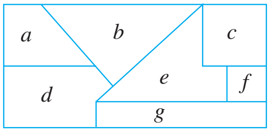
46. Imagine that the diagram shown below is a map with countries labeled a–g. Is it possible to color the map with only three colors so that no two adjacent countries have the same color? To answer this question, draw and analyze a graph in which each country is represented by a vertex and two vertices are connected by an edge if, and only if, the countries share a common border.



H **47.** In this exercise a graph is used to help solve a scheduling problem. Twelve faculty members in a mathematics department serve on the following committees:

Undergraduate Education: Tenner, Peterson, Kashina, Cohen

Graduate Education: Gatto, Yang, Cohen, Catoiu

Colloquium: Sahin, McMurry, Ash

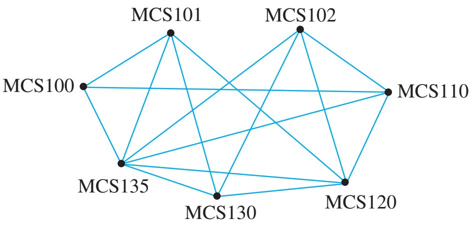
Library: Cortzen, Tenner, Sahin

Hiring: Gatto, McMurry, Yang, Peterson

Personnel: Yang, Wang, Cortzen

The committees must all meet during the first week of classes, but there are only three time slots available. Find a schedule that will allow all faculty members to attend the meetings of all committees on which they serve. To do this, represent each committee as the vertex of a graph, and draw an edge between two vertices if the two committees have a common member. Find a way to color the vertices using only three colors so that no two committees have the same color, and explain how to use the result to schedule the meetings.

48. A department wants to schedule final exams so that no student has more than one exam on any given day. The vertices of the graph below show the courses that are being taken by more than one student, with an edge connecting two vertices if there is a student in both courses. Find a way to color the vertices of the graph with only four colors so that no two adjacent vertices have the same color and explain how to use the result to schedule the final exams.



Answers for Test Yourself

1. a finite, nonempty set of vertices; a finite set of edges; one or two vertices called its endpoints 2. an edge with a single endpoint 3. they have the same set of endpoints 4. they are connected by an edge 5. each of its endpoints 6. adjacent 7. isolated 8. an ordered pair of vertices called its endpoints 9. a graph with no loops or parallel edges 10. simple graph with n vertices whose set of edges contains exactly one edge for each pair of vertices 11. connected by an edge; any other vertex in V_1; any other vertex in V_2 12. every vertex in H is also a vertex in G; every edge in H is also an edge in G; every edge in H has the same endpoints as it has in G 13. the number of edges that are incident on the vertex, with an edge that is a loop counted twice 14. the sum of the degrees of all the vertices of the graph 15. equal to twice the number of edges of the graph 16. an even number

10.2 Trails, Paths, and Circuits

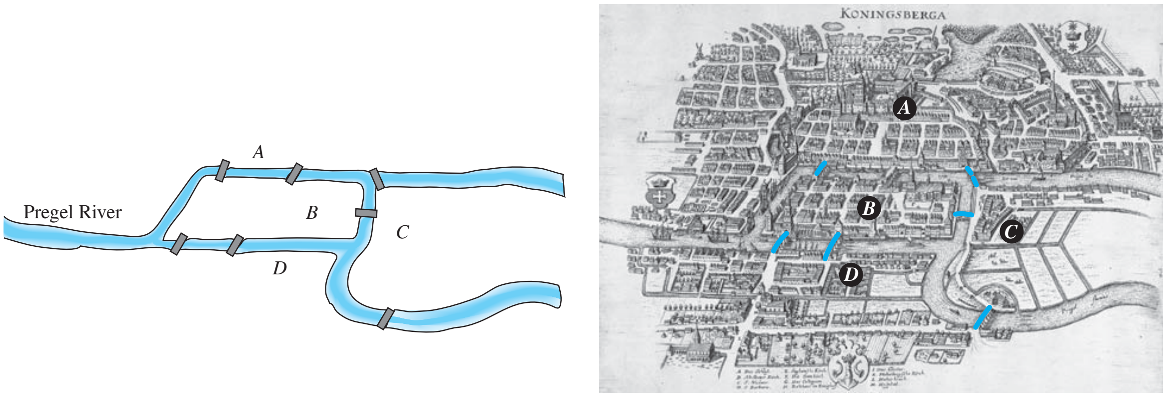
One can begin to reason only when a clear picture has been formed in the imagination.

— W. W. Sawyer, *Mathematician's Delight,* 1943

The subject of graph theory began in the year 1736 when the great mathematician Leonhard Euler published a paper giving the solution to the following puzzle:

> The town of Königsberg in Prussia (now Kaliningrad in Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks. These were connected by seven bridges as shown in Figure 10.2.1.
>
> The question is this: Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?*

*In his original paper, Euler did not require the walk to start and end at the same point. The analysis of the problem is simplified, however, by adding this condition. Later in the section, we discuss walks that start and end at different points.



Merian-Erben

Figure 10.2.1 **The Seven Bridges of Königsberg**

Bettmann/CORBIS

Leonhard Euler
(1707–1783)

To solve this puzzle, Euler translated it into a graph theory problem. He noticed that all points of a given land mass can be identified with each other since a person can travel from any one point to any other point of the same land mass without crossing a bridge. Thus for the purpose of solving the puzzle, the map of Königsberg can be identified with the graph shown in Figure 10.2.2, in which the vertices A, B, C, and D represent land masses and the seven edges represent the seven bridges.

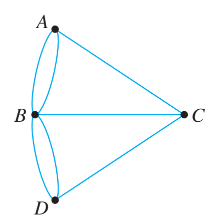


Figure 10.2.2 **Graph Version of Königsberg Map**

In terms of this graph, the question becomes the following:

> Is it possible to find a route through the graph that starts and ends at some vertex, one of A, B, C, or D, and traverses each edge exactly once?

Equivalently:

> Is it possible to trace this graph, starting and ending at the same point, without ever lifting your pencil from the paper?

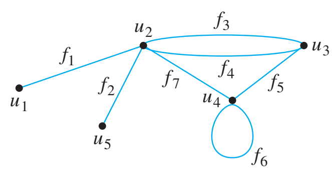
Take a few minutes to think about the question yourself. Can you find a route that meets the requirements? Try it!

Looking for a route is frustrating because you continually find yourself at a vertex that does not have an unused edge on which to leave, while elsewhere there are unused edges that must still be traversed. If you start at vertex A, for example, each time you pass through vertex B, C, or D, you use up two edges because you arrive on one edge and depart on a different one. So, if it is possible to find a route that uses all the edges of the graph and starts and ends at A, then the total number of arrivals and departures from each vertex B, C, and D must be a multiple of 2. Or, in other words, the degrees of the vertices B, C, and D must be even. But they are not: $\deg(B) = 5$, $\deg(C) = 3$, and $\deg(D) = 3$. Hence there is no route that solves the puzzle by starting and ending at A. Similar reasoning can be used to show that there are no routes that solve the puzzle by starting and ending at B, C, or D. Therefore, it is impossible to