CHAPTER 10

GRAPHS AND TREES

Graphs and trees have appeared previously in this book as convenient visualizations. For instance, a possibility tree shows all possible outcomes of a multistep operation with a finite number of outcomes for each step, the directed graph of a relation on a set shows which elements of the set are related to which a Hasse diagram illustrates the relations among elements in a partially ordered set, and a PERT diagram shows which tasks must precede which in executing a project.

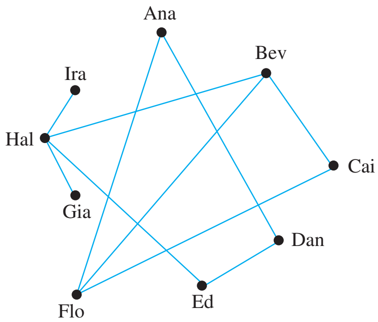
In this chapter we present some of the mathematics of graphs and trees, discussing concepts such as the degree of a vertex, connectedness, Euler and Hamiltonian circuits, representation of graphs by matrices, isomorphisms of graphs, the relation between the number of vertices and the number of edges of a tree, properties of rooted trees spanning trees, and shortest paths in graphs. Applications include uses of graphs and trees in the study of artificial intelligence, chemistry, scheduling problems, and transportation systems.

10.1 Graphs: Definitions and Basic Properties

The whole of mathematics consists in the organization of a series of aids to the imagination in the process of reasoning. — Alfred North Whitehead, 1861–1947

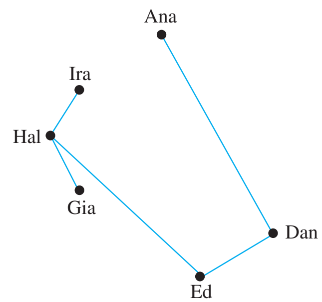
Imagine an organization that wants to set up teams of three to work on some projects. In order to maximize the number of people on each team who had previous experience working together successfully, the director asked the members to provide names of their past partners. This information is displayed below both in a table and in a diagram.

Name	Past Partners
Ana	Dan, Flo
Bev	Cai, Flo, Hal
Cai	Bev, Flo
Dan	Ana, Ed
Ed	Dan, Hal
Flo	Cai, Bev, Ana
Gia	Hal
Hal	Gia, Ed, Bev, Ira
Ira	Hal



From the diagram, it is easy to see that Bev, Cai, and Flo are a group of three past partners, and so they should form one of these teams. The figure on the next page shows the result when these three names are removed from the diagram.

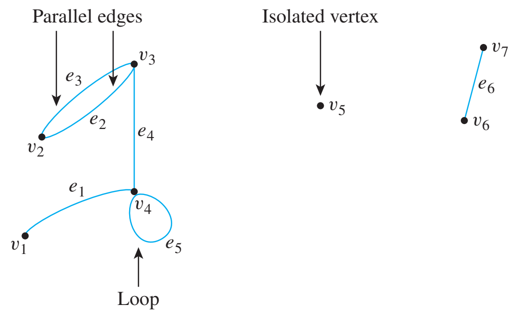
Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



This drawing shows that placing Hal on the same team as Ed would leave Gia and Ira on a team containing no past partners. However, if Hal is placed on a team with Gia and Ira, then the remaining team would consist of Ana, Dan, and Ed, and both teams would contain at least one pair of past partners.

Drawings such as those shown previously are illustrations of a structure known as a *graph*. The dots are called *vertices* (plural of *vertex*) and the line segments joining vertices are called *edges*. As you can see from the drawing, it is possible for two edges to cross at a point that is not a vertex. Note also that the type of graph described here is quite different from the "graph of an equation" or the "graph of a function."

In general, a graph consists of a set of vertices and a set of edges connecting various pairs of vertices. The edges may be straight or curved and should either connect one vertex to another or a vertex to itself, as shown below.



In this drawing, the vertices have been labeled with v's and the edges with e's. When an edge connects a vertex to itself (as e_5 does), it is called a *loop*. When two edges connect the same pair of vertices (as e_2 and e_3 do), they are said to be *parallel*. It is quite possible for a vertex to be unconnected by an edge to any other vertex in the graph (as v_5 is), and in that case the vertex is said to be *isolated*. The formal definition of a graph follows.

• Definition

A **graph** G consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges,** where each edge is associated with a set consisting of either one or two vertices called its **endpoints.** The correspondence from edges to endpoints is called the **edge-endpoint function.**

An edge with just one endpoint is called a **loop,** and two or more distinct edges with the same set of endpoints are said to be **parallel.** An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent;** and a vertex that is an endpoint of a loop is said to be **adjacent to itself.**

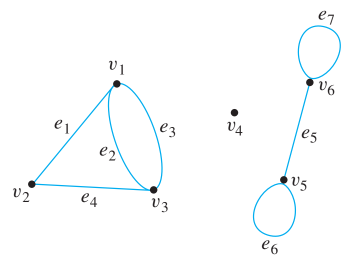
An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent.** A vertex on which no edges are incident is called **isolated.**

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Graphs have pictorial representations in which the vertices are represented by dots and the edges by line segments. A given pictorial representation uniquely determines a graph.

Example 10.1.1 Terminology

Consider the following graph:



a. Write the vertex set and the edge set, and give a table showing the edge-endpoint function.

b. Find all edges that are incident on v_1, all vertices that are adjacent to v_1, all edges that are adjacent to e_1, all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

Solution

a. vertex set $= \{v_1, v_2, v_3, v_4, v_5, v_6\}$
edge set $= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

Note that the isolated vertex v_4 does not appear in this table. Although each edge must have either one or two endpoints, a vertex need not be an endpoint of an edge.

b. e_1, e_2, and e_3 are incident on v_1.
v_2 and v_3 are adjacent to v_1.
e_2, e_3, and e_4 are adjacent to e_1.
e_6 and e_7 are loops.
e_2 and e_3 are parallel.
v_5 and v_6 are adjacent to themselves.
v_4 is an isolated vertex. ■

As noted earlier, a given pictorial representation uniquely determines a graph. However, a given graph may have more than one pictorial representation. Such things as the lengths or curvatures of the edges and the relative position of the vertices on the page may vary from one pictorial representation to another.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Example 10.1.2 Drawing More Than One Picture for a Graph

