On the role of limsup in the definition of topological entropy: An alternative view of the construction

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For the purpose of this talk, a dynamical system is a pair \((X, F)\), where \(X\) is a compact metric space with distance function \(D\) and \(F : X \to X\) is a homeomorphism. Roughly speaking:

- **A (forward) trajectory** is a sequence \((F^t(x))_{t=0}^{\infty}\) for some \(x \in X\).

- **In chaotic systems**, for sufficiently small \(\varepsilon > 0\), the number \(N_T(\varepsilon, D)\) of trajectories that are distinguishable at resolution \(\varepsilon\) within \(T\) time steps scales like \(B(\varepsilon)^T\) for some \(B(\varepsilon) > 1\).

Thus we can use the growth rate of \(N_T(\varepsilon, D)\) to define a measure for how chaotic the system is:

\[
h(X, F) = \lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln N_T(\varepsilon, D)}{T}.
\]

This measure is called the **topological entropy** of \((X, F)\).
Let \((X, d)\) be metric space, and let \(\varepsilon > 0\).

We define the **separation number** \(\text{sep}(X, \varepsilon, d)\) as the **largest** size of a subset of \(Y \subset X\) such that \(d(x, x') \geq \varepsilon\) for all \(x, x' \in Y\),

and the **spanning number** \(\text{span}(X, \varepsilon, d)\) as the **smallest** size of a subset of \(Y \subset X\) such that for all \(x \in X\) there exists \(y \in Y\) with \(d(x, y) < \varepsilon\).

\[\text{A} = \{p1, p2, p3, p4\}\] is 3-separated of size 4 = \(\text{sep}(X, 3, d)\)

\[\text{B} = \{p5, p6\}\] is 3-spanning of size 2 = \(\text{span}(X, 3, d)\)
Two definitions of $N_T(\varepsilon, D)$

We could define $N_T(\varepsilon, D)$ as the largest size $\text{sep}(X, \varepsilon, D_T)$ of a $(T, \varepsilon)$-separated subset of $X$, that is, of a set $Y \subseteq X$ such that for all $x, x' \in Y$ there exists a $0 \leq t < T$ with $D(F^t(x), F^t(x')) \geq \varepsilon$.

Or we could define $N_T(\varepsilon, D)$ as the smallest size $\text{span}(X, \varepsilon, D_T)$ of a $(T, \varepsilon)$-spanning subset of $X$, that is, of a set $Y \subseteq X$ such that for all $x \in X$ there exists $y \in Y$ such that for all $0 \leq t < T$ we have $D(F^t(x), F^t(y)) < \varepsilon$.

The separation numbers $\text{sep}(X, \varepsilon, D_T)$ and spanning numbers $\text{span}(X, \varepsilon, D_T)$ are always finite and satisfy the inequality $\text{sep}(X, \varepsilon, D_T) \geq \text{span}(X, \varepsilon, D_T)$. 
But why limsup?

So we can define the topological entropy $h(X, F)$ as:

$$\lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_T)}{T} = \lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_T)}{T}.$$

But why lim sup? Could we use lim instead?

**Chorus of the experts:** It doesn’t really matter. There is another definition, based on covering numbers, where you can.

But for the definitions based on separation or spanning numbers?

**Chorus:** Then you cannot.

But why?

**Chorus:** Because there is no obvious reason why you could.

Any counterexamples?

**Chorus:** Hmmm. Not that we know of.

(This was what we heard about 3 years ago.)
Our main theorem

So we went ahead and constructed one.

**Theorem**

*There exists a system \((X, F)\) with a metric \(D\) on \(X\) such that for \(\varepsilon > 0\) we have:*

\[
\liminf_{T \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_T)}{T} < \limsup_{T \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_T)}{T},
\]

\[
\liminf_{T \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_T)}{T} < \limsup_{T \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_T)}{T}.
\]

*The system \((X, F)\) is minimal; i.e., every forward trajectory is dense in \(X\).*

**Remark:** We have more results along these lines. They are included in Ying’s dissertation. But the blue part is very new.

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What did we do with these results?

1. Included as a chapter in Ying’s dissertation.
3. Submitted to journals.

Chorus of Potential Reviewers:
( Remarkable lack of enthusiasm for reading the proof. )

How can we make the presentation of this construction more palatable?
A normal way to present such a construction

1. Define the space $X$.
2. Define the function $F$.
3. Define the metric $D$ on $X$.
4. Prove that these objects have all the desired properties.

