

Discrete Dynamical Systems: Introduction to Symbolic Dynamics

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Review: Discrete-Time Dynamical Systems

A (deterministic) **discrete-time dynamical system** is a pair (X, F) such that

- The **state space** X is a topological space.
- $F : X \rightarrow X$ is a **continuous map**.

The **time evolution function** is then given by

$$\varphi(x, t) = F^t(x).$$

When F is a homeomorphism we will write (X, T) instead of (X, F) . The system then becomes time-reversible and we take $\mathbb{T} = \mathbb{Z}$ as the **time line**.

When we write (X, F) , we implicitly assume that $\mathbb{T} = \mathbb{N}$ is the **time line**.

Some examples

We have seen examples where the state space X is:

- The set of positive integers (the system for the Collatz Conjecture),
- the set of pairs of positive integers (the system for Euclid's algorithm),
- \mathbb{R}^n for some n (linear and affine systems),
- $\{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\}$ (directions of vectors for a linear system),
- $[0, 1]$ (the discrete logistic system).

Here I will introduce systems with a **compact** state space

(like in the last two examples on the list)

where the elements of X are sequences of symbols

(unlike in the above examples).

The study of these systems is known as **symbolic dynamics**.

Alphabets, words, and sequences

Let A be a finite set, called an **alphabet**, with at least 2 elements.

Let A^* be the set of **words** that can be written in A , that is, the set of all finite sequences $w = (a_0, a_1, \dots, a_{\ell-1})$ with $a_i \in A$ for all $i < \ell$.

Let $X = A^{\mathbb{N}}$ be the set of all sequences $a = (a_0, a_1, \dots, a_i, \dots)$ of elements of A .

Let $d(a, b) = 2^{-\min\{i: a_i \neq b_i\}}$, where $\min(\emptyset) = \infty$ and $2^{-\infty} = 0$.

- d is a metric on X .
- The family $\{U_w : w \in A^*\}$ is a basis for the topology on X , where $U_w = \{a \in A^{\mathbb{N}} : a \text{ starts with } w\}$.
- Each U_w is **clopen** (simultaneously closed and open).
- (X, d) is a zero-dimensional compact metric space.

Full (one-sided) shifts

Define the **one-sided shift operator** $\sigma : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$:

$$\sigma(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots)$$

Then σ is continuous wrt d :

If $w = (b_0, \dots, b_{\ell-1}) \in A^*$ and $\sigma(a) \in U_w$, then

$$a = (a_0, a_1, a_2, \dots) \text{ is such that } (b_0, \dots, b_{\ell-1}) = (a_1, \dots, a_{\ell})$$

and σ maps $U_{(a_0, a_1, \dots, a_{\ell})}$ into U_w .

The system (X, σ) is called a **(one-sided) full shift**.

There is also a **two-sided full shift**, but it will not be covered here. I will drop the adjective “one-sided” for the remainder of this talk.

Properties of $(A^{\mathbb{N}}, \sigma)$: Sensitive dependence

Recall the following definition.

Definition

Let (X, F) be a discrete dynamical system and let $K \subseteq X$. We say that the system is **sensitive on K** if

$$\exists \varepsilon > 0 \forall x \in K \forall \delta > 0 \exists y \in B_\delta(x) \cap K \exists t \in \mathbb{N} d(F^t(x), F^t(y)) > \varepsilon.$$

The full shift $(A^{\mathbb{N}}, \sigma)$ is always sensitive on $A^{\mathbb{N}}$, because each basic open neighborhood U_w of any a contains b with $a_t \neq b_t$ for some t that exceeds the length of w .

For such a, b, t we will have $d(F^t(a), F^t(b)) = 2^0$.

Properties of $(A^{\mathbb{N}}, \sigma)$: Transitivity

Recall the following definition:

Definition

(X, F) is **transitive** if for all nonempty open $U, V \subseteq X$ there exist infinitely many $t > 0$ with $F^t(U) \cap V \neq \emptyset$.

The full shift $(A^{\mathbb{N}}, \sigma)$ is always transitive, because for each basic open neighborhood U_w, U_v we can find $a \in A^{\mathbb{N}}$ such that a starts with the concatenation wv of the words w and v .

- Then $a \in U_w$,
- thus $F^t(a) \in F^t(U_w)$ if t denotes the length of w ,
- $F^t(a) \in U_v$.

Properties of $(A^{\mathbb{N}}, \sigma)$: Chaos

We have shown that full shifts are chaotic according to our first (more stringent) definition of chaos.

Note that this was **relatively easy** compared with proving the analogous property for systems like the discrete logistic system with $r = 4$.

(How) can we use symbolic dynamics (full shifts and related systems) to study dynamical systems with more familiar-looking state spaces?

Consider a discrete dynamical system (X, F) with X compact T_2 .

Consider a family \mathcal{V} such that

- 1 $\mathcal{V} = \{V_1, \dots, V_k\}$,
- 2 V_i is **regular closed**, that is, $V_i = cl(int(V_i))$,
- 3 \mathcal{V} is a **cover** of X , that is $V_1 \cup \dots \cup V_k = X$,
- 4 the sets $int(V_1), \dots, int(V_k)$ are pairwise disjoint.