Consider the graph specified as follows:

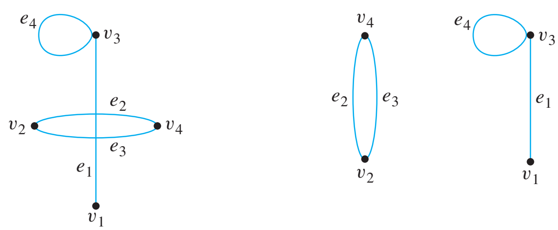
vertex set $= \{v_1, v_2, v_3, v_4\}$

edge set $= \{e_1, e_2, e_3, e_4\}$

edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_2, v_4\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_3\}$

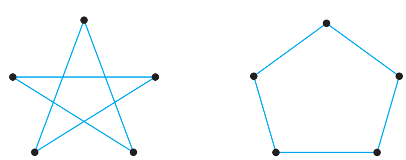
Both drawings (a) and (b) shown below are pictorial representations of this graph.



(a) (b) ■

Example 10.1.3 Labeling Drawings to Show They Represent the Same Graph

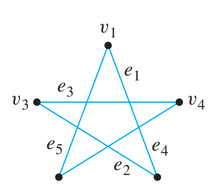
Consider the two drawings shown in Figure 10.1.1. Label vertices and edges in such a way that both drawings represent the same graph.



(a) (b)

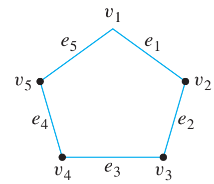
Figure 10.1.1

Solution Imagine putting one end of a piece of string at the top vertex of Figure 10.1.1(a) (call this vertex v_1), then laying the string to the next adjacent vertex on the lower right (call this vertex v_2), then laying it to the next adjacent vertex on the upper left (v_3), and so forth, returning finally to the top vertex v_1. Call the first edge e_1, the second e_2, and so forth, as shown below.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Now imagine picking up the piece of string, together with its labels, and repositioning it as follows:



This is the same as Figure 10.1.1(b), so both drawings are representations of the graph with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$, edge set $\{e_1, e_2, e_3, e_4, e_5\}$, and edge-endpoint function as follows:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_3, v_4\}$
e_4	$\{v_4, v_5\}$
e_5	$\{v_5, v_1\}$

■

In Chapter 8 we discussed the directed graph of a binary relation on a set. The general definition of directed graph is similar to the definition of graph, except that one associates an *ordered pair* of vertices with each edge instead of a *set* of vertices. Thus each edge of a directed graph can be drawn as an arrow going from the first vertex to the second vertex of the ordered pair.

• Definition

A **directed graph,** or **digraph,** consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $D(G)$ of directed edges, where each is associated with an ordered pair of vertices called its **endpoints.** If edge e is associated with the pair (v, w) of vertices, then e is said to be the (**directed**) **edge** from v to w.

Note that each directed graph has an associated ordinary (undirected) graph, which is obtained by ignoring the directions of the edges.

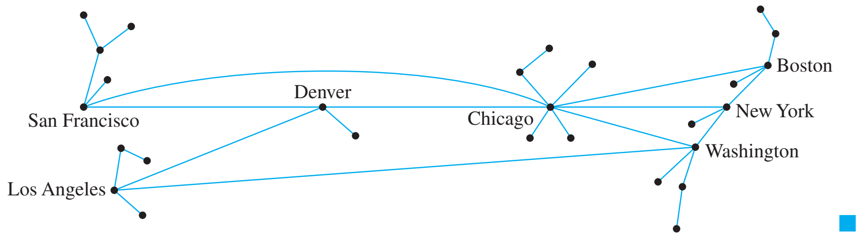
Examples of Graphs

Graphs are a powerful problem-solving tool because they enable us to represent a complex situation with a single image that can be analyzed both visually and with the aid of a computer. A few examples follow, and others are included in the exercises.

Example 10.1.4 Using a Graph to Represent a Network

Telephone, electric power, gas pipeline, and air transport systems can all be represented by graphs, as can computer networks—from small local area networks to the global Internet system that connects millions of computers worldwide. Questions that arise in the design of such systems involve choosing connecting edges to minimize cost, optimize a certain type of service, and so forth. A typical network, called a hub and spoke model, is shown on the next page.

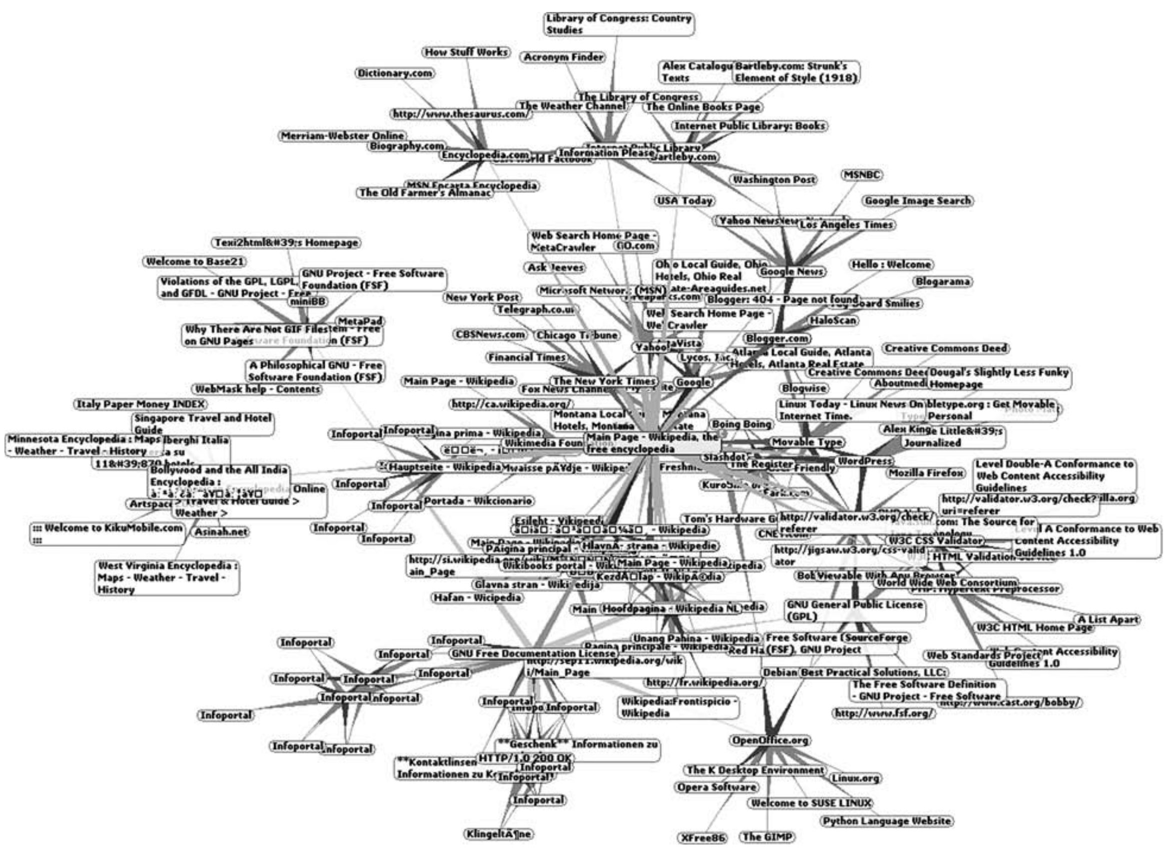
Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



