# Lecture 8: Images and inverse Images

Winfried Just
Department of Mathematics, Ohio University

Companion to Advanced Calculus

# The (forward) image of a set of under a function: Definition and notation

**Definition 3.4.1:** If $f : X \to Y$ is a function from $X$ to $Y$ and $S$ is a set in $X$, then we define the *(forward) image of $S$ (under the map $f$)* as

$$f(S) := \{f(x) : \ x \in S\}.$$

- The symbols $f(S)$ for an image and $f(x)$ for a function value are the same, but they denote different kind of objects, depending on whether $S$ is a subset of $X$ (then $f(S)$ is a subset of $Y$) or $x$ is an element of $X$ (then $f(x)$ is an element of $Y$). So we need to pay careful attention to the kind of objects that are considered in the given context.
- The adjective "forward" is optional.
- $f(X)$ is what most mathematicians (including your instructor) call the *range* of $f$.

# The (forward) image of a set of under a function: Examples

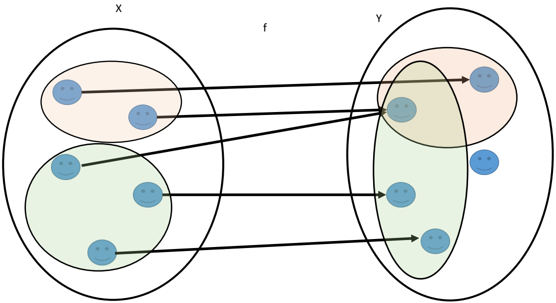
**Definition 3.4.1:** If $f : X \to Y$ is a function from $X$ to $Y$ and $S$ is a set in $X$, then we define the *(forward) image of $S$ (under the map $f$)* as

$$f(S) := \{f(x) : \ x \in S\}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then:

- $f(\mathbb{R}) = [0, \infty)$ is the range of $f$.
- $f(\{-2, -1, 0\}) = \{0, 1, 4\} = f(\{0, 1, 2\})$.
- $f(\emptyset) = \emptyset$ (this is always true for any function).
- Note that, for example, $f(\{5\}) = \{25\} \neq f(5) = 25$.
  The former is a set of function values, while the latter is a single number.

# Behavior of images of unions of sets



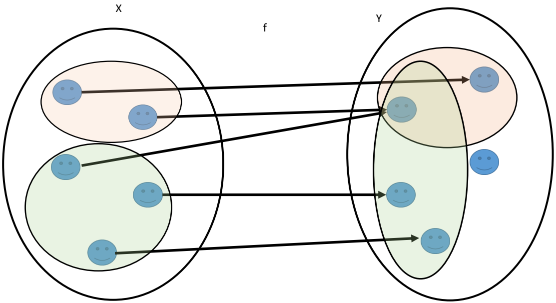

A set $S \subseteq X$ and its forward image $f(S)$ are shown in light orange; a set $T \subseteq X$ and its forward image $f(T)$ are shown in light green.

We can see that $f(S \cup T) = f(S) \cup f(T)$.

This is true in general:

**The image of a union of sets is the union of their images.**

# Misbehavior of images of intersection of sets



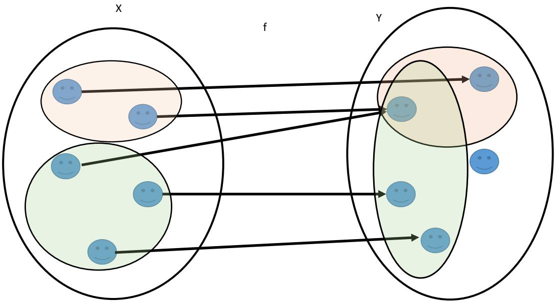

A set $S \subseteq X$ and its forward image $f(S)$ are shown in light orange; a set $T \subseteq X$ and its forward image $f(T)$ are shown in light green.

We can see that $f(S \cap T) = f(\emptyset) = \emptyset \neq f(S) \cap f(T)$.

**Question L8.1:** What causes the observed misbehavior?

The fact that $f$ is not one-to-one.

# Misbehavior of images of complements of sets




A set $S \subseteq X$ and its forward image $f(S)$ are shown in light orange; a set $T \subseteq X$ and its forward image $f(T)$ are shown in light green.

We can see that $f(X \backslash (S \cup T)) = f(\emptyset) = \emptyset \neq Y \backslash (f(S \cup T)$.

We will study the (mis)behavior of images of complements in more detail in Module 8.

# The inverse image of a set of under a function: Definition and notation

**Definition 3.4.5:** If $f : X \to Y$ is a function from $X$ to $Y$ and $U \subseteq Y$, then the *inverse image of $U$ (under the map $f$)* is the set

$$f^{-1}(U) := \{x \in X : \; f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all $x \in X$ which map into U:

$$f(x) \in U \Longleftrightarrow x \in f^{-1}(U).$$

- The adjective "inverse" is obligatory here.
- The symbols $f^{-1}(U)$ for an inverse image and $f^{-1}(y)$ for a function value of the inverse function of $f$ (if $f$ is a bijection) are the same, but they denote different kind of objects, depending on whether $U$ is a subset of $Y$ (then $f^{-1}(U)$ is a subset of $U$) or $y$ is an element of $Y$ (then $f^{-1}(y)$ is an element of $X$). So we need to pay careful attention to the kind of objects that are considered in the given context.

**Definition 3.4.5:** If $f : X \to Y$ is a function from $X$ to $Y$ and $U \subseteq Y$, then the *inverse image of $U$ (under the map $f$)* is the set

$$f^{-1}(U) := \{x \in X : \ f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all $x \in X$ which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

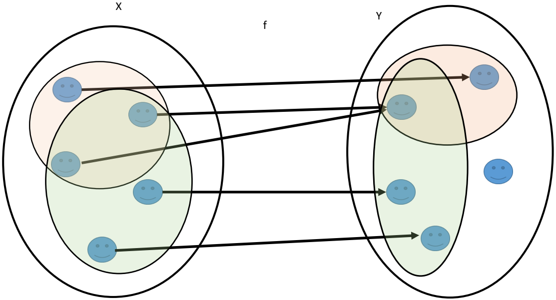
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then:

- $f^{-1}(\{2\}) = \{-\sqrt{2}, \sqrt{2}\}$.
- $f^{-1}(2)$ is undefined.

**Question L8.2:** What can you say about $f^{-1}(\{-2\})$ for the above function $f$?

**Answer:** $f^{-1}(\{-2\}) = \emptyset$.

# Behavior of inverse images under set-theoretic operations




A set $U \subseteq Y$ and its inverse image $f^{-1}(U)$ are shown in light orange; a set $V \subseteq Y$ and its inverse image $f^{-1}(V)$ are shown in light green.

**For inverse images we always have:** $f^{-1}(Y) = X$,

$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, $\quad f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$,

$f^{-1}(U \backslash V) = f^{-1}(U) \backslash f^{-1}(V)$, $\quad f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$.