**Problem with this approach:** We tried that one and tended to get a lot of discussion and feedback on steps 1 and 2. But with a few notable exceptions, our audience didn’t make it through step 3.
An alternative way of presenting our construction

Here is our proposed remedy:

1. **Don’t** define $X$ and $D$ until half way into the proof.

2. Start with some relatively easy properties that our system $(X, F)$ and the metric $D$ will have, and then prove the inequalities (1) and (2) of the theorem from these properties.

3. Then formally define $(X, F)$ and $D$ with the help of certain parameters and show that they also satisfy the more pedestrian requirements of the theorem, like compactness of $(X, D)$.

4. Finally show that suitable parameters for the definition of $X$ and $D$ actually exist.
The first ingredient of our construction

We will need positive integers $T(n)$ and $T^+(n)$ with

$$1 < T(0) < T^+(0) < \cdots < T(n) < T^+(n) < T(n+1) < T^+(n+1) < \ldots$$

These sequences will be parameters in our construction and will satisfy a number of technical conditions that mostly boil down to requiring that $T(n+1)$ is sufficiently large relative to $T^+(n)$, and $T^+(n)$ is sufficiently large, but not too large, relative to $T(n)$.

We will then prove, for a suitably chosen parameter $\varepsilon > 0$, the following inequalities:

$$\liminf_{n \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_{T^+(n)})}{T^+(n)} > \limsup_{n \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_{2T(n)})}{2T(n)}.$$

$$\liminf_{n \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_{T^+(n)})}{T^+(n)} > \limsup_{n \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_{2T(n)})}{2T(n)}.$$
More precisely, we will construct things so that for some $\lambda < 0.9$:

$$\liminf_{n \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_{T^+(n)})}{T^+(n)} \geq 0.9 \ln 2.$$  

$$\limsup_{n \to \infty} \frac{\ln \text{sep}(X, \varepsilon, D_{2T(n)})}{2T(n)} \leq \lambda \ln 2.$$  

$$\liminf_{n \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_{T^+(n)})}{T^+(n)} \geq 0.9 \ln 2.$$  

$$\limsup_{n \to \infty} \frac{\ln \text{span}(X, \varepsilon, D_{2T(n)})}{2T(n)} \leq \lambda \ln 2.$$
What do we need to know about $X$ and $F$?

In order to achieve this, we will construct $(X, F)$ so that with every $x \in X$ and $n \in \mathbb{N}$ we can associate an integer $k_n(x) \in T^+(n) := \{0, \ldots, T^+(n) - 1\}$ and a function $\psi_n(x) \in T^+(n)\{0, 1\}$ such that for all $x \in X$ and $n \in \mathbb{N}$:

(Fk) \quad k_n(F(x)) = k_n(x) + 1 \mod T^+(n)

(Fx) \quad \text{If } k_n(x) < T^+(n) - 1, \text{ then } \psi_n(x) = \psi_n(F(x)).

We will require that $\psi_n \in \mathcal{X}_n$, where $\mathcal{X}_n$ is a subset of $T^+(n)\{0, 1\}$ that satisfies certain conditions, in particular, is of size at least $|\mathcal{X}_n| \geq 2^{0.9T^+(n)}$. 
The key property of the metric $D$

The metric $D$ on $X$ is then defined in terms of a sequence of conditions $Cond_n$ on triplets $(\varphi, \psi, k)$ such that

$$(D_\varepsilon) \quad D(x, x') \geq \varepsilon$$

if, and only if,

$$\forall n \in \mathbb{N} \ (k_n(x) \neq k_n(x') \lor Cond_n(\psi_n(x), \psi_n(x'), k_n(x))).$$
Next we want to assure that for some fixed $\lambda < 0.9$

$$\exists N \in \mathbb{N} \forall n > N \quad \text{span}(X, \varepsilon, D_{2T(n)}) \leq \text{sep}(X, \varepsilon, D_{2T(n)}) < 2^{\lambda 2T(n)},$$

$$\exists N \forall n > N \quad \frac{\ln \text{span}(X, \varepsilon, D_{2T(n)})}{2T(n)} \leq \frac{\ln \text{sep}(X, \varepsilon, D_{2T(n)})}{2T(n)} < \lambda \ln 2.$$