Example 10.1.5 Using a Graph to Represent the World Wide Web

The World Wide Web, or Web, is a system of interlinked documents, or webpages, contained on the Internet. Users employing Web browsers, such as Internet Explorer, Google Chrome, Apple Safari, and Opera, can move quickly from one webpage to another by clicking on hyperlinks, which use versions of software called hypertext transfer protocols (HTTPs). Individuals and individual companies create the pages, which they transmit to servers that contain software capable of delivering them to those who request them through a Web browser. Because the amount of information currently on the Web is so vast, search engines, such as Google, Yahoo, and Bing, have algorithms for finding information very efficiently.

The picture below shows a minute fraction of the hyperlink connections on the Internet that radiate in and out from the Wikipedia main page.



Wikipedia/Chris 73

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Example 10.1.6 Using a Graph to Represent Knowledge

In many applications of artifical intelligence, a knowledge base of information is collected and represented inside a computer. Because of the way the knowledge is represented and because of the properties that govern the artificial intelligence program, the computer is not limited to retrieving data in the same form as it was entered; it can also derive new facts from the knowledge base by using certain built-in rules of inference. For example, from the knowledge that the *Los Angeles Times* is a big-city daily and that a big-city daily contains national news, an artifical intelligence program could infer that the *Los Angeles Times* contains national news. The directed graph shown in Figure 10.1.2 is a pictorial representation for a simplified knowledge base about periodical publications.

According to this knowledge base, what paper finish does the *New York Times* use?

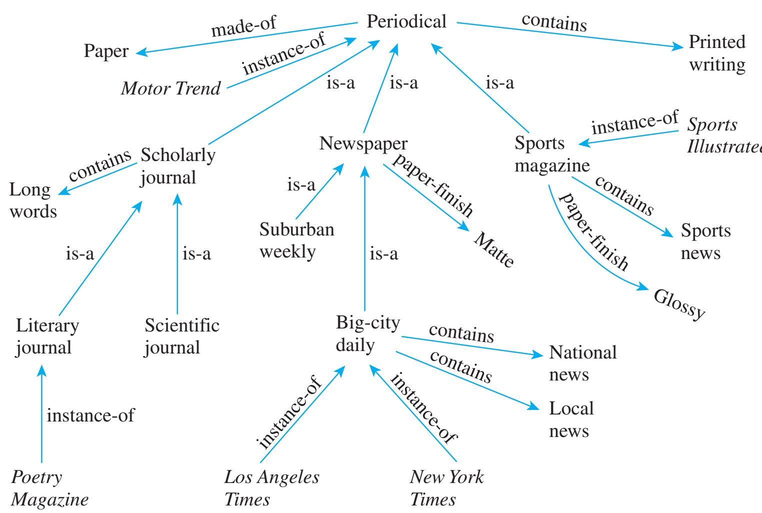


Figure 10.1.2

Solution The arrow going from *New York Times* to big-city daily (labeled "instance-of") shows that the *New York Times* is a big-city daily. The arrow going from big-city daily to newspaper (labeled "is-a") shows that a big-city daily is a newspaper. The arrow going from newspaper to matte (labeled "paper-finish") indicates that the paper finish on a newspaper is matte. Hence it can be inferred that the paper finish on the *New York Times* is matte. ■

Example 10.1.7 Using a Graph to Solve a Problem: Vegetarians and Cannibals

The following is a variation of a famous puzzle often used as an example in the study of artificial intelligence. It concerns an island on which all the people are of one of two types, either vegetarians or cannibals. Initially, two vegetarians and two cannibals are on the left bank of a river. With them is a boat that can hold a maximum of two people. The aim of the puzzle is to find a way to transport all the vegetarians and cannibals to the right bank of the river. What makes this difficult is that at no time can the number of cannibals on either bank outnumber the number of vegetarians. Otherwise, disaster befalls the vegetarians!

Solution A systematic way to approach this problem is to introduce a notation that can indicate all possible arrangements of vegetarians, cannibals, and the boat on the banks of

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

the river. For example, you could write (vvc/Bc) to indicate that there are two vegetarians and one cannibal on the left bank and one cannibal and the boat on the right bank. Then $(vvccB/)$ would indicate the initial position in which both vegetarians, both cannibals, and the boat are on the left bank of the river. The aim of the puzzle is to figure out a sequence of moves to reach the position $(/Bvvcc)$ in which both vegetarians, both cannibals, and the boat are on the right bank of the river.