Now let $A = \{1, \dots, k\}$. Call $a = (a_0, a_1, \dots) \in A^{\mathbb{N}}$ **an itinerary** for $x \in X$ if $F^t(x) \in V_{a_t}$ for all $t \in \mathbb{N}$.

When \mathcal{V} is a **generating cover** each $a \in A^{\mathbb{N}}$ can be the itinerary for **at most one** $x \in X$.

We will from now on **assume that \mathcal{V} is a generating cover**, and let $\pi(a)$ denote this unique x .

Disclaimer: Not all systems do have generating covers.

Now consider the inverse image $\Sigma : \{a \in A^{\mathbb{N}} : \exists x \in X \pi(a) = x\}$.

- Σ is a closed subspace of $A^{\mathbb{N}}$.
- Σ is **closed under** σ , that is $\sigma(a) \in \Sigma$ for all $a \in \Sigma$.

Subspaces Σ of $A^{\mathbb{N}}$ with the above properties (or systems $(\Sigma, \sigma \upharpoonright \Sigma)$) are called **subshifts**.

For simplicity we will write (Σ, σ) and call this system the **subshift of (X, F) (wrt \mathcal{V})**.

Subshifts of finite type

Let $\Sigma \subseteq A^{\mathbb{N}}$. A system (Σ, σ) is a subshift iff

- Σ is a closed subspace of $A^{\mathbb{N}}$.
- Σ is **closed under** σ , that is $\sigma(a) \in \Sigma$ for all $a \in \Sigma$.

For each subshift there exists a countable set

$FW = \{w_1, w_2, \dots\}$ of **forbidden words**

such that $a = (a_0, a_1, \dots) \in \Sigma$ iff

for all $k \leq \ell$ the word $(a_k, a_{k+1}, \dots, a_\ell) \notin FW$.

An important example are **subshifts of finite type**, where FW is finite.

The dynamics of (X, F) vs. the dynamics of (Σ, σ)

Let (Σ, σ) be the subshift of some system (X, F) .

Then $F(\pi)(a) = \pi(\sigma(a))$, that is, the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \pi \downarrow & & \pi \downarrow \\ X & \xrightarrow{F} & X \end{array}$$

Moreover, π is continuous. It is a **semiconjugacy** aka **factor map**.

(X, F) is called a **factor** of (Σ, σ) and (Σ, σ) is called an **extension** or **cover** of (X, F) .

When π is invertible, then (by compactness) it must be a homeomorphism.

In this case π is a **conjugacy** and the systems (X, F) and (Σ, σ) are **the same** from the point of view of dynamical systems.

We can then study the properties of (X, F) as properties of its subshift (Σ, σ) , which is usually easier.

When are (X, F) and (Σ, σ) similar?

Note that π is a conjugacy iff each of its **fibres** $\pi^{-1}\{x\}$ are singletons.

In this case the dynamical properties of (X, F) and (Σ, σ) are **the same**, but this can **only** happen if X is a zero-dimensional space.

If $\dim(X) > 0$, then π must have at least some fibres that contain more than one point.

When all fibres are **small**, then (X, F) and (Σ, σ) will still have similar properties.

When should we consider the fibres “small”?

Which $x \in X$ have fibres that are **not singletons, anyway?**

When does $x \in X$ have multiple fibres?

To simplify notation, let us consider this question only when F is a homeomorphism. Then we can consider the system (X, T) instead, where $T = F^{-1}$.

Key observation: If $\pi(a) = \pi(b) = x$ and $a_t \neq b_t$, then $T^t(x) \in bd(V_{a_t}) \cap bd(V_{b_t})$.

This follows from our assumption that $int(V_{a_t}) \cap int(V_{b_t}) = \emptyset$.

Let $J(x, \mathcal{V}) = \{t \geq 0 : T^t(x) \in bd(V_1) \cup \dots \cup bd(V_k)\}$ be the **hitting set** of x .

- If there exists some $m \in \mathbb{N}$ such that $|J(x, \mathcal{V})| \leq m$ for all $x \in X$, then all fibres have size at most k^m .
- If each $J(x, \mathcal{V})$ is finite, then all fibres are finite.

In either of the above cases we would consider the fibres “small.”

Could there be a less stringent but still meaningful notion of “smallness”?

An application of smallness

Assume that if there exists some $m \in \mathbb{N}$ such that $|J(x)| \leq m$ for all $x \in X$. Then all fibres have size at most k^m .

Of great interest in the study of dynamical systems is the **topological entropy** $h(X, F)$.