As long as $T^+(n) \leq T(n)2^{0.01T(n)}$ it suffices to show that

$$\text{sep}(X, \varepsilon, D_{2T(n)}) \leq T^+(n)2^{1.75T(n)},$$

since for all $\lambda > 0.88$ and all sufficiently large $n$ we will then have

$$\text{sep}(X, \varepsilon, D_{2T(n)}) \leq T^+(n)2^{1.75T(n)} \leq T(n)2^{1.76T(n)} < 2^{\lambda 2T(n)}.$$

In order to achieve this goal, we need to design the conditions $\text{Cond}_n$ in the definition of the metric $D$ in a certain way that is based on so-called colorings.
Let $C(n) = T^+(n)/T(n)$. We partition the interval $T^+(n) := \{0, \ldots, T^+(n) - 1\}$ into consecutive subintervals $I^n_j$ of length $T(n)$ each, where $j$ ranges from 1 to $C(n)$.

Let $T^+(n)\{0, 1\}$ denote the set of all functions with domain $T^+(n)$ that take values in the set $\{0, 1\}$. For a subset $S \subseteq T^+(n)\{0, 1\}$, let $[S]^2$ denote the set of all unordered pairs $\{\varphi, \psi\}$ of different functions from $S$. Moreover, let $[C(n)] = \{1, 2, \ldots, C(n)\}$. For our purposes, a coloring will be a function $c : [S]^2 \rightarrow [C(n)]$, where $S \subseteq T^+(n)\{0, 1\}$ for some $n \in \mathbb{N}$.

**Definition**

A subset $S^- \subseteq S$ is $\leq 2$-chromatic for $c$ if the restriction of $c$ to $[S^-]^2$ takes at most 2 values from the set $[C(n)]$. 
In our construction we use a sequence \((c_n)_{n \in \mathbb{N}}\) of suitable colorings as parameters. In particular, the domain of \(c_n\) will be \(\mathcal{X}_n^2\), and the colorings \(c_n\) will not admit large \(\leq 2\)-chromatic subsets of size \(\geq 2^{0.75T(n)}\).

We prove existence of suitable colorings using the probabilistic method.

The conditions \(\text{Cond}_n\) for the definition of the metric \(D\) will then take the form:

\[
\text{Cond}_n(\varphi, \psi, k_n) \iff (\varphi(k_n) \neq \psi(k_n) \& k_n \in I_{c_n(\varphi, \psi)}^n).
\]

Let us next illustrate how this works for keeping some separation numbers small.
Consider a \((2T, \epsilon)\)-separated subset \(A \subset X\) such that \(k_n(x) = \tau\) for all \(x \in A\). Since there are only \(T^+(n)\) possible choices for \(\tau\), for the inequality \(\text{sep}(X, \epsilon, D_{2T(n)}) \leq T^+(n)2^{1.75T(n)}\) it will suffice to show that such \(A\) has size at most \(2^{1.75T(n)}\).

Let us focus in this illustration on the simplest case where \(\tau\) with \(0 \leq \tau < \tau + 2T(n) < T^+(n)\).

Then by (Fk) and (Fx), for all \(t \in 2T\) and \(x \in A\) we have \(k_n(F^t(x)) = \tau + t\) and \(\psi_n(F^t(x)) = \psi_n(x)\).

Since \(A\) was assumed to be \((2T, \epsilon)\)-separated, for every \(x \neq x' \in A\) there exists \(t < 2T\) such that \(D(F^t(x), F^t(x')) \geq \epsilon\).

By (D\(\epsilon\)), this implies that for every \(x \neq x' \in A\) there exists \(t < 2T\) such that \(\text{Cond}_n(\psi_n(F^t(x)), \psi_n(F^t(x')), \tau + t)\) holds. In view of (16) this now implies that for every \(x \neq x' \in A\) there exists \(t < 2T\) such that \(\psi_n(x)(t + \tau) \neq \psi_n(x')(t + \tau)\) and \(\tau + t \in I^n_c(\psi_n(x), \psi_n(x'))\).
Now assume towards a contradiction that $A$ has more than $2^{1.75T(n)}$ elements and for every $x \neq x' \in A$ there exists $t < 2T$ such that $\psi_n(x)(t + \tau) \neq \psi_n(x')(t + \tau)$ and $\tau + t \in I^n_{cn(\psi_n(x), \psi_n(x'))}$.