Construct a graph whose vertices are the various arrangements that can be reached in a sequence of legal moves starting from the initial position. Connect vertex x to vertex y if it is possible to reach vertex y in one legal move from vertex x. For instance, from the initial position there are four legal moves: one vegetarian and one cannibal can take the boat to the right bank; two cannibals can take the boat to the right bank; one cannibal can take the boat to the right bank; or two vegetarians can take the boat to the right bank. You can show these by drawing edges connecting vertex $(vvccB/)$ to vertices (vc/Bvc), (vv/Bcc), $(vvcBc)$, and (cc/Bvv). (It might seem natural to draw directed edges rather than undirected edges from one vertex to another. The rationale for drawing undirected edges is that each legal move is reversible.) From the position (vc/Bvc), the only legal moves are to go back to $(vvccB/)$ or to go to $(vvcB/c)$. You can also show these by drawing in edges. Continue this process until finally you reach $(/Bvvcc)$. From Figure 10.1.3 it is apparent that one successful sequence of moves is $(vvccB/) \to (vc/Bvc) \to (vvcB/c) \to (c/Bvvc) \to (ccB/vv) \to (/Bvvcc)$.

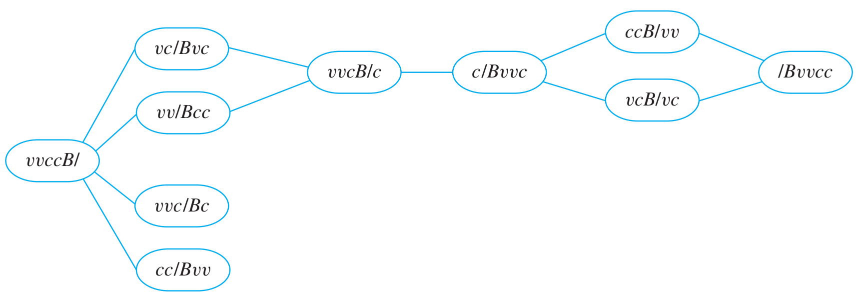


Figure 10.1.3 ■

Special Graphs

One important class of graphs consists of those that do not have any loops or parallel edges. Such graphs are called *simple*. In a simple graph, no two edges share the same set of endpoints, so specifying two endpoints is sufficient to determine an edge.

• Definition and Notation

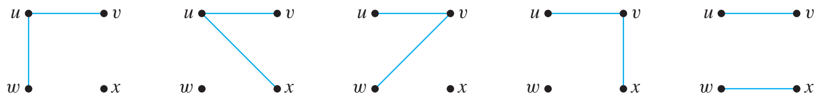
A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{v, w\}$.

Example 10.1.8 A Simple Graph

Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$.

Solution Each possible edge of a simple graph corresponds to a subset of two vertices. Given four vertices, there are $\binom{4}{2} = 6$ such subsets in all: $\{u, v\}$, $\{u, w\}$, $\{u, x\}$, $\{v, w\}$, $\{v, x\}$, and $\{w, x\}$. Now one edge of the graph is specified to be $\{u, v\}$, so any of the remaining five from this list can be chosen to be the second edge. The possibilities are shown on the next page.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



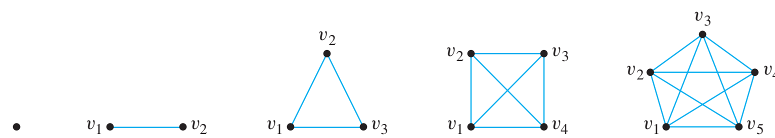
Another important class of graphs consists of those that are "complete" in the sense that all pairs of vertices are connected by edges.

Note The K stands for the German word *komplett*, which means "complete."

> **• Definition**
>
> Let n be a positive integer. A **complete graph on n vertices,** denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

Example 10.1.9 Complete Graphs on n Vertices: K_1, K_2, K_3, K_4, K_5

The complete graphs K_1, K_2, K_3, K_4, and K_5 can be drawn as follows:



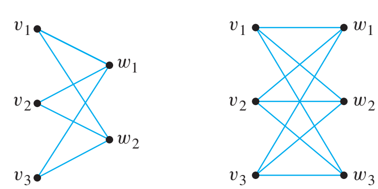
K_1 K_2 K_3 K_4 K_5

In yet another class of graphs, the vertex set can be separated into two subsets: Each vertex in one of the subsets is connected by exactly one edge to each vertex in the other subset, but not to any vertices in its own subset. Such a graph is called *complete bipartite.*

> **• Definition**
>
> Let m and n be positive integers. A **complete bipartite graph on (m, n) vertices,** denoted $K_{m,n}$, is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties: For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,
>
> 1. There is an edge from each vertex v_i to each vertex w_j.
> 2. There is no edge from any vertex v_i to any other vertex v_k.
> 3. There is no edge from any vertex w_j to any other vertex w_l.

Example 10.1.10 Complete Bipartite Graphs: $K_{3,2}$ and $K_{3,3}$

The complete bipartite graphs $K_{3,2}$ and $K_{3,3}$ are illustrated below.



$K_{3,2}$ $K_{3,3}$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

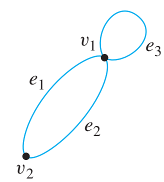
• Definition

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Example 10.1.11 Subgraphs

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2, and e_3 is a loop at v_1.

Solution G can be drawn as shown below.



There are 11 subgraphs of G, which can be grouped according to those that do not have any edges, those that have one edge, those that have two edges, and those that have three edges. The 11 subgraphs are shown in Figure 10.1.4.

