On the role of limsup in definitions of topological entropy: What we proved, how we proved it, and what remains open

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This talk aims at giving a reasonably accessible introduction to rather abstract and technical results that we have been working on for the last couple of years.

Therefore we will omit most of the details and sacrifice some rigor.

The complete and formal presentation of our work can be found in

W. Just and Y. Xin (2017). On the role of limsup in the definition of topological entropy via spanning or separation numbers. Part I: Basic examples. *Preprint.* arXiv:1707.09052

Topological entropy of a dynamical system: The idea

For the purpose of this talk, a dynamical system is a pair (X, F), where X is a compact metric space with distance function D and $F : X \to X$ is a homeomorphism. Roughly speaking:

- A (forward) trajectory is a sequence $(F^t(x))_{t=0}^{\infty}$ for some $x \in X$.
- **In chaotic systems,** for sufficiently small $\varepsilon > 0$, the number $N_T(\varepsilon, D)$ of trajectories that are distinguishable at resolution ε within T time steps scales like $B(\varepsilon)^T$ for some $B(\varepsilon) > 1$.
- Thus we can use the growth rate of $N_T(\varepsilon, D)$ to define measure for how chaotic the system is:

  $$h(X, F) = \lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln N_T(\varepsilon, D)}{T}.$$

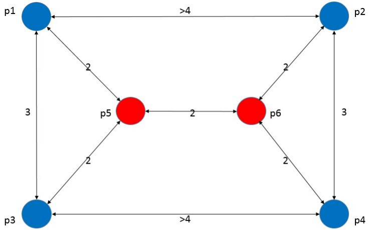
- This measure is called the topological entropy of (X, F).

Separation numbers and spanning numbers

Let (X, d) be metric space, and let $\varepsilon > 0$.

We define the separation number $sep(X, \varepsilon, d)$ as the **largest** size of a subset of $Y \subset X$ such that $d(x, x') \geq \varepsilon$ for all $x, x' \in Y$,

and the spanning number $span(X, \varepsilon, d)$ as the **smallest** size of a subset of $Y \subset X$ such that for all $x \in X$ there exists $y \in Y$ with $d(x, y) < \varepsilon$.



A = {p1, p2, p3, p4} is 3-separated of size 4 = sep(X, 3, d)
B = {p5, p6} is 3-spanning of size 2 = span(X, 3, d)

Two definitions of $N_T(\varepsilon, D)$

We could define $N_T(\varepsilon, D)$ as the largest size $sep(X, \varepsilon, D_T)$ of a (T, ε)-separated subset of X, that is, of a set $Y \subseteq X$ such that for all $x, x' \in Y$ there exists a $0 \leq t < T$ with $D(F^t(x), F^t(x')) \geq \varepsilon$.

Or we could define $N_T(\varepsilon, D)$ as the smallest size $span(X, \varepsilon, D_T)$ of a (T, ε)-spanning subset of X, that is, of a set $Y \subseteq X$ such that for all $x \in X$ there exists $y \in Y$ such that for all $0 \leq t < T$ we have $D(F^t(x), F^t(y)) < \varepsilon$.

The separation numbers $sep(X, \varepsilon, D_T)$ and spanning numbers $span(X, \varepsilon, D_T)$ are always finite and satisfy the inequality $sep(X, \varepsilon, D_T) \geq span(X, \varepsilon, D_T)$.

But why limsup?

So we can define the topological entropy $h(X, F)$ as:

$$\lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln sep(X, \varepsilon, D_T)}{T} = \lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln span(X, \varepsilon, D_T)}{T}.$$

But why lim sup? Could we use lim instead?

Chorus of the experts: It doesn't really matter. There is another definition, based on covering numbers, where you can.

But for the definitions based on separation or spanning numbers?

Chorus: Then you cannot.

But why?

Chorus: Because there is no obvious reason why you could.

Any counterexamples?

Chorus: Hmmm. Not that we know of.

Let's construct one!!

Our main theorem

Theorem

There exists a system (X^-, F) with a metric D on X^- such that for every ε of the form $\varepsilon = 3^{-j}$ for some $j \in \mathbb{N}$ we have:

$$\liminf_{T\to\infty} \frac{\ln sep(X^-, \varepsilon, D_T)}{T} < \limsup_{T\to\infty} \frac{\ln sep(X^-, \varepsilon, D_T)}{T}, \qquad (1)$$

$$\liminf_{T\to\infty} \frac{\ln span(X^-, \varepsilon, D_T)}{T} < \limsup_{T\to\infty} \frac{\ln span(X^-, \varepsilon, D_T)}{T}. \qquad (2)$$

Remark: We have also shown that neither does (1) imply (2) nor does (2) imply (1).

