Alice in Weatherland: A Breezy Trajectory through some Cool Topics in Dynamical Systems

Winfried Just

Department of Mathematics, Ohio University

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Alice: Grandma, may I go play outside?

Grandma: No, my child. Not after dark.

All these strange old men lurking in the past will make a little girl's head spin and lead her into chaos.

Alice: (Nibbles at her mushroom. Grows.)

Look at me and stop talking to me like that!!!

Can you give me a rational, grown-up kind of reason?

Grandma: The weather is going to be nasty.

Alice: (Keeps angrily nibbling at her mushroom.)

Can one predict the weather?

Alice: (Fully grown now and towering over poor grandma.)
You cannot predict the weather!

Grandma: (Calmly.) Yes I can.

Alice: Then give me any prediction about the weather that will be true for certain!

Grandma: It will keep changing.

Alice: (Feels the wind taken out of her sails.)

Grandma: And since you want the fully grown-up version:
The weather is a dynamical system.

This means that its state at any time can be described as a vector of variables that change over time.

Grandma: Now go to your cozy warm cubicle and do your homework.

If only Alice had listened . . .

Alice: (Gets up, leaves the house, and angrily slams the door.)

Immediately a howling wind sweeps her off the feet, twirls her through the air, carries her somewhere, sometime . . .

(wind howling and twirling)

. . . and finally deposits her right in front of a nasty old man with an impossible hairstyle who seems rather full of himself.[1]

[1] **Disclaimer:** Any similarities to living people as alleged by the biased media are purely coincidental and unintended.

What is the weather?

Alice: Good evening!

SIN: (Icy stare.)

Alice: . . . Sir! (Courtsies.)

SIN: (Merely chilly stare.)

Alice: What was that???

SIN: Wind.

Alice: But what on earth, or rather, what in the sky, is wind?

SIN: Movement of molecules.

Alice: Do you mean that what we think of as weather is nothing else but the positions of all molecules in the air at a given time?

SIN: Not quite, but had you said "positions and momenta" you would be right.

Alice: So the vector of all these positions and momenta is the state of the weather at a given time?

Sir Isaac Newton (1643–1727)

SIN: Yes.

Alice: And how does the weather change over time?

SIN: According to the laws of mechanics.
MY laws of mechanics, that I alone discovered.

Alice: How are these laws written?

SIN: As ordinary differential equations (ODEs), a fundamental mathematical notion that I, and I alone, discovered.

Speaking at a very general level, if x_i is one of the variables in the dynamical system, then its rate of change will satisfy an ODE of the form

$$\frac{d\,x_i}{d\,t} = f_i(\vec{x}),$$

where the f_is are continuous functions and $\vec{x}$ is the state of all variables, that is, the state of the entire system.

Alice: Do your laws actually allow us to predict the weather???

Pierre-Simone Laplace (1749-1827)

Before Sir Isaac Newton could answer, another gust of wind took hold of Alice and she found herself right at the feet of a congenial Frenchman wearing an insipid Halloween costume.

PSL: Mais oui! Bien sure!

It occurred to Alice that she ought to be surprised that the Frenchman knew what her question was and that she understood his answer, but so many strange things occurred tonight ...

PSL: We may regard the present state of the weather as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set weather in motion, and all positions and velocities of all items of which weather is composed, for such an intellect nothing about the weather would be uncertain and the future just like the past would be present before its eyes.

Deterministic dynamical systems

Alice: Wow!

(It dawns upon Alice that Grandma might have been onto something.)

PSL: (Somehow reading her mind.)

Your grandma would say that the weather is a deterministic dynamical system.

Alice: So Grandma could actually predict the weather with certainty?

PSL: Your grandma? No.

I meant to add: "if this intellect were also vast enough to submit these data to analysis,"

Alice: (Deeply offended turns around and walks away ...

... right into the lab of an enigmatic Englishman.)

AT: (Gently.) He wasn't commenting on your grandma, but rather on the limitations of us humans in general.

Alice: Whom did he have in mind then with his "vast intellect"?

AT: Not whom but what.

Perhaps a kind of abstract computer that would later be called a Turing machine.

A Turing machine is also a deterministic dynamical system, except that time now is incremented in discrete steps and the current state is mapped to the next state by the rules that define the machine.

Alice: And your Turing machines are vast enough to submit the weather data to analysis?