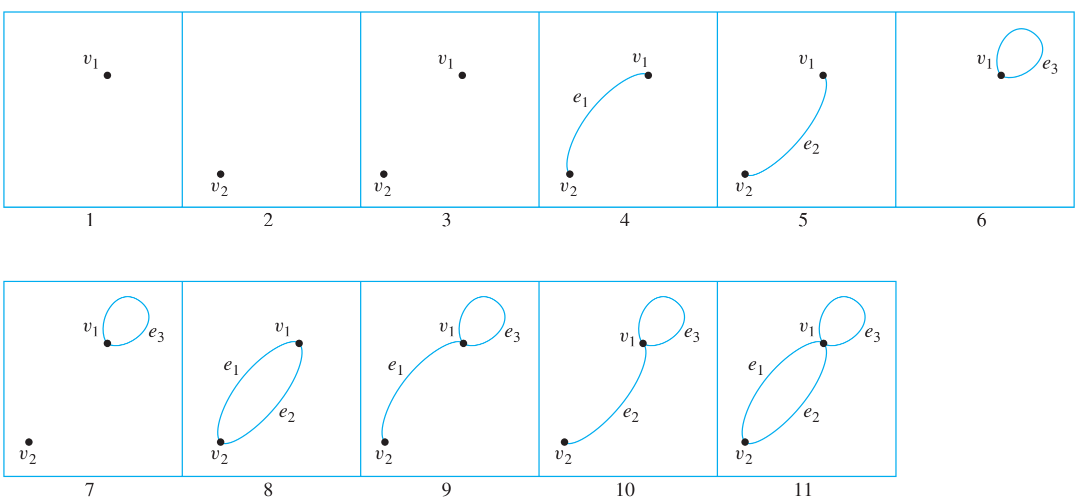


Figure 10.1.4 ■

The Concept of Degree

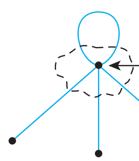
The *degree of a vertex* is the number of end segments of edges that "stick out of" the vertex. We will show that the sum of the degrees of all the vertices in a graph is twice the number of edges of the graph.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

• Definition

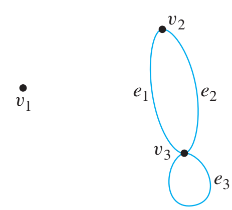
Let G be a graph and v a vertex of G. The **degree of v,** denoted **deg(v),** equals the number of edges that are incident on v, with an edge that is a loop counted twice. The **total degree of G** is the sum of the degrees of all the vertices of G.

Since an edge that is a loop is counted twice, the degree of a vertex can be obtained from the drawing of a graph by counting how many end segments of edges are incident on the vertex. This is illustrated below.



Example 10.1.12 Degree of a Vertex and Total Degree of a Graph

Find the degree of each vertex of the graph G shown below. Then find the total degree of G.



Solution

$\deg(v_1) = 0$ since no edge is incident on v_1 (v_1 is isolated).

$\deg(v_2) = 2$ since both e_1 and e_2 are incident on v_2.

$\deg(v_3) = 4$ since e_1 and e_2 are incident on v_3 and the loop e_3 is also incident on v_3 (and contributes 2 to the degree of v_3).

total degree of $G = \deg(v_1) + \deg(v_2) + \deg(v_3) = 0 + 2 + 4 = 6$. ■

Note that the total degree of the graph G of Example 10.1.12, which is 6, equals twice the number of edges of G, which is 3. Roughly speaking, this is true because each edge has two end segments, and each end segment is counted once toward the degree of some vertex. This result generalizes to any graph.

In fact, for any graph without loops, the general result can be explained as follows: Imagine a group of people at a party. Depending on how social they are, each person shakes hands with various other people. So each person participates in a certain number of handshakes—perhaps many, perhaps none—but because each handshake is experienced by two different people, if the numbers experienced by each person are added together, the sum will equal twice the total number of handshakes. This is such an attractive way of understanding the situation that the following theorem is often called the *handshake lemma* or the *handshake theorem*. As the proof demonstrates, the conclusion is true even if the graph contains loops.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

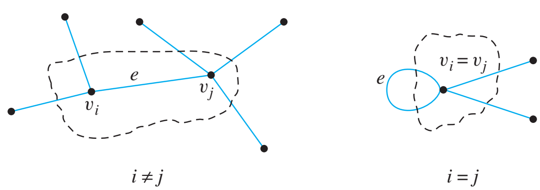
Theorem 10.1.1 The Handshake Theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then

$$\begin{aligned}\text{the total degree of } G &= \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n)\\ &= 2 \cdot (\text{the number of edges of } G).\end{aligned}$$

Proof:

Let G be a particular but arbitrarily chosen graph, and suppose that G has n vertices $v_1, v_2, \ldots, v_n$ and m edges, where n is a positive integer and m is a nonnegative integer. We claim that each edge of G contributes 2 to the total degree of G. For suppose e is an arbitrarily chosen edge with endpoints v_i and v_j. This edge contributes 1 to the degree of v_i and 1 to the degree v_j. As shown below, this is true even if $i = j$, because an edge that is a loop is counted twice in computing the degree of the vertex on which it is incident.



Therefore, e contributes 2 to the total degree of G. Since e was arbitrarily chosen, this shows that *each* edge of G contributes 2 to the total degree of G. Thus

$$\text{the total degree of } G = 2 \cdot (\text{the number of edges of } G).$$

The following corollary is an immediate consequence of Theorem 10.1.1.

Corollary 10.1.2

The total degree of a graph is even.

Proof:

By Theorem 10.1.1 the total degree of G equals 2 times the number of edges, which is an integer, and so the total degree of G is even.

Example 10.1.13 Determining Whether Certain Graphs Exist

Draw a graph with the specified properties or show that no such graph exists.

a. A graph with four vertices of degrees 1, 1, 2, and 3

b. A graph with four vertices of degrees 1, 1, 3, and 3

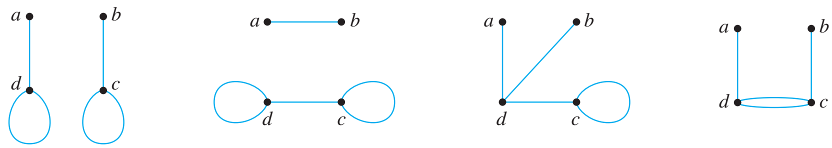
c. A simple graph with four vertices of degrees 1, 1, 3, and 3

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Solution

a. No such graph is possible. By Corollary 10.1.2, the total degree of a graph is even. But a graph with four vertices of degrees 1, 1, 2, and 3 would have a total degree of $1 + 1 + 2 + 3 = 7$, which is odd.

b. Let G be any of the graphs shown below.

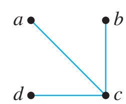


In each case, no matter how the edges are labeled, $\deg(a) = 1$, $\deg(b) = 1$, $\deg(c) = 3$, and $\deg(d) = 3$.

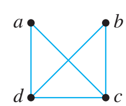
c. There is no simple graph with four vertices of degrees 1, 1, 3, and 3.

Proof (by contradiction):

Suppose there were a simple graph G with four vertices of degrees 1, 1, 3, and 3. Call a and b the vertices of degree 1, and call c and d the vertices of degree 3. Since $\deg(c) = 3$ and G does not have any loops or parallel edges (because it is simple), there must be edges that connect c to a, b, and d.