Towards the proof of our main theorem: A preliminary result

Lemma

There exists a positive constant R^* such that for each positive integer T there exist a finite dynamical system (X_0, F) such that for some $\varepsilon_0 > 0$:

(i) $sep\left(\varepsilon_0, D^0_{3T}\right) = span\left(\varepsilon_0, D^0_{3T}\right) = 3T2^T = |X_0|$.

(ii) $span\left(\varepsilon_0, D^0_{T}\right) \leq sep\left(\varepsilon_0, D^0_{T}\right) \leq R^* T^2$.

Sketch of the proof of the lemma

Let $0 < \delta_0 < \varepsilon_0 < 2\delta_0$.

We construct a finite X_0 and D^0 so that $D^0(x, x') \in \{\delta_0, \varepsilon_0\}$ for all $x \neq x' \in X_0$.

Then (X_0, D^0) will automatically be a compact metric space.

We let X_0 be the set of all functions $x \in {}^{\mathbb{Z}}\{0, 1\}$ that are periodic with period T.

(For this rough sketch we ignore a second coordinate of the actual construction.)

Now let $F = \sigma$ be the shift operator defined by

$\sigma(x)(t) = x(t + 1)$ for all $x \in X_0$ and $t \in \mathbb{Z}$.

Then F will automatically be a homeomorphism of X_0.

The set X_0 in this example is a two-sided subshift, that is, a subset $\Sigma \subset {}^{\mathbb{Z}}A$ for a finite alphabet A that is closed both topologically and under the subshift operator σ.

Subshift systems

The set X_0 in this example is a two-sided subshift, that is, a subset $\Sigma \subset {}^{\mathbb{Z}}A$ for **some** finite alphabet A that is closed both topologically and under the subshift operator σ.

If we define a subshift metric D on Σ by

$$D(x, y) = d\left(x_{\Delta(x,y)}, y_{\Delta(x,y)}\right) 2^{-\Delta(x,y)},$$

for some metric d on A, where $\Delta(x, y)$ is the first n where $x_n \neq y_n$, then (Σ, σ) becomes a subshift system.

When d takes only values in the set $\{0, \varepsilon\}$ for some $\varepsilon > 0$, then we get a standard subshift metric.

But for subshift systems we know that . . .

$$\forall\, T > 0 \quad cov(\Sigma, \varepsilon, D_T) = sep(\Sigma, \varepsilon, D_T) = span(\Sigma, \varepsilon, D_T), \quad (3)$$

which is inconsistent with the inequalities in our lemma because the function $T \mapsto \ln cov(X, \varepsilon, D_T)$ is always subadditive.

The proof of (3) assumes a standard subshift metric, which does not apply to our example.

But our $(X_0, F) = (X_0, \sigma)$ is an example of a subshift system with a nonstandard subshift metric D^0, as the underlying d can take values in the set $\{0, \delta_0, \varepsilon_0\}$.

Question: Are there examples as in our theorems that are subshift systems with nonstandard subshift metrics?

Sketch of the proof of the lemma, continued

We need to define D^0. Let $T^+ = 3T$ and partition the interval $T^+ := \{0, \dots, T^+ - 1\}$ into three consecutive subintervals I_j^0 of length T each, where $j \in \{1, 2, 3\}$.

Let S be the set of all restrictions of $x \in X_0$ to the interval T^+, and let $[S]^2$ denote the set of all unordered pairs of of elements of S. We choose a function $c : [S]^2 \to \{1, 2, 3\}$ that we will refer to as a coloring of $[S]^2$ with 3 colors.

We will need certain special properties of c. About this later.

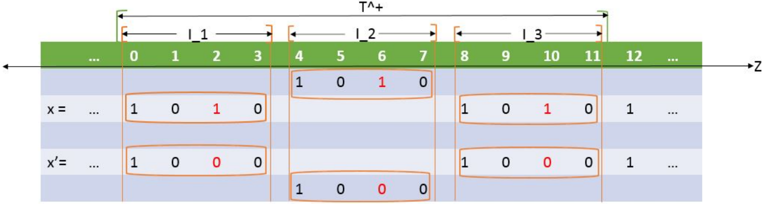
Right now let us define, for $x \neq x' \in X_0$ and $0 \leq t < T^+$:
$D^0(\sigma^t(x), \sigma^t(x')) = \varepsilon_0$ if $x(t) \neq x'(t)$ and $t \in I_{c(\{x \restriction T^+, x' \restriction T^+\})}$,
$D^0(\sigma^t(x), \sigma^t(x')) = \delta_0$ otherwise.

