset of B* , denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

In Question L6.5, A was a subset of B , but not a proper subset.

Caution: The literature often uses the symbol $A \subset B$, which depending on the source may mean either $A \subseteq B$ or $A \subsetneq B$. Here we avoid using this symbol to eliminate the ambiguity.

More about subsets

We often write or say “ A is contained in B ” to express that $A \subseteq B$. It is bad style to write or say “ A is in B ” instead, as this would normally mean $A \in B$.

Proposition

Let A, B be any sets. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be sets and assume that both $A \subseteq B$ and $B \subseteq A$. Let x be any object. Then both implications $x \in A \implies x \in B$ and $x \in B \implies x \in A$ hold.

As we have seen in Module 1, the conjunction of these two implications is equivalent to $x \in A \iff x \in B$, so that $A = B$ in view of the definition of equality of sets. \square

The above proposition provides a key tool for proving that two sets A, B are equal. Instead of proving equality directly, we can split up our work into proving the two *inclusions* $A \subseteq B$ and $B \subseteq A$.

Specifying subsets

When we are given a set A and a property $P(x)$ that some elements of A may have, then we can form the subset B of A that consist of all elements of A that do have this property. In symbols:

$$B = \{x \in A : P(x)\}.$$

For example, for the set \mathbb{N} of natural numbers we can form

$P = \{n \in \mathbb{N} : n \neq 0\}$, which is the set of positive natural numbers.

Question L6.6: Let \mathbb{R} be the set of real numbers. Find

$$\{x \in \mathbb{R} : x^2 < x\}.$$

This is the open interval $(0, 1)$, the set of all real numbers x that satisfy the inequalities $0 < x < 1$.

Take-home message

Thus set theory provides an important language for our work in this course. We need to become fluent in this language if we want to study the concepts defined in this terminology.

Your textbook develops set theory as an axiomatic theory. We will take advantage of this approach by being very clear of what our undefined notions are:

- *Object*: Anything that we want to talk about.
Sets are objects.
Objects that are not sets are called *urelements* here.
- *Set*: Any collection of objects.
- *Membership relation*: \in .

Beyond that, for the most part we will work here in the framework of *naive set theory*, somewhat optimistically assuming that if a set can be defined or constructed from previously given sets, then it exists.

Take-home message, continued

We introduced the following notions:

- The *empty set* \emptyset has no elements.
- The *union* $A \cup B$ of two sets.
- The *intersection* $A \cap B$ of two sets.
- The notions of a *subset* $A \subseteq B$ and of a *proper subset* $A \subsetneq B$.
- The notation $\{x \in A : P(x)\}$ for *specifying* a subset of a given set.

We saw that two sets are equal iff they have the same elements, and that no urelement can be equal to a set.

We also saw that if $A \subseteq B$ and $B \subseteq A$, then $A = B$.