By the same reasoning, there must be edges connecting d to a, b, and c.



But then $\deg(a) \geq 2$ and $\deg(b) \geq 2$, which contradicts the supposition that these vertices have degree 1. Hence the supposition is false, and consequently there is no simple graph with four vertices of degrees 1, 1, 3, and 3. ■

Example 10.1.14 Application to an Acquaintance Graph

Is it possible in a group of nine people for each to be friends with exactly five others?

Solution The answer is no. Imagine constructing an "acquaintance graph" in which each of the nine people represented by a vertex and two vertices are joined by an edge if, and only if, the people they represent are friends. Suppose each of the people were friends with exactly five others. Then the degree of each of the nine vertices of the graph would be five, and so the total degree of the graph would be 45. But this contradicts Corollary 10.1.2, which says that the total degree of a graph is even. This contradiction shows that the supposition is false, and hence it is impossible for each person in a group of nine people to be friends with exactly five others. ■

The following proposition is easily deduced from Corollary 10.1.2 using properties of even and odd integers.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Proposition 10.1.3

In any graph there are an even number of vertices of odd degree.

Proof:

Suppose G is any graph, and suppose G has n vertices of odd degree and m vertices of even degree, where n is a positive integer and m is a nonnegative integer. *[We must show that n is even.]* Let E be the sum of the degrees of all the vertices of even degree, O the sum of the degrees of all the vertices of odd degree, and T the total degree of G. If $u_1, u_2, \ldots, u_m$ are the vertices of even degree and $v_1, v_2, \ldots, v_n$ are the vertices of odd degree, then

$$E = \deg(u_1) + \deg(u_2) + \cdots + \deg(u_m),$$
$$O = \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n), \quad \text{and}$$
$$T = \deg(u_1) + \cdots + \deg(u_m) + \deg(v_1) + \cdots + \deg(v_n) = E + O.$$

Now T, the total degree of G, is an even integer by Corollary 10.1.2. Also E is even since either E is zero, which is even, or E is a sum of the numbers $\deg(u_i)$, each of which is even. But

$$T = E + O,$$

and therefore

$$O = T - E.$$

Hence O is a difference of two even integers, and so O is even.

By assumption, $\deg(v_i)$ is odd for all $i = 1, 2, \ldots, n$. Thus O, an even integer, is a sum of the n odd integers $\deg(v_1), \deg(v_2), \ldots, \deg(v_n)$. But if a sum of n odd integers is even, then n is even. (See exercise 32 at the end of this section.) Therefore, n is even *[as was to be shown]*.

Example 10.1.15 Applying the Fact That the Number of Vertices with Odd Degree Is Even

Is there a graph with ten vertices of degrees 1, 1, 2, 2, 2, 3, 4, 4, 4, and 6?

Solution No. Such a graph would have three vertices of odd degree, which is impossible by Proposition 10.1.3.

Note that this same result could have been deduced directly from Corollary 10.1.2 by computing the total degree $(1 + 1 + 2 + 2 + 2 + 3 + 4 + 4 + 4 + 6 = 29)$ and noting that it is odd. However, use of Proposition 10.1.3 gives the result without the need to perform this addition. ■

Test Yourself

Answers to Test Yourself questions are located at the end of each section.

1. A graph consists of two finite sets: _____ and _____, where each edge is associated with a set consisting of _____.
2. A loop in a graph is _____.
3. Two distinct edges in a graph are parallel if, and only if, _____.
4. Two vertices are called adjacent if, and only if, _____.
5. An edge is incident on _____.
6. Two edges incident on the same endpoint are _____.
7. A vertex on which no edges are incident is _____.
8. In a directed graph, each edge is associated with _____.
9. A simple graph is _____.
10. A complete graph on n vertices is a _____.
11. A complete bipartite graph on (m, n) vertices is a simple graph whose vertices can be partitioned into two disjoint sets

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

V_1 and V_2 in such a way that (1) each of the m vertices in V_1 is _____ to each of the n vertices in V_2, no vertex in V_1 is connected to _____, and no vertex in V_2 is connected to _____.

12. A graph H is a subgraph of a graph G if, and only if, (1) _____, (2) _____, and (3) _____.

13. The degree of a vertex in a graph is _____.

14. The total degree of a graph is defined as _____.

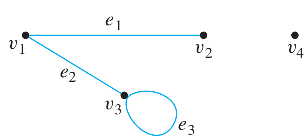
15. The handshake theorem says that the total degree of a graph is _____.

16. In any graph the number of vertices of odd degree is _____.

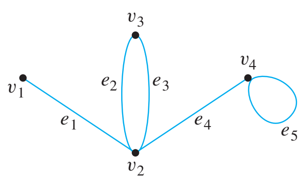
*Exercise Set 10.1**

In 1 and 2, graphs are represented by drawings. Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function.

1.



2.



In 3 and 4, draw pictures of the specified graphs.

3. Graph G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follows:

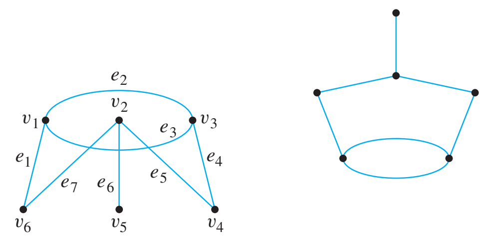
Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2\}$

4. Graph H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge-endpoint function as follows:

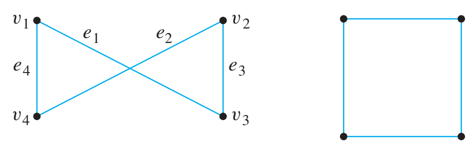
Edge	Endpoints
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

In 5–7, show that the two drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

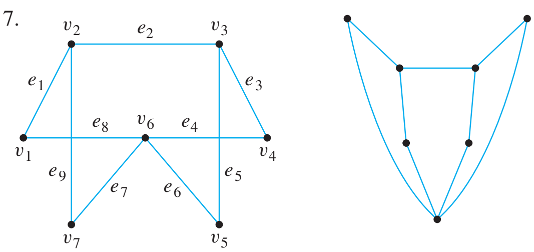
5.



