Dynamics of cooperative discrete systems

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Discrete systems

Let $\Pi = \{0, 1, \ldots, p-1\}^n$, where $n \geq 1,\ p > 1$. A map $g : \Pi \to \Pi$ defines an n-**dimensional,** p-**discrete system** dynamical system (Π, g).

A state in the system at time t will be denoted by $s(t) = [s_1(t), \ldots, s_n(t)]$.

The dynamics is given by

$$s(t+1) = g(s(t)), \quad s(t) \in \Pi.$$

Note that all trajectories eventually reach a periodic orbit or a fixed point.

The letter n will always stand for the dimension of the system.

Cooperative discrete systems

Define the **cooperative (partial) order** on Π by $r \leq s$ if $r_i \leq s_i$ for $i = 1, \ldots, n$. A discrete system is *cooperative* if $r(0) \leq s(0)$ implies $r(t) \leq r(t)$ for every $t \geq 0$, which is equivalent to the implication

$$r \leq s \to g(r) \leq g(s).$$

Why do we care about such systems?

Discrete cooperative systems were recently proposed as a tools to study genetic networks by Sontag, Laubenbacher, and others.

Of particular interest is the case of **Boolean systems (Boolean networks)** when $p = 2$. Boolean networks (not necessarily cooperative ones) were already proposed as models of gene regulation by Kauffman in the late 1960's. In this framework, the components s_i are considered discretized concentrations of individual gene products and the components g_i of $g = [g_1, \ldots, g_n]$ are considered **regulatory functions** of the corresponding genes.

Long periodic orbits

Kauffman distinguishes two basic types of dynamics of Boolean networks: an **ordered regime** and a **chaotic regime.** In particular, very long periodic orbits are one hallmark of the chaotic regime and tend not to be observed in the ordered regime.

Very long periodic orbits tend not to be observed in simulations of random Boolean networks if:

- All regulatory function have only a small number of inputs, or

- all regulatory functions are nested canalyzing, or

- **there are few negative feedback loops.**

How long is “long?”

Note that all periodic orbits in a p-discrete dynamical system have length at most p^n.

Our research was guided by the question:

Which conditions guarantee that for some or all c with $1 < c < p$ and sufficiently large n, an n-dimensional cooperative system cannot have periodic orbits of length $\geq c^n$?

Note that we are aiming at rigorously proving the absence of exponentially long orbits rather than showing that they occur infrequently.

Smale's Theorem

Smale's Theorem states that any compactly supported, $(n-1)$-dimensional, C^1 dynamical system defined on

$$H = \{x \in \mathbb{R}^n \,|\, x_1 + \ldots + x_n = 0\}$$

can be embedded into some cooperative C^1 system. Equivalently, the dynamics of cooperative systems can be completely arbitrary on unordered hyperplanes such as H.

Does this theorem have a counterpart discrete cooperative systems?

An "almost" Smale Theorem

1. For every $n, p > 1$ there exists an n-dimensional p-discrete system that **cannot** be embedded into a cooperative p-discrete system of dimension $n + 1$.

2. For every $n > 0$, there exists p_0 such that for every $p > p_0$ every n-dimensional p-discrete system **can** be embedded into a cooperative p-discrete system of dimension $n + 2$.

Hirsch's Theorem

A C^1-cooperative system is *strongly cooperative* if for every two different initial conditions $x(0) \leq y(0)$ we have $x_i(t) < y_i(t)$ for all $i = 1, \ldots, n$ and $t > 0$.

Hirsch's theorem states that almost every bounded solution of a C^1-strongly cooperative system converges towards the set of equilibria. This result rules out stable periodic orbits and chaotic attractors.

Is there a discrete counterpart of strong cooperativity that rules out exponentially long orbits?

Strongly cooperative discrete systems

Let $r(t), s(t)$ be states in a p-discrete system (Π, g). We will write $r(t) < s(t)$ if $r(t) \leq s(t)$ and $r_i < s_i$ for at least one i.

We will say that a (Π, g) is **strongly cooperative** if

$$r(0) < s(0) \to r(1) < s(1)$$

for all $r(0), s(0) \in \Pi$.

A discrete version of Hirsch's Theorem

Theorem: Suppose (Π, g) is an n-dimensional strongly cooperative p-discrete system, Then each periodic orbit in (Π, g) has length at most

$$e^{\sqrt{(p-1)n \ln(p-1)n}(1+o(1))}.$$

Note that $e^{\sqrt{(p-1)n \ln(p-1)n}(1+o(1))} < c^n$ for each $c > 1$ and sufficiently large n; hence strongly cooperative discrete systems cannot have exponentially long periodic orbits.