We could get the same element $x \in X_0$ of the form $x = \sigma^{t_1}(x_1) = \sigma^{t_2}(x_2)$ for $t_1 \neq t_2$ and $x_1 \neq x_2$.
The second coordinates of $x \in X_0$ that we ignored here eliminate these ambiguities.

But if we sweep these coordinates under the rug . . .

$D^0(\sigma^t(x), \sigma^t(x')) = \varepsilon_0$ if $x(t) \neq x'(t)$ and $t \in I_{c(\{x \upharpoonright T^+, x' \upharpoonright T^+\})}$,
$D^0(\sigma^t(x), \sigma^t(x')) = \delta_0$ otherwise.

Example: Suppose $T^+ = 3 \times 4 = 12$,
$c(\{101010101010, 100010001000\}) = 2$

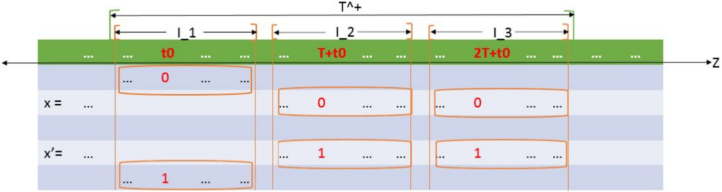


$t = 2 \notin I_c = I_2 \Rightarrow D^0(\sigma^2(x), \sigma^2(x')) = \delta_0$

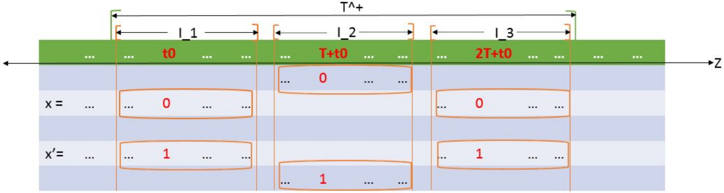
$t = 6 \in I_c = I_2 \Rightarrow D^0(\sigma^6(x), \sigma^6(x')) = \varepsilon_0$

The set X_0 is (T^+, ε_0)-separated

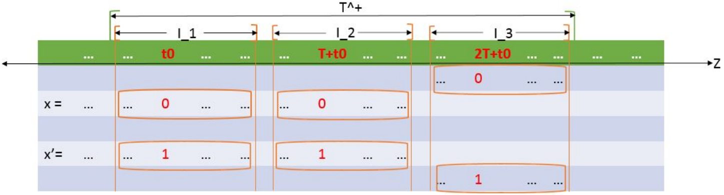
If $c(\{x \restriction T^+, x' \restriction T^+\}) = 1$, then $D^0(\sigma^{t_0}(x), \sigma^{t_0}(x')) = \varepsilon_0$



If $c(\{x \restriction T^+, x' \restriction T^+\}) = 2$, then $D^0(\sigma^{T+t_0}(x), \sigma^{T+t_0}(x')) = \varepsilon_0$



If $c(\{x \restriction T^+, x' \restriction T^+\}) = 3$, then $D^0(\sigma^{2T+t_0}(x), \sigma^{2T+t_0}(x')) = \varepsilon_0$



Part (i) of the lemma follows:

$$sep\left(\varepsilon_0, D^0_{3T}\right) = span\left(\varepsilon_0, D^0_{3T}\right) = |X_0| = 3T2^T.$$

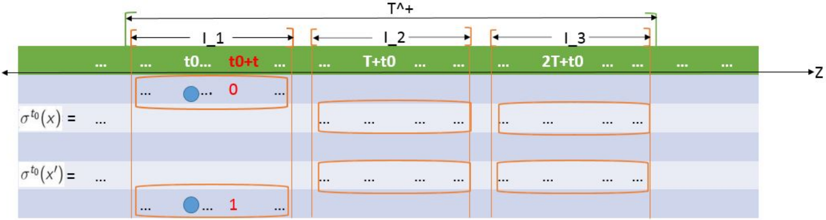
(T, ε_0)-separated subsets of X_0

Suppose $Y \subset X_0$ is such that for some t_0:

$$\forall x \neq x' \in Y \, \exists 0 \leq t < T \quad D^0(\sigma^{t_0+t}(x), \sigma^{t_0+t}(x')) = \varepsilon_0.$$

Example: $t_0 \in I_1$. Note that we always have $|I_1| = |I_2| = |I_3| = T$, and we need $0 \leq t < T$.