6.



7.



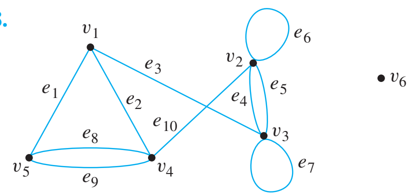
*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

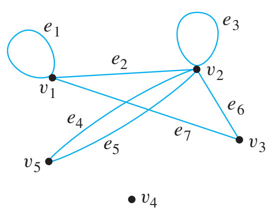
For each of the graphs in 8 and 9:

(i) Find all edges that are incident on v_1.
(ii) Find all vertices that are adjacent to v_3.
(iii) Find all edges that are adjacent to e_1.
(iv) Find all loops.
(v) Find all parallel edges.
(vi) Find all isolated vertices.
(vii) Find the degree of v_3.
(viii) Find the total degree of the graph.

8.



9.



10. Use the graph of Example 10.1.6 to determine
    a. whether *Sports Illustrated* contains printed writing;
    b. whether *Poetry Magazine* contains long words.

11. Find three other winning sequences of moves for the vegetarians and the cannibals in Example 10.1.7.

12. Another famous puzzle used as an example in the study of artificial intelligence seems first to have appeared in a collection of problems, *Problems for the Quickening of the Mind*, which was compiled about A.D. 775. It involves a wolf, a goat, a bag of cabbage, and a ferryman. From an initial position on the left bank of a river, the ferryman is to transport the wolf, the goat, and the cabbage to the right bank. The difficulty is that the ferryman's boat is only big enough for him to transport one object at a time, other than himself. Yet, for obvious reasons, the wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage. How should the ferryman proceed?

13. Solve the vegetarians-and-cannibals puzzle for the case where there are three vegetarians and three cannibals to be transported from one side of a river to the other.

H **14.** Two jugs A and B have capacities of 3 quarts and 5 quarts, respectively. Can you use the jugs to measure out exactly 1 quart of water, while obeying the following restrictions? You may fill either jug to capacity from a water tap; you may empty the contents of either jug into a drain; and you may pour water from either jug into the other.

15. A graph has vertices of degrees 0, 2, 2, 3, and 9. How many edges does the graph have?

16. A graph has vertices of degrees 1, 1, 4, 4, and 6. How many edges does the graph have?

In each of 17–25, either draw a graph with the specified properties or explain why no such graph exists.

17. Graph with five vertices of degrees 1, 2, 3, 3, and 5.

18. Graph with four vertices of degrees 1, 2, 3, and 3.

19. Graph with four vertices of degrees 1, 1, 1, and 4.

20. Graph with four vertices of degrees 1, 2, 3, and 4.

21. Simple graph with four vertices of degrees 1, 2, 3, and 4.

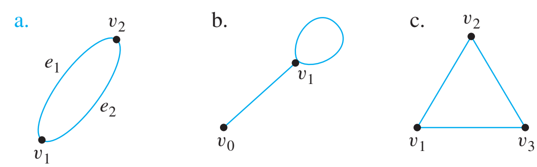
22. Simple graph with five vertices of degrees 2, 3, 3, 3, and 5.

23. Simple graph with five vertices of degrees 1, 1, 1, 2, and 3.

24. Simple graph with six edges and all vertices of degree 3.

25. Simple graph with nine edges and all vertices of degree 3.

26. Find all subgraphs of each of the following graphs.



27. **a.** In a group of 15 people, is it possible for each person to have exactly 3 friends? Explain. (Assume that friendship is a symmetric relationship: If x is a friend of y, then y is a friend of x.)
    b. In a group of 4 people, is it possible for each person to have exactly 3 friends? Why?

28. In a group of 25 people, is it possible for each to shake hands with exactly 3 other people? Explain.

29. Is there a simple graph, each of whose vertices has even degree? Explain.

30. Suppose that G is a graph with v vertices and e edges and that the degree of each vertex is at least $d_{\min}$ and at most $d_{\max}$. Show that

$$\frac{1}{2}d_{\min} \cdot v \leq e \leq \frac{1}{2}d_{\max} \cdot v.$$

31. Prove that any sum of an odd number of odd integers is odd.

H **32.** Deduce from exercise 31 that for any positive integer n, if there is a sum of n odd integers that is even, then n is even.

33. Recall that K_n denotes a complete graph on n vertices.
    a. Draw K_6.
    H **b.** Show that for all integers $n \geq 1$, the number of edges of K_n is $\dfrac{n(n-1)}{2}$.

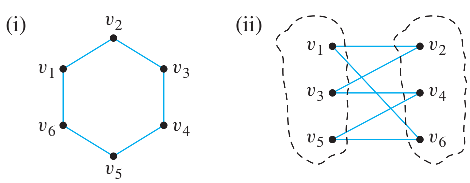
34. Use the result of exercise 33 to show that the number of edges of a simple graph with n vertices is less than or equal to $\dfrac{n(n-1)}{2}$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

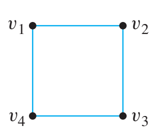
35. Is there a simple graph with twice as many edges as vertices? Explain. (You may find it helpful to use the result of exercise 34.)

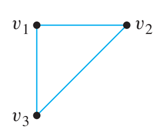
36. Recall that $K_{m,n}$ denotes a complete bipartite graph on (m, n) vertices.
   a. Draw $K_{4,2}$
   b. Draw $K_{1,3}$
   c. Draw $K_{3,4}$
   d. How many vertices of $K_{m,n}$ have degree m? degree n?
   e. What is the total degree of $K_{m,n}$?
   f. Find a formula in terms of m and n for the number of edges of $K_{m,n}$. Explain.

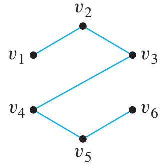
37. A **bipartite graph** G is a simple graph whose vertex set can be partitioned into two disjoint nonempty subsets V_1 and V_2 such that vertices in V_1 may be connected to vertices in V_2, but no vertices in V_1 are connected to other vertices in V_1 and no vertices in V_2 are connected to other vertices in V_2. For example, the graph G illustrated in (i) can be redrawn as shown in (ii). From the drawing in (ii), you can see that G is bipartite with mutually disjoint vertex sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$.

