Conversation 21: Marvin and Marilyn Go on a Diet

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

A health-conscious couple

This dialogue will serve as a review of some important notions that were covered earlier in Chapter 3.

It features our guest performers Marilyn and Marvin.

They are a health-conscious couple and decide to go on Dr. Losit's new scientifically proven diet. According to this expert, they should eat exactly 50 grams of sugar, exactly 300 grams of protein, and exactly 100 grams of fat per day, and restrict themselves to one meal per day prepared from a mixture of a wide range of products offered by his company.

Being on a budget and having somewhat different tastes, they decide to purchase Losit-Quick (strawberry taste, which they both like), Losit-Fast (Marylin's beloved watermelon taste that Marvin cannot stand), and Losit-Easy (beer-flavored for Marvin, detested by Marylin).

The products are in, now what?

After receiving the products in the mail and reading the fine print on the labels, they discover that:

- One serving of Losit-Quick contains 20 grams of sugar, 200 grams of protein, and 20 grams of fat.
- One serving of Losit-Fast contains 15 grams of sugar, 50 grams of protein, and 40 grams of fat.
- One serving of Losit-Easy contains 25 grams of sugar, 150 grams of protein, and 60 grams of fat.

How should they compose their meals if they want to strictly follow the guidelines?

Each of our protagonists sits down with a pencil and paper and tries to figure it out.

Marilyn's reasoning

Marilyn reasons as follows: I will need x_1 servings of Losit-Quick and x_2 servings of Losit-Fast.

The numbers x_1, x_2 need to be chosen in such a way that the total amounts of sugar, protein, and fat add up to exactly the recommended values.

They must be solutions of the following system of linear equations, where all quantities are expressed in grams:

$$\begin{aligned} 20x_1 + 15x_2 &= 50 \quad \text{(the total amount of sugar)} \\ 200x_1 + 50x_2 &= 300 \quad \text{(the total amount of protein)} \\ 20x_1 + 40x_2 &= 100 \quad \text{(the total amount of fat)} \end{aligned}$$

Question C21.1: Did Marilyn get this right up to this point?

Yes.

Marilyn's solution

She is really good at Gaussian elimination and uses it to quickly reduce the augmented matrix $[\mathbf{A}, \vec{b}]$ of this system to row-echelon form:

$$[\mathbf{A}, \vec{b}] = \begin{bmatrix} 20 & 15 & 50 \\ 200 & 50 & 300 \\ 20 & 40 & 100 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 10R1} \begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 20 & 40 & 100 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 20 & 40 & 100 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1} \begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix} \xrightarrow{R1 \mapsto R1/20} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix}$$

Marilyn's solution, continued

$$\begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix} \xrightarrow{R2 \mapsto R2/(-100)} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 25 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 25 & 50 \end{bmatrix} \xrightarrow{R3 \mapsto R3/25} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R2} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Marilyn's starts cooking

Marilyn has found the augmented matrix

$$[\mathbf{A}, \vec{b}] = \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

of the equivalent system

$$\begin{aligned} x_1 + 0.75x_2 &= 2.5 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

Question C21.2: How many servings x_1 of Losit-Quick and how many servings x_2 of Losit-Fast does Marilyn need for her meal?

This tells her that she needs to mix $x_2 = 2$ servings of Losit-Fast with $x_1 = 1$ serving of Losit-Quick.

She starts cooking.

In the meantime Marvin ...

who isn't a big fan of Gaussian elimination tried randomly guessing an appropriate mixture of Losit-Quick and Losit-Easy, without much success.

He likes vector notation though, and reasons as follows:
Let $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E, \vec{\mathbf{v}}_M$ be the vectors of grams of sugar, protein, and fat in one serving of Losit-Quick, Losit-Easy, and in the meal that I need to compose. Then

$$\vec{\mathbf{v}}_Q = \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} \qquad \vec{\mathbf{v}}_E = \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} \qquad \vec{\mathbf{v}}_M = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix}$$

I want to find scalars d_Q, d_E such that $\vec{\mathbf{v}}_M$ is the **linear combination**
$d_Q\vec{\mathbf{v}}_Q + d_E\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$,
where d_Q, d_E are the numbers of servings of Losit-Quick and Losit-Easy that I need for my meal.

Question C21.3: Did Marvin get it right up to this point?

Yes.

Marvin is getting hungry

It remains to find the actual numbers d_Q, d_E such that $\vec{\mathbf{v}}_M$ is the linear combination

$d_Q\vec{\mathbf{v}}_Q + d_E\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$.   Hmmm.   ????

Eventually hunger prevails over pride and Marvin decides that his notation and the phrase **linear combination** look sufficiently impressive so that it wouldn't be too embarrassing to ask:

Marvin: Hey, sweetheart, I have practically figured it out. Could you just give me a hand with these boring calculations?

Marilyn: Sorry, bud. You will need to add some of my Losit-Fast.

Marvin: (taken aback) Why do you say this?

Now we have a bit of reality TV here.

Is Marilyn right or just being nasty?

Let's use linear algebra to derive an impartial answer to this question.

Marvin explains his notation

Marilyn: Er, well, ... I don't really understand your linear combination stuff. Can you explain it to me?