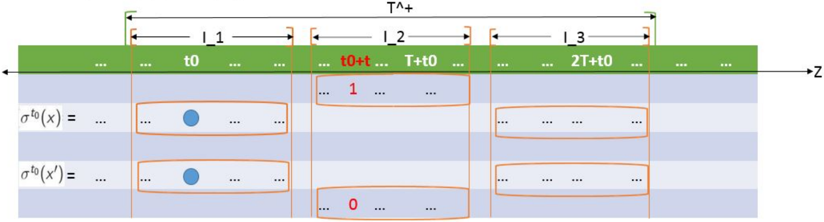
If $c(\{x \restriction T^+, x' \restriction T^+\}) = 1$ and $t_0 + t \in I_1$ then $D^0(\sigma^{t_0+t}(x), \sigma^{t_0+t}(x')) = \varepsilon_0$



(T, ε_0)-separated subsets of X_0

Example: $t_0 \in I_1$. Note that we always have $|I_1| = |I_2| = |I_3| = T$, and we need $0 \le t < T$.

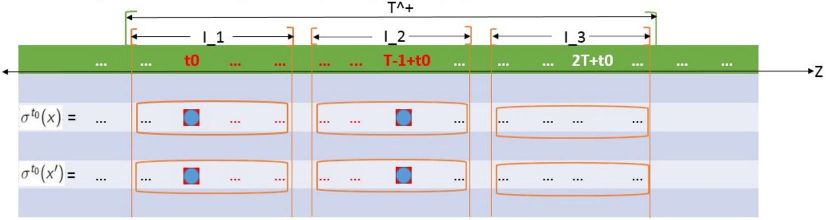
If $c(\{x \restriction T^+, x' \restriction T^+\}) = 2$ and $t_0 + t \in I_2$ then $D^0(\sigma^{t_0+t}(x), \sigma^{t_0+t}(x')) = \varepsilon_0$



(T, ε_0)-separated subsets of X_0

Example: $t_0 \in I_1$. Note that we always have $|I_1| = |I_2| = |I_3| = T$, and we need $0 \leq t < T$.

If $t_0 \in I_1$, then for all $0 \leq t < T$ we have $t_0 + t \in I_1 \cup I_2$.
Then if $c(\{x \upharpoonright T^+, x' \upharpoonright T^+\}) = 3$, $D^0(\sigma^{t_0+t}(x), \sigma^{t_0+t}(x')) < \varepsilon_0$.



The figures show that the restriction of c to the set of pairs $\{x, x'\} \subset Y$ can take at most two values; that is,
Y is $\leq$ 2-chromatic for c.

Thus $sep(X_0, \varepsilon_0, D^0_T)$ does not exceed the largest size of a $\leq$ 2-chromatic subset for c,

and if we use colorings c without a $\leq$ 2-chromatic subset of size $> 3TR\ln 2$ for some fixed R, part (ii) of the lemma follows:

$span\left(\varepsilon_0, D^0_T\right) \leq sep\left(\varepsilon_0, D^0_T\right) \leq 3R\ln 2\, T^2 := R^* T^2$.

One can prove their existence for $R = \frac{1}{\ln\sqrt{3} - \ln\sqrt{2}}$ with the probabilistic method.

For random colorings one can estimate the probability of existence of a $\leq$ 2-chromatic subset of size $> 3TR\ln 2$ as less than 1, so that there must exist at least one coloring with the desired properties.

This proves the lemma. $\square$

Dr. Young: Take infinite products of these systems.

Let's develop Dr. Young's suggestion, see how far it takes us, and where we will need to modify our construction.

What do you mean by “these systems”?

For every n we choose positive integers $T(n), C(n)$ and $T^+(n) = T(n)C(n)$. There is nothing magic about the number $C(n) = 3$ in this construction.

We also pick $0 < \delta_n < \varepsilon_n$ (with some additional properties) and let $\varepsilon = \sum_{n=0}^{\infty} \varepsilon_n$.

