YOUR COMPANION TO ADVANCED CALCULUS
MODULE 12: CARDINALITY

WINFRIED JUST, OHIO UNIVERSITY

This module is based on textbook Section 3.6 and Lecture 12.
Recall the following definitions and facts:

Definition 3.6.1: (Equal cardinality) *We say that two sets X and Y have equal cardinality if, and only if, there exists a bijection $f : X \to Y$ from X to Y.*

When there exists such a bijection from X to the set $\{1, \ldots, n\}$, then we say that *n is the number $\#(X)$ of elements of X.* Similarly, when $X = \emptyset$, then we say that $\#(X) = 0$.

When $\#(X)$ is defined by one of the above clauses, then we say that *X is finite.*

The textbook gives a list of basic properties of the numbers $\#(X)$:

Proposition 3.6.14: (Cardinal arithmetic)

(a) Let X be a finite set, and let x be an object that is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

(b) Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint (i.e., $X \cap Y = \emptyset$), then $\#(X \cup Y) = \#(X) + \#(Y)$.

(c) Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$ (i.e., Y is a proper subset of X), then we have $\#(Y) < \#(X)$.

(d) If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.

(e) Let X and Y be finite sets. Then the Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

(f) Let X and Y be finite sets. Then the set Y^X is finite and $\#(Y^X) = \#(Y)^{\#(X)}$.

All these properties, except possibly the last one, are already very familiar to us, or at least we recognize them intuitively as true. The reason the textbook lists them explicitly is that each of them can be formally proved in the axiomatic theory developed in Chapter 3. Here we will not overly dwell on this formal development, but we will study a proof of the last property.

Question 12.1: Consider the following proof of part (f) of Proposition 3.6.14. Identify the mistakes or gaps in this proof and explain how to fix them.

Proof of (f): It suffices to prove that the following property holds for all natural numbers n:

$P(n)$: If X is any set such that $\#(X) = n$ and Y is any set such that $\#(Y) = m$ for some $m \in \mathbb{N}$, then $\#(Y^X) = m^n$.

We prove $P(n)$ by induction over n. That is, we assume that $P(n)$ is true and we prove $P(n+1)$, which translates into:

$P(n+1)$: If X is any set such that $\#(X) = n + 1$ and Y is any set such that $\#(Y) = m$ for some $m \in \mathbb{N}$, then $\#(Y^X) = m^{n+1}$.

So let X be any set such that $\#(X) = n + 1$, let Y be any finite set, and let m be such that $\#(Y) = m$.
We need to show that there is a bijection $f : Y^X \to \{1, \ldots, m^{n+1}\}$.
By the choice of m, there exists a bijection $g : Y \to \{1, \ldots, m\}$.
Moreover, by the inductive assumption, there exists a bijection $g : Y^{X^-} \to \{1, \ldots, m^n\}$. Fix such bijections.
Let $x \in X$. Then Proposition 3.6.14(a) says that for $X^- = X \setminus \{x\}$ we have $\#(X^-) = n$.
Now we construct f as follows: For every function $\varphi \in Y^X$, the function value $\varphi(x) \in Y$ and the restriction $\varphi \restriction X^-$ is an element of Y^{X^-}. Thus we can define

$$f(\varphi) = g(\varphi)h(\varphi \restriction X^-). \tag{1}$$

Then $f : Y^X \to \{1, \ldots, m^{n+1}\}$ is a bijection, as required.
Now the result follows from the Principle of Mathematical Induction. $\square$

Next let us prove the following useful result:

The Pigeonhole Principle: Let $A_1, \ldots, A_n$ be finite sets such that $\#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) > n$. Then there exists $i \in \{1, \ldots, n\}$ such that $\#(A_i) \geq 2$.

There are many ways to prove this. Here we will construct a *proof by contraposition,* which is also a type of *indirect proof.* It works as follows: We consider arbitrary finite sets $A_1, \ldots, A_n$, where n a positive natural number. Then we want to prove the following implication:

$$\#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) > n \implies \exists i \in \{1, \ldots, n\} \;\; \#(A_i) \geq 2. \tag{2}$$

Instead of proving (2) directly, we will prove its *contrapositive*

$$\sim (\exists i \in \{1, \ldots, n\} \;\; \#(A_i) \geq 2) \implies \sim \left(\#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) > n\right). \tag{3}$$

This then also proves (2), as every implication is logically equivalent to its contrapositive.

Question 12.2: Prove that (3) for arbitrary finite sets $A_1, \ldots, A_n$, where n a positive natural number. *Hint:* Use induction over $n \geq 1$ and relevant parts of Proposition 3.6.14.