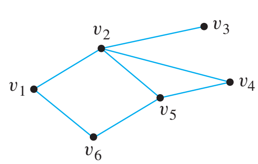


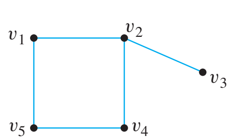
Find which of the following graphs are bipartite. Redraw the bipartite graphs so that their bipartite nature is evident.

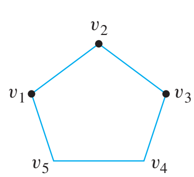
a.


b.


c.


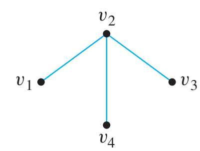
d.


e.


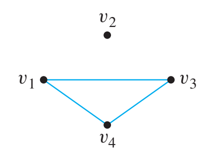
f.


38. Suppose r and s are any positive integers. Does there exist a graph G with the property that G has vertices of degrees r and s and of no other degrees? Explain.

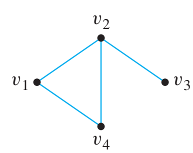
Definition: If G is a simple graph, the **complement of G,** denoted $\boldsymbol{G'}$, is obtained as follows: The vertex set of G' is identical to the vertex set of G. However, two distinct vertices v and w of G' are connected by an edge if, and only if, v and w are not connected by an edge in G. For example, if G is the graph

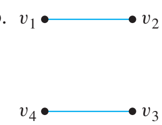


then G' is



39. Find the complement of each of the following graphs.

a.


b.


40. a. Find the complement of the graph K_4, the complete graph on four vertices. (See Example 10.1.9.)
   b. Find the complement of the graph $K_{3,2}$, the complete bipartite graph on (3, 2) vertices. (See Example 10.1.10.)

41. Suppose that in a group of five people A, B, C, D, and E the following pairs of people are acquainted with each other:
   A and C, A and D, B and C, C and D, C and E.
   a. Draw a graph to represent this situation.
   b. Draw a graph that illustrates who among these five people are *not* acquainted. That is, draw an edge between two people if, and only if, they are not acquainted.

***H* 42.** Let G be a simple graph with n vertices. What is the relation between the number of edges of G and the number of edges of the complement G'?

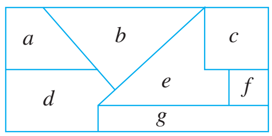
43. Show that at a party with at least two people, there are at least two mutual acquaintances or at least two mutual strangers.

44. a. In a simple graph, must every vertex have degree that is less than the number of vertices in the graph? Why?
   b. Can there be a simple graph that has four vertices each of different degrees?
   ***H* ✶ c.** Can there be a simple graph that has n vertices all of different degrees?

***H* ✶ 45.** In a group of two or more people, must there always be at least two people who are acquainted with the same number of people within the group? Why?

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

46. Imagine that the diagram shown below is a map with countries labeled a–g. Is it possible to color the map with only three colors so that no two adjacent countries have the same color? To answer this question, draw and analyze a graph in which each country is represented by a vertex and two vertices are connected by an edge if, and only if, the countries share a common border.



H **47.** In this exercise a graph is used to help solve a scheduling problem. Twelve faculty members in a mathematics department serve on the following committees:

Undergraduate Education: Tenner, Peterson, Kashina, Cohen

Graduate Education: Gatto, Yang, Cohen, Catoiu

Colloquium: Sahin, McMurry, Ash

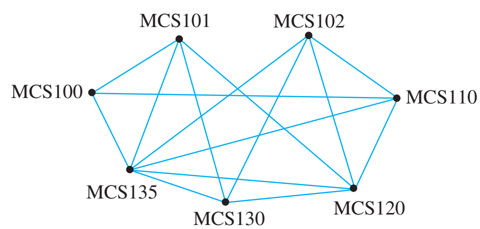
Library: Cortzen, Tenner, Sahin

Hiring: Gatto, McMurry, Yang, Peterson

Personnel: Yang, Wang, Cortzen

The committees must all meet during the first week of classes, but there are only three time slots available. Find a schedule that will allow all faculty members to attend the meetings of all committees on which they serve. To do this, represent each committee as the vertex of a graph, and draw an edge between two vertices if the two committees have a common member. Find a way to color the vertices using only three colors so that no two committees have the same color, and explain how to use the result to schedule the meetings.

48. A department wants to schedule final exams so that no student has more than one exam on any given day. The vertices of the graph below show the courses that are being taken by more than one student, with an edge connecting two vertices if there is a student in both courses. Find a way to color the vertices of the graph with only four colors so that no two adjacent vertices have the same color and explain how to use the result to schedule the final exams.



Answers for Test Yourself

1. a finite, nonempty set of vertices; a finite set of edges; one or two vertices called its endpoints 2. an edge with a single endpoint 3. they have the same set of endpoints 4. they are connected by an edge 5. each of its endpoints 6. adjacent 7. isolated 8. an ordered pair of vertices called its endpoints 9. a graph with no loops or parallel edges 10. simple graph with n vertices whose set of edges contains exactly one edge for each pair of vertices 11. connected by an edge; any other vertex in V_1; any other vertex in V_2 12. every vertex in H is also a vertex in G; every edge in H is also an edge in G; every edge in H has the same endpoints as it has in G 13. the number of edges that are incident on the vertex, with an edge that is a loop counted twice 14. the sum of the degrees of all the vertices of the graph 15. equal to twice the number of edges of the graph 16. an even number

10.2 Trails, Paths, and Circuits

One can begin to reason only when a clear picture has been formed in the imagination.
— W. W. Sawyer, *Mathematician's Delight,* 1943

The subject of graph theory began in the year 1736 when the great mathematician Leonhard Euler published a paper giving the solution to the following puzzle:

> The town of Königsberg in Prussia (now Kaliningrad in Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks. These were connected by seven bridges as shown in Figure 10.2.1.
>
> The question is this: Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?*

*In his original paper, Euler did not require the walk to start and end at the same point. The analysis of the problem is simplified, however, by adding this condition. Later in the section, we discuss walks that start and end at different points.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.