Then we can think of $X_n \subseteq {}^{\mathbb{Z}}\{0,1\}$, with a metric D^n that takes values in $\{0, \delta_n, \varepsilon_n\}$ and is defined, analogously as D^0 above, in terms of a coloring c_n without large $\leq$ 2-chromatic subsets.

ECn-systems

We define *ECn-systems* (X_n, F_n) for $n \in \mathbb{N}$ as follows:

- The set X_n consists of triples (y, n, k), where $y \in {}^{\mathbb{Z}}\{0, 1\}$ and $k \in \{0, 1, \dots, T^+(n) - 1\}$.
- The function F_n is defined by

$$F_n((y, n, k)) = (\sigma(y), n, F_n(k)), \text{ where}$$
$$F_n(k) = (k + 1) \; mod \; T^+(n).$$

Thus ECn-systems (X_n, F_n) are products of a subshift systems $({}^{\mathbb{Z}}\{0, 1\}, \sigma)$ with a cyclic permutations of $T^+(n)$.

Next we define an auxiliary function $\Phi : \bigcup_{n \in \mathbb{N}} X_n \to {}^{T^+(n)}\{0, 1\}$:

$$\Phi((y, n, k)) = (y(-k), y(-k + 1), \dots, y(-k + T^+(n) - 1))$$

and an auxiliary function Δ on $\left({}^{\mathbb{Z}}\{0, 1\}\right)^2$, such that $\Delta(y, z)$ marks the first place i where $y(i) \neq z(i)$.

Let $\beta_n \in \{\varepsilon_n, \delta_n\}$ and let c_n be a coloring. We define an ECn-metric $D^n : (X_n)^2 \to [0, \infty)$ as follows:

(Dn1) If $k \neq k'$, then $D^n((y, n, k), (y', n, k')) = \beta_n$.

(Dn2) If $k = k'$ and $y = y'$, then $D^n((y, n, k), (y', n, k')) = 0$.

(Dn3) If $k = k'$ and $y \neq y'$, then

- (Dn31) If $0 < \Delta(y, y') < \infty$, then $D^n((y, n, k), (y', n, k')) = \varepsilon_n 3^{-\Delta(y,y')}$.
- (Dn32) If $\Delta(y, y') = 0$, then we let $\varphi = \Phi((y, n, k))$ and $\psi = \Phi((y', n, k'))$ and define:
  - $D^n((y, n, k), (y', n, k')) = \varepsilon_n$ if $k \in I_j^n$ and $c_n(\varphi, \psi) = j$.
  - $D^n((y, n, k), (y', n, k')) = \delta_n$ if $k \in I_j^n$ and $c_n(\varphi, \psi) \neq j$.

EC-systems and EC-metrics

Let $(X_n, F_n)_{n\in\mathbb{N}}$ be a sequence of EC-systems with EC-metrics D^n. Then an EC-system (X, F) with EC-metric D is defined as follows:

- X: Let $X = \prod_{n\in\mathbb{N}} X_n$. That is, we let X consist of all sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $x_n \in X_n$ for each $n \in \mathbb{N}$.
- F: For $x \in X$, define $F(x)_n = F_n(x_n)$ for all $n \in \mathbb{N}$.
- D: The function $D : X^2 \to [0, \infty)$ is defined as:

$$D(x, x') = \sum_{n\in\mathbb{N}} D^n(x_n, x'_n).$$

Then (X, D) will be compact and F will be a homeomorphism.

Upper bounds on $sep(X, \varepsilon, D_{2T(n)})$ and $span(X, \varepsilon, D_{2T(n)})$

The really cool property of EC-systems is that

$$D_T(x, x') = \varepsilon \;\Leftrightarrow\; \exists 0 \leq t < T \, \forall n \in \mathbb{N} \quad D^n(F_n^t(x_n), F_n^t(x_n')) = \varepsilon_n.$$

Thus (T, ε)-separated subsets of X must be (T, ε_n)-separated on each coordinate X_n,

and upper bounds on $sep(X_n, \varepsilon_n, T)$ translate into upper bounds on $sep(X, \varepsilon, T)$.

This gives upper bounds on $sep(X, \varepsilon, 2T(n))$

that allow us to keep $\liminf_{T \to \infty} \frac{\ln sep(X, \varepsilon, D_T)}{T}$ small.

How about lower bounds on $sep(X, \varepsilon, D_{T+(n)})$?