AT: Since they have infinite tapes to record data, in principle, yes.

(Enter a boisterous Hungarian-born American.)

Johnny: Principle–Schminciple. If you want to predict anything, you need to build these machines. I did.

Alice: With infinite tapes?

Johnny: (Laughs.) Not exactly. The machines themselves are finite. But you could let them run on as long strings of input data as you want.

Alice: What if you don't feed one of your machines any data?

Johnny: Then you get a finite-state discrete deterministic dynamical system. Let it start in any initial state. Then it goes to the next state, the next, and so on. The sequence of these states is called the trajectory. Eventually, it must reach a state that it has already visited before. If it gets stuck there, we have a steady state. If not, it will keep cycling through the same sequence of states, which form a cyclic attractor. The states that are visited only once in the trajectory are called the transient (part of the trajectory).

Can computers predict the weather?

Alice: Are your computers big enough to predict the weather?

Johnny: Not precisely.

Alice: So the answer to my question is "no."

Johnny: There's no sense in being precise when you don't even know what you're talking about. If you wanted to use those famous equations of Newton to precisely predict the weather, you would need to measure the exact positions and velocities of all the molecules of the atmosphere at a given time.

Alice: And this is not possible?

Johnny: Of course not. You would need a lot of funding. I can give you a quick back-of-my-mind estimate of all the hours that would need to be spent in writing the grant applications ...

(Enter a bearded Austrian.)

LB: Lass das sein, Johnny! Back in the old days of Vienna coffee houses we did not write grant applications. We discussed ideas!

Ludwig Boltzmann (1844–1906)

LB: When an icy wind blows, do you feel molecules dancing around, or do you feel temperature and air pressure?

Alice: But Sir Isaac Newton said the weather is really the movement of molecules!

LB: Position, momentum, temperature, pressure—all these are just concepts that we use to understand the real world. They are variables that we can measure and use to build mathematical models, such as dynamical systems models of the real world.

Alice: But which of these models is the correct one?

LB: None of them is entirely correct. Some do a better job at making useful predictions than others.

My theory, called statistical mechanics, relates the fine-grained but intractable models of Newton to coarser-grained models based on macroscopic variables, such as temperature and pressure, that are easier to study.

Approximating dynamical systems

Alice: So your theory relates one dynamical system to another one?

LB: Right.

Alice: Cool! Where can I learn about statistical mechanics?

Grandma: (Whispers in the background.) From your physics homework. (Chuckles and quickly disappears.)

Johnny: Well, my dear Ludwig, nothing that special about your theory. When they run a simulation on one of my machines, in effect, they also approximate one dynamical system by another. And when an algorithm runs too slowly, they sometimes approximate its dynamical system with another algorithm that runs faster.

Alice: (Feels her head spinning.)

Grandma: (Chuckles in the background.)

Alice: Let me see whether I get this straight:

You take one of Sir Newton's dynamical systems with the horrendously large number of variables and then use Professor Boltzmann's theory to reduce it to a simpler system with, let's say, three variables?

LB: For example, like this one:

$$\frac{dx}{dt} = 10(y - x) \qquad \frac{dy}{dt} = 28x - y - xz \qquad \frac{dz}{dt} = -\frac{8}{3}z + xy$$

Alice: Then you discretize both time and the state space to simulate the system on a computer, and the simulations will give you a pretty good idea of what happens in the above system of ODEs,

LB: and therefore in the original high-dimensional system of Isaac Newton.

Alice continues

Alice: As Professor von Neumann told us,

Johnny: Call me Johnny!

Alice: each simulation in the computer would need to eventually reach a steady state or periodic attractor, and if our discretizations utilize smaller and smaller time steps it would appear that the corresponding attractors should converge to a steady state or periodic attractor of the system that was mentioned by Professor Boltzmann, and therefore the same should happen in the original high-dimensional system of Sir Isaac Newton.

Johnny: Bravo, sweetie!

Alice: (Sends him a devastating look.)

(Enter E. L. and defuses the situation.)

EL: It doesn't work like this.

Edward Lorenz (1917–2008)

Johnny: Come on, Ed! Don't give me any of this PC stuff!

EL: What PC? I did this in 1963 on a mainframe.

Johnny: Did what?

EL: Simulated the system that Ludwig mentioned.

The trajectories do not converge to any kind of periodic attractor.