Then there exists a subset $A^- \subset A$ of size $> 2^{0.75T(n)}$ such that $\psi_n(x), \psi_n(x')$ take the same values on $\tau + t$ for all $\tau + T(n) \leq t < \tau + 2T(n)$, so that for $x \neq x' \in A^-$ there exists $t < T$ such that $\psi_n(x)(t + \tau) \neq \psi_n(x')(t + \tau)$ and $\tau + t \in I^n_{cn(\psi_n(x), \psi_n(x'))}$.

But note that since each of the intervals $I^n_j$ has length $T$, the values $\tau, \tau + 1, \ldots, \tau + T - 1$ can belong to at most two of the intervals $I^n_j$.

It follows that the set $S := \{\psi_n(x) : x \in A^-\}$ has the same size as the set $A^-$ and that the restriction of $c_n$ to the set $[S]^2$ takes at most 2 values from the set $C(n)$. Thus the size of $A^-$ cannot exceed the maximal size of a $\leq 2$-chromatic subset of $c_n$, which contradicts our choices of $A^-$ and $c_n$. 

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The role of limsup: Alternative view of the construction
Large \((T^+(n), \varepsilon)\)-separated subsets of \(X\)

For each \(n\), we will construct a set \(W_n\) such that:

\[
\forall \psi \in X_n \exists x \in W_n \quad \psi_n(x) = \psi.
\]

Thus \(|W_n| \geq 2^{0.9T^+(n)}\).

\[
\forall m \in \mathbb{N} \forall x \in W_n \quad k_m(x) = 0.
\]

\[
\forall x \neq x' \in W_n \exists t \in T^+(n) \forall m \in \mathbb{N} \quad \text{Cond}_m(\psi_m(F^t(x)), \psi_m(F^t(x')), k_m),
\]

where \(k_m = t \mod T^+(m)\).

By Property (Fk), for \(x, x' \in W_n\) we will then have

\[
k_m(F^t(x)) = k_m(F^t(x')) = t \mod T^+(m)
\]

for all \(m\), and Property (D\(\varepsilon\)) will imply that

\[
\text{sep}(X, \varepsilon, D_{T^+(n)}) \geq 2^{0.9T^+(n)}, \quad \text{so that} \quad \frac{\ln \text{sep}(X, \varepsilon, D_{T^+(n)})}{T^+(n)} \geq 0.9 \ln 2.
\]

We will get the analogue of the last inequality for spanning numbers by showing that for any \(x' \in X\) the inequality \(D_{T^+(n)}(x', x) < \varepsilon\) can hold for at most one \(x \in W_n\).
How do we get large spanning numbers?

We will not define the sets $W_n$ here. Instead, as we go, we will list additional properties that we need to get large spanning numbers.

Fix $n \in \mathbb{N}$ and let $x' \in X$. We will need the following property:

$$(X1) \quad x \neq x' \in W_n \implies \psi_n(x) \neq \psi_n(x').$$

Then there exists at most one $x \in W_n$ with $\psi_n(x) = \psi_n(x')$.

So assume $x \in W_n$ is such that $\psi_n(x) \neq \psi_n(x')$. It suffices to show that $D_{T^+(n)}(x, x') \geq \varepsilon$.

For that we need some $t \in T^+(n)$ with $D(F^t(x), F^t(x')) \geq \varepsilon$.

Note that we need to find such a $t$ that works at all levels $m$ simultaneously.

Note also that the first clause of $(D\varepsilon)$ essentially tells us that levels $m$ with $k_m(x) \neq k_m(x')$ are unproblematic, so we will restrict our illustration here to the most interesting situation where $k_m(x') = k_m(x) = 0$ for all $m$. Then we also have $k_m(F^t(x')) = k_m(F^t(x))$ for all $m$ and all $t$ by Property (Fk).
Now we need some \( t \) with \( t = k_n(F^t(x)) = k_n(F^t(x')) \) such that

- \( \psi_n(x)(t) \neq \psi_n(x')(t) \).
- \( t \in I^n_{\psi_n(x),\psi_n(x')} \).