If we want to construct a (T, ε)-separated subset $B \subset X$, then the cool property

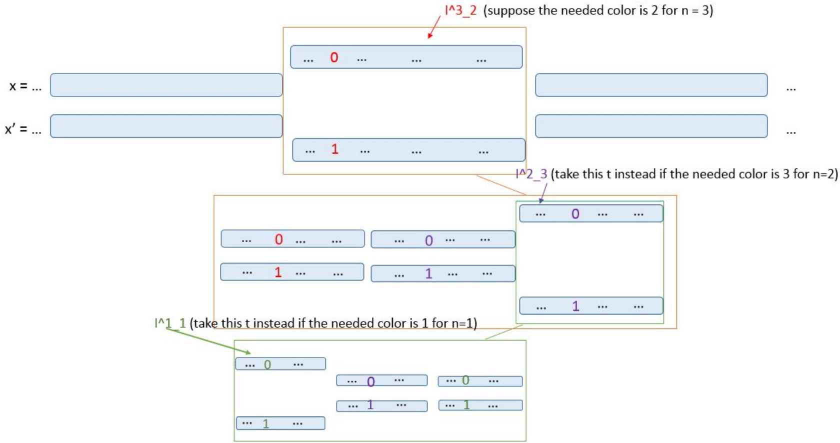
$$D_T(x, x') = \varepsilon \;\Leftrightarrow\; \exists 0 \leq t < T \, \forall n \in \mathbb{N} \quad D^n(F_n^t(x_n), F_n^t(x_n')) = \varepsilon_n$$

forces us to produce, for each $x \neq x' \in B$,

one t with $0 \leq t < T$ such that

$D^n(F_n^t(x_n), F_n^t(x_n')) = \varepsilon_n$ for all $n \in \mathbb{N}$ simultaneously.

One more figure



We ran into serious obstacles . . .

- There aren't enough periodic functions to give us $\limsup_{n\to\infty} \frac{\ln sep(X,\varepsilon,D_{T+(n)})}{T} > 0$ from this construction.
- For non-periodic functions he argument from the proof of part (i) of the lemma breaks down.
- To overcome these obstacles, we need to find another type of colorings c_n with special properties.

Colorings with nicer properties

For each $n \in \mathbb{N}$ we choose a coloring c_n such that:

(cC1) Assume
$\varphi \restriction (0, \ldots, T^+(n-1)-1) \neq \psi \restriction (0, \ldots, T^+(n-1)-1)$
while $\varphi(i) = \psi(i)$ for all
$i \in \{T^+(n-1), T^+(n-1)+1, \ldots, T^+(n)-1\}$.
Then, $c_n(\varphi, \psi) = 1$.

(cC2) Let $C(\varphi, \psi) \subset [C(n)]$ denote the set of j such that
$\varphi \restriction I^n_j \neq \psi \restriction I^n_j$.
If $|C(\varphi, \psi)| \geq 3$, then $c_n(\varphi, \psi) \in C(\varphi, \psi)$.

(cC3) For every subset $S \subset {}^{T^+(n)}\{0, 1\}$ of size $|S| \geq 2^{0.75T(n)}$ the restriction of c_n to $[S]^2$ takes on at least three colors.

Condition (cC2) here essentially says: If two functions x_n, x'_n differ often enough on $\{0, \ldots, T^+ - 1\}$, they must differ in at least one place **"with the right color."**

Do colorings with these properties exist?

Yes, as long as our parameters satisfy the following conditions:

(PCn): $\prod_{i=0}^{n}(C(i)-2) > 0.95\prod_{i=0}^{n} C(i).$

(PKn1): $K(n)$ is a positive integer multiple of 100.

(PKn2): $2^{0.05T^+(n)} > \left(\frac{C(n)^2}{2}\right)\prod_{m=0}^{n-1}\left(\frac{C(m)^2}{2}\right)^{\prod_{i=m+1}^{n}(C(i)-2)K(i)}.$

(PKn3): $\left(2^{0.7T^+(n-1)K(n)}\right)! > \frac{C(n)^2}{2}.$

(PKn4): $\left(\log_2\sqrt{\frac{3}{2}}\right)\left(2^{0.7T^+(n-1)K(n)}-1\right) > T^+(n-1)C(n)K(n).$

(PKn5): $2^{0.01T(n)} = 2^{0.01K(n)T^+(n-1)} \geq C(n).$

Parameters that satisfy all of the above exist.