Moreover, we show that small perturbations of initial conditions don't amplify, an **analogue of Lyapunov stability of all attractors.**

Irreducible C^1 systems

We can associate directed graphs G_x on $\{1, \ldots, n\}$ with an n-dimensional cooperative C^1-system by including an arc $< i, j >$ iff $\partial f_j / \partial x_i(x) > 0$.

We called the system **irreducible** if G_x is strongly connected for every x.

Irreducible cooperative C^1-systems are strongly cooperative and Hirsch's Theorem applies to them.

Irreducible discrete systems

For $s \in \Pi$ and $i \in \{1, \dots, n\}$ define $s^{i+} \in \Pi$ by $(s^{i+})_i = \min\{s_i + 1, p - 1\}$ and $(s^{i+})_j = x_j$ for $j \neq i$. Similarly, define $x^{i-} \in \Pi$.

Define G_s^* by including an arc $< i, j >$ iff $g(s)_j < g(s^{i+})_j$ or $g(s^{i-})_j < g(s)_j$. Moreover, define G_s by removing the arcs $< i, j >$ from G_s^* for which $0 < s_i < p - 1$ and $g(s^{i-})_j = g(s)_j$ or $g(s)_j < g(s^{i+})_j$.

Call (Π, g) **irreducible** if G_s is strongly connected for all s and **semi-irreducible** if G_s^* is strongly connected for all s. We call (Π, g) **strongly irreducible along an attractor** S if the intersection of all G_s for $s \in S$ is strongly connected.

Irreducible cooperative discrete systems

1. Irreducible cooperative p-discrete systems are strongly cooperative and have orbits of length at most n.

2. Semi-irreducible cooperative non-Boolean p-discrete systems can have orbits of length $\geq c^n$ for every $0 < c < p$.

3. Strong irreducibility along an attractor S in a cooperative Boolean system implies $|S| \leq n$.

4. Non-Boolean cooperative p-discrete systems can be strongly irreducible along attractors of length $\geq c^n$ for every $0 < c < p$.

The empirical results revisited

Very long periodic orbits tend not to be observed in simulations of random Boolean networks if:

- **All regulatory function have only a small number of inputs,** or

- all regulatory functions are nested canalyzing, or

- there are few negative feedback loops.

Bi-quadratic cooperative systems

Can cooperative Boolean systems with limited numbers of inputs have very long orbits?

Let us call a Boolean system **quadratic** if each regulatory function takes input from at most two variables. The only regulatory functions allowed in cooperative quadratic Boolean systems are the **strictly quadratic** Boolean functions

$s_k(t+1) = s_i(t) \wedge s_j(t)$ and $s_k(t+1) = s_i(t) \vee s_j(t)$ and the **monic** functions $s_k(t+1) = s_i(t)$.

All these permissible functions are nested canalyzing.

We call a quadratic Boolean system **bi-quadratic** if, in addition, each variable can act as input only to at most two variables.

Long orbits in bi-quadratic cooperative Boolean systems

Theorem: Let $0 < c < 2$. Then for all sufficiently large n there exist na n-dimensional bi-quadratic cooperative Boolean systems (Π, g) with orbits of length $> c^n$.

The system constructed in the proof is akin to a small (relative to n) Turing machine that acts on several "tapes" of variables with monic regulatory functions.

Can one prove the theorem with a radically different construction?

Turing systems

We call an n-dimensional Boolean system an (M, n)-*Turing system* if at least $n - M$ of the regulatory functions are monic.

Cooperative (M, n)-Turing systems behave like a Turing machine with at most M internal variables that acts on one or several 'tapes' that contain the values of the remaining variables. One can also conceptualize (M, n) Turing systems as M-dimensional Boolean delay systems.

The systems constructed for the proof of our theorem are $(M(n), n)$-Turing systems such that

$$\lim_{n \to \infty} \frac{M(n)}{n} = 0.$$

Turing systems are the only examples

Theorem: Let $\alpha > 0$. Then there exists a positive constant $c < 2$ such that for sufficiently large n, every n-dimensional bi-quadratic cooperative Boolean system with an orbit of length at least c^n is an $(\alpha n, n)$-Turing system.