Marvin: (becomes animated again) This is really quite simple. It works like this: If you have any vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ and another vector $\vec{\mathbf{w}}$, then $\vec{\mathbf{w}}$ is a **linear combination** of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if, and only if, there exist numbers (or as they say, **scalars**) $d_1, d_2, \ldots, d_n$ such that

$$d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n = \vec{\mathbf{w}}.$$

Marilyn: Can you give me an example?

Marvin: Sure. Let $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E, \vec{\mathbf{v}}_M$ be the vectors of amounts of sugar, protein, and fat in one serving of Losit-Quick, Losit-Fast, Losit-Easy, and in the meal that each of us needs to compose.

Meals as linear combinations of the ingredients

Marvin: Let $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E, \vec{\mathbf{v}}_M$ be the vectors of amounts of sugar, protein, and fat in one serving of Losit-Quick, Losit-Fast, Losit-Easy, and in the meal that each of us needs to compose.

Marilyn: I found that I need $x_1 = 1$ serving of Losit-Quick and $x_2 = 2$ servings of Losit-Fast. Then I can compose my meal without using any of your Losit-Easy.

Marvin: So you can compose the meal $\vec{\mathbf{v}}_M$ as a linear combination of $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F$ with coefficients $x_1 = 1$ and $x_2 = 2$:

$\vec{\mathbf{v}}_M = x_1\vec{\mathbf{v}}_Q + x_2\vec{\mathbf{v}}_F = \vec{\mathbf{v}}_Q + 2\vec{\mathbf{v}}_F.$

Since I don't like your Losit-Fast, for my meal I would need coefficients d_Q, d_E such that

$\vec{\mathbf{v}}_M = d_Q\vec{\mathbf{v}}_Q + d_E\vec{\mathbf{v}}_E.$

Marilyn: I see. But can you explain to me how, exactly, all this is related to the amounts of sugar, fat, and protein in each of these ingredients?

Marvin's meal

Marvin: I need coefficients d_Q, d_E such that $d_Q\vec{\mathbf{v}}_Q + d_E\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$.
We can look at this coordinate by coordinate:

$$d_Q \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} + d_E \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} = \begin{bmatrix} d_Q 20 + d_E 25 \\ d_Q 200 + d_E 150 \\ d_Q 20 + d_E 60 \end{bmatrix} = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix} = \vec{\mathbf{v}}_M$$

Marilyn: So your coefficients d_Q and d_E must then satisfy the following system of linear equations:

$$\begin{aligned} 20d_Q + 25d_E &= 50 \quad \text{(the total amount of sugar in grams)} \\ 200d_Q + 150d_E &= 300 \quad \text{(the total amount of protein in grams)} \\ 20d_Q + 60d_E &= 100 \quad \text{(the total amount of fat in grams)} \end{aligned}$$

Marvin: Right!

Using different names for the variables

Marilyn: Could I use x_1, x_2 instead of d_Q, d_E for the coefficients of your linear combination?

Marvin: Absolutely! It doesn't matter which letters you use. These coefficients would then need to satisfy

$x_1\vec{\mathbf{v}}_Q + x_2\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$.

Marilyn: So the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must be a solution of

the following system of linear equations:

$$\begin{aligned} 20x_1 + 25x_2 &= 50 \\ 200x_1 + 150x_2 &= 300 \\ 20x_1 + 60x_2 &= 100 \end{aligned}$$

She now finds the augmented matrix of this system and performs Gaussian elimination on it.

Gaussian elimination for this system

$$\begin{bmatrix} 20 & 25 & 50 \\ 200 & 150 & 300 \\ 20 & 60 & 100 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 10R1} \begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 20 & 60 & 100 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 20 & 60 & 100 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1} \begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix} \xrightarrow{R1 \mapsto R1/20} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix}$$

Gaussian elimination for this system, completed

$$\begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix} \xrightarrow{R2 \mapsto R2/(-100)} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 35 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 35 & 50 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 35R2} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & -20 \end{bmatrix} \xrightarrow{R3 \mapsto R3/(-20)} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So how about Marvin's meal?

Marilyn has found the augmented matrix

$$[\mathbf{A}, \vec{b}] = \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

of the equivalent system

$$\begin{aligned} x_1 + 1.25x_2 &= 2.5 \\ x_2 &= 2 \\ 0 &= 1 \end{aligned}$$

Question C21.4: How many servings of Losit-Quick and Losit-Easy would Marvin need for his meal?

Marilyn: This system is *inconsistent.* Marvin, you cannot compose your meal without adding some Losit-Fast.

Marvin's meal, Plan B

(A moment of heavy silence.)

Marvin: Perhaps we could, maybe . . .

Marilyn: (gently) Could . . . yes?

Marvin: Find a linear combination that doesn't exactly satisfy Dr. Losit's constraints, but would be least bad in some sense? But I don't know what this would mean.

Marilyn: At the very end of the course we will learn about such "least bad" or *least square solutions* of overconstrained linear systems.

Marvin: So what shall we do until then?

Marilyn: Let's postpone the diet and have some burgers with fries

Marvin: and a couple of beers!

Take-home message

This conversation illustrates that determining whether a given vector is a linear combination of a given set of vectors and finding coefficients of the linear combination if it is essentially boils down to solving corresponding systems of linear equations.

In Lecture 22 we studied in detail the general method for setting up these corresponding systems.

Moreover, the conversation illustrates that for an overconstrained system of linear equations it may be of interest to find a vector that is not exactly a solution, but violates the constraints as little as possible. We will return to this problem at the very end of this course.