For a generating cover \mathcal{V} as above, this can be computed as

$$h(X, F) = \lim_{n \rightarrow \infty} \frac{\ln |\mathcal{V}^n|}{n},$$

where $|\mathcal{V}^n|$ is the smallest size of any subcover of \mathcal{V}^n . The precise definition of \mathcal{V}^n is not really needed here. But one can show that if $\mathcal{U} = \{\pi^{-1}V_1, \dots, \pi^{-1}V_k\}$, then $|\mathcal{V}^n| \leq |\mathcal{U}^n| \leq |\mathcal{V}^n|k^m$, where k^m is the above bound on the size of the fibres. Now

$$\begin{aligned} h(X, F) &\leq h(\Sigma, \sigma) = \lim_{n \rightarrow \infty} \frac{\ln |\mathcal{U}^n|}{n} \leq \lim_{n \rightarrow \infty} \frac{\ln (|\mathcal{V}^n|k^m)}{n} \\ &\leq \lim_{n \rightarrow \infty} \frac{\ln |\mathcal{V}^n|}{n} + \lim_{n \rightarrow \infty} \frac{\ln k^m}{n} = h(X, F) + 0. \end{aligned}$$

Thus if all fibres are “small” (in the most stringent sense), then the computation of entropy for the system (F, X) reduces to (usually easier) computation of entropy for its subshift $h(\Sigma, \sigma)$.

Ideals as families of small sets

Let \mathcal{I} be a family of subsets of \mathbb{N} . Then \mathcal{I} is a **proper ideal** if

- (a) \mathcal{I} is closed under subsets, that is $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ whenever $A \subset B$.
- (b) \mathcal{I} is closed under finite unions, that is $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.
- (c) $\mathbb{N} \notin \mathcal{I}$.

Example 1: The family \mathcal{I}_{fin} of all finite subsets of \mathbb{N} is a proper ideal.

Example 2: The family of all subsets of \mathbb{N} is an ideal (it satisfies conditions (a) and (b)), but not a proper one (it does not satisfy condition (c)).

Example 3: For each positive integer k , let \mathcal{I}_k denote the family of subsets of \mathbb{N} that have at most k elements. This family satisfies (a) and (c), but not (b). So it is **not an ideal**, but still a **family of small subsets of \mathbb{N}** .

Where does the name “ideal” come from?

Let \mathcal{I} be a family of subsets of \mathbb{N} . Then \mathcal{I} is a **proper ideal** if

- (a) \mathcal{I} is closed under subsets, that is $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ whenever $A \subset B$.
- (b) \mathcal{I} is closed under finite unions, that is $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.
- (c) $\mathbb{N} \notin \mathcal{I}$.

Consider the family $\mathcal{P}(\mathbb{N})$ of all subsets of \mathbb{N} .

Then the structure $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$, where Δ denotes symmetric difference, is an algebraic ring.

A family \mathcal{I} is an ideal in this ring (in the algebraic sense) iff it satisfies conditions (a) and (b) above.

Two more examples of proper ideals

A proper subset $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is a proper ideal iff \mathcal{I} is closed under subsets and finite unions.

Let $[n] = \{0, 1, \dots, n-1\}$, and let $|a|$ denote the size of a finite set.

For $a \subset \mathbb{N}$ define:

$d^+(a) = \limsup_{n \rightarrow \infty} \frac{|a \cap [n]|}{n}$ the upper density of a ,

$d_{\log}^+(a) = \limsup_{n \rightarrow \infty} \frac{\sum_{k \in a \cap [n]} \frac{1}{k+1}}{\ln n}$ the upper logarithmic density of a .

- $\mathcal{I}_d := \{a \subset \mathbb{N} : d^+(a) = 0\}$ is a proper ideal.
- $\mathcal{I}_{d\log} := \{a \subset \mathbb{N} : d_{\log}^+(a) = 0\}$ is a proper ideal.
- $\mathcal{I}_d \subset \mathcal{I}_{d\log}$ and the inclusion is strict.

The small boundary property

Definition (Shub and Weiss, 1991)

A system (X, F) has the **small boundary property (SBP)** if it has a basis of open sets such that for each U in this basis:

$$\forall x \in X \{t \geq 0 : F^t(x) \in bd(U)\} \in \mathcal{I}_d.$$

If (X, T^{-1}) has **any** generating cover and (X, T) has the SBP, then (X, T^{-1}) **also** has a generating cover \mathcal{V} such that $J(x, \mathcal{V}) \in \mathcal{I}_d$ for all $x \in X$.

Thus the SBP gives us a (potentially) less stringent notion of “small fibres” than the assumption that the sizes of all fibres are bounded from above by a fixed constant.

The SBP is sufficient for the proof that the entropy of the subshift of the system is the same as the entropy of the system itself.

Our generalization of the SBP

Definition

Let \mathcal{I} be a family of small subsets of \mathbb{N} .

A system (X, F) has the \mathcal{I} -**SBP** if it has a basis of open sets such that for each U in this basis:

$$\forall x \in X \{t \geq 0 : F^t(x) \in bd(U)\} \in \mathcal{I}.$$

Problem 1: For which systems does the SBP imply the \mathcal{I}_k -SBP (for some fixed k)?

Problem 2: For which systems does the SBP imply the \mathcal{I}_{fin} -SBP?

Problem 3: Are there any system with the SBP that do **not** have the \mathcal{I} -SBP for any ideal \mathcal{I} that is strictly smaller than \mathcal{I}_d ?

Problem 4: Are there any system with the \mathcal{I}_{dlog} -SBP that do **not** have the SBP?