We get such \( t \) from the following property of our colorings:

\[(\text{CD}) \quad \psi \neq \varphi \in \mathcal{X}_n, \text{ then there exists at least one } k \in I^n_{\psi,\varphi} \text{ such that } \psi(k) \neq \varphi(k).\]

Note then any \( t \in I^n_{\psi_n(x),\psi_n(x')} \) with \( \psi_n(x)(t) \neq \psi_n(x')(t) \) will work for satisfying the clause in (D\( \varepsilon \)) that deals with level \( n \).

We still need to find a \( t \), in this same interval, that covers the clauses of (D\( \varepsilon \)) that deal with levels \( m < n \) and with levels \( m > n \).
Dealing with levels $m < n$

We need the following properties:

(TC) $T^+(n)$ is an integer multiple of $T^+(m)$ for $m < n$.

(XC) For $x \in X$, $m < n$, and $\ell T^+(m) < T^+(n)$, the restriction of $\psi_n(x)$ to $\{\ell T^+(m), \ldots, (\ell + 1) T^+(m) - 1\}$ is equal to $\psi_m(F\ell T^+(m)(x))$.

Assume by induction that

- $\psi_n(x) \upharpoonright I^n_{cn(\psi_n(x), \psi_n(x'))} \neq \psi_n(x') \upharpoonright I^n_{cn(\psi_n(x), \psi_n(x'))}$.

Let $m = n - 1$. Then we find $\tau = \ell T^+(m)$ with

- $\psi_m(F\tau(x)) \neq \psi_m(F\tau(x'))$.
- $k_m(x) = k_m(x') = 0$.

Now we can find $t' \in I^m_{cm(\psi_m(F\tau(x)), \psi_m(F\tau(x'))}$ as on the previous slide, and $t = (c_n(\psi_n(x), \psi_n(x')) - 1)C(n) + \ell T^+(m) + t'$ will work at both levels $m$ and $n$.

By iterating this argument, we find $t$ that works at all levels $m \leq n$. 

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The role of limsup: Alternative view of the construction
Dealing with levels $m > n$

The $t$ we have found so far is an element of $l_1^m$ for $m > n$.

This $t$ will automatically work at levels $m > n$ if we have the following properties:

(WS) For $x \in W_m$ and $m < n$, the function $\psi_n(x)$ is “special.”

(C1) If $\psi \neq \varphi \in \mathcal{X}_n$ and at least one of $\psi, \varphi$ is special, then $c_n(\psi, \varphi) = 1$. 
So what about that definition of \((X, F)\)?

We let \(X\) consist of pairs \(x = (y, \kappa)\), where \(y \in \mathbb{Z}\{0, 1\}\) and \(\kappa : \mathbb{N} \to \mathbb{N}\) is a function such that \(\kappa(n) \in T^+(n)\) for all \(n\).

\(X\) will not consist of all such pairs, only of the pairs that are allowed by our conditions.

\[ F(y, \kappa) = (\sigma(y), \kappa \oplus 1), \]

where:

- \(\sigma\) is the usual shift operator,
- \((\kappa \oplus 1)(n) = \kappa(n) + 1 \mod T^+(n)\) for all \(n \in \mathbb{N}\).

We let \(k_n(x) = \kappa(n)\) and \(\psi_n(x) = \sigma^{-\kappa(n)}(y) \upharpoonright T^+(n)\).

We then define the sets \(W_n\) so that they satisfy all relevant conditions.
And what about that definition of $D$?

We define $D$ so that:

- $(D\varepsilon)$ holds, and

For each $x = (y, \kappa) \in X$ and $N \in \mathbb{N}$ the sets $U_N(x)$ of all $x' = (y', \kappa')$ such that $\forall i \mid i < N \implies y(i) = y'(i) \& \kappa(i) = \kappa'(i)$ are open and form a basis for the topology.

The proof that $X$ is compact and $F$ a homeomorphism then becomes completely standard and independent from the more technical parts of the argument.

Minimality of $(X, F)$ will result from the following properties:

(FC) When $x \in X$ and $m < n$, then $k_m(x) = k_n(x) \mod T^+(m)$.

(XM) For all $m < n$ and all $\psi, \varphi \in \mathcal{X}_m$, every $\psi^+ \in \mathcal{X}_n$ contains a block of the form $\psi \prec \varphi$. 

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