We got a result!

Then we can show that

$$\liminf_{T\to\infty} \frac{\ln sep(X,\varepsilon,D_T)}{T} < \limsup_{T\to\infty} \frac{\ln sep(X,\varepsilon,D_T)}{T}.$$

The proof uses the ideas that we have described so far.

How about the spanning numbers?

Getting an EC-system with

$$\liminf_{T\to\infty} \frac{\ln span(X,\varepsilon,D_T)}{T} < \limsup_{T\to\infty} \frac{\ln span(X,\varepsilon,D_T)}{T}$$

seems a lot harder.

For obtaining a lower bound on $span(X, \varepsilon, D_T)$ we not only need to produce a large (T, ε)-separated subset B of X,

but also need to show that here is no other subset $B' \subset X$ of smaller size that would be (T, ε)-spanning.

Let's consider a subspace X^-

For the EC-space (X, D) we define certain subsets $W^n \subset X$ of sufficiently large sizes.

Then we define:

$$X^- = \overline{\bigcup_{t \in \mathbb{Z}} F^t \left(\bigcup_{n \geq 0} W^n \right)},$$

Similar arguments as before show that the system (X^-, F) has the properties stated in our main theorem.

First we need different kinds of colorings

For each $n \in \mathbb{N}$ we choose a coloring c_n such that:

(cCi) Assume $\varphi(i) = 0$ for all $T^+(n-1) \leq i \leq T^+(n) - 1$.
Then, $c_n(\varphi, \psi) = 1$ for all $\psi \neq \varphi \in {}^{T^+(n)}\{0,1\}$.

(cC) Assume there exist $T^+(n-1) \leq i, j \leq T^+(n) - 1$ such that $\varphi(i) = \psi(j) = 1$. Let $C(\varphi, \psi) \subset [C(n)]$ denote the set of j such that $\varphi \restriction I^n_j \neq \psi \restriction I^n_j$.
If $|C(\varphi, \psi)| \geq 3$, then $c_n(\varphi, \psi) \in C(\varphi, \psi)$.

(cC3) For every subset $S \subset {}^{T^+(n)}\{0,1\}$ of size $|S| \geq 2^{0.75\,T(n)}$ the restriction of c_n to $[S]^2$ takes on at least three colors.

The existence of such colorings c_n can again be proved by the probabilistic method.

To obtain such bounds I proved the following result.

Lemma

Fix any $n \in \mathbb{N}$. Then for all $x \in X^-$ and all $u \neq v \in W^n$,

$$\max\{D_{T^+(n)}(x, u), D_{T^+(n)}(x, v)\} = \varepsilon.$$

It then follows that $span(X^-, \varepsilon, D_{T^+(n)}) \geq |W^n|$ for all $n \in \mathbb{N}$.

Proof of the lemma

Fix any $n \in \mathbb{N}$ and $u \neq v \in W^n$. For $x \in X^-$, we distinguish the following seven cases:

Case 1: $x \in W^n$.

Case 2: $x \in W^m$ for some $0 \leq m < n$ (if $n \neq 0$).

Case 3: $x \in W^m$ for some $m > n$.

Case 4: $x \in F^\tau(W^m)$ for some $\tau > 0$ and $0 \leq m \leq n$.

Case 5: $x \in F^\tau(W^m)$ for some $\tau > 0$ and $m > n$.

Case 6: $x \in F^\tau(W^m)$ for some $\tau < 0$ and $m \in \mathbb{N}$.

Case 7: $x \in X^- \setminus \left[\bigcup_{t \in \mathbb{Z}} F^t(\bigcup_{n \geq 0} W^n)\right]$.

Two more open problems

Recall that a dynamical system is topologically transitive if there exists at least one point with a dense (forward) orbit, and minimal if every point has a dense (forward) orbit.

Question

Can analogues of our theorems be obtained for topologically transitive or even for minimal systems?

This question remains largely open, but we have a partial positive result on topologically transitive systems.

The following fascinating question remains completely open:

Question

Consider (X, F) with (X, d) a compact metric space. Is there always a metric D on X that is equivalent to d, for which $\lim_{T\to\infty} \frac{\ln sep(\varepsilon, D_T)}{T}$ and/or $\lim_{T\to\infty} \frac{\ln span(\varepsilon, D_T)}{T}$ exist for all $\varepsilon > 0$?