LB: So where do they go?

EL: They seem to be all over the place, well, not over all the state space, but over all this thing I call a strange attractor.

Jackson Pollock: It's my lost painting of a butterfly.

EL: And the trajectories are extremely sensitive to the initial conditions. As if a butterfly butting its wings somewhere in Rio could cause a major hurricane three weeks later in Miami.

Told you so!

Johnny: Impossible! These IBM guys messed up your hardware.

LB: Impossible! Did you carefully check your assumptions?

AT: Impossible! Did you double-check your code?

SPL: Impossible! You should have used an integral transform!

SIN: Impossible! Brush up on your calculus, man!

HP[2]**:** Mais c'est très possible! Strange attractors!
Sensitive dependence! I told you so!

Alice: This is chaos!!! (Faints.)

Grandma: I told you so!

[2] Henri Poincaré (1854–1912)

When Alice comes to she finds herself in the company of an ugly pauper wrapped in a kind of blanket

Alice: Is there anything, anything steady to hold on to in these chaotic dynamical weather systems?

S: How would I know about all these modern notions?
But I'm sure you can help me understand.

Alice: I just don't know!!!

S: And what is it that you don't know?

Alice: Where, exactly, the trajectory will be at a future time.

S: And why don't you know this?

Alice: Because I don't know, where, exactly, it is at the present time!

S: So you know that you don't know, and why you don't know.
Could this be the steady thing to hold on to?

Alice: This sounds Greek to me!

S: It is called $\Sigma o\varphi\iota\alpha$. Wisdom.

Alice: But this is not a mathematical concept!

S: I cannot be sure about this, but perhaps our three Russian friends over there could help us with our uncertainty.
How do mathematicians say "I don't know?"

AK: "With probability *p*."

Alice: What does this have to do with dynamical systems?

AK: We can consider systems on state spaces with a so-called probability measure. If this measure is invariant under the transformation that defines the system, then the level of uncertainty of whether or not the state of the system is in a given measurable set will not change over time.

Alice: This does look like something steady, something at least, to hold on to.

S: You are saying that there is virtue in these invariant measures.

Nikolay Krylov (1879–1955), Nikolay Bogolyubov (1909–1992), George Birkhoff (1884–1944)

Alice: But do these invariant probability measures exist?

NK and NB: Yes. Always. At least one, sometimes more.

Alice: But how would I find out the probability p of any given set for some such measure?

GB: Simple. Pick (almost) any initial condition, iterate the system many times on a computer, and then calculate the proportion p of iterates that hit your set.

Alice: Cool! . . . But this only will inform me about the level of my ignorance, not about the actual trajectories of the real system.

S: $\Sigma o\varphi\iota\alpha$!

(Embarrassed silence among the others.)

Max Planck (1858–1947) and Albert Einstein (1879–1955)

MP: Aber meine Herren, these positions and momenta do not really exist! They cannot be measured with arbitrary precision, at least not at the same time. At the bottom of reality there are only probability distributions.

(A clerk of the Swiss patent office interrupts.)

AE: Aber Herr Kollege, God does not play dice![3]

MP: (Ignores him.) So we should think of the real dynamical system for the weather as transforming probability distributions.

Alice: Can we think about it as changing the levels of our uncertainty about states over time?

MP: This is one way to look at it.

S: $\Sigma o \varphi \iota \alpha$!

[3]This upstanding defender of the truth was silenced by the corrupt establishment with a Nobel price for his discovery of the alleged photoelectric effect that tainted him with shady contributions to the quantum conspiracy.

Oscar Perron (1880–1975) and Ferdinand Georg Frobenius (1849–1917)

OP: And if you look at it in this way, the system can be treated as defined by a linear transformation of probability measures, known as the Perron-Frobenius operator.

FGF: Alle diese Probleme mit den Systemen, Chaos und so weiter, are really caused by their nonlinearities, you know.
So our operator really makes everything nice and linear.

It is getting stuffy in the room and Alice opens the window.

Immediately a howling wind sweeps her away, twirls her through the air, carries her somewhere, sometime . . .

(wind howling and twirling)

The attractor

. . . and finally deposits her right in front of her desk in her cubicle.

In the meantime, Grandma has strategically placed on Alice's desk stacks of books and papers on dynamical systems.

Now Alice is in a pickle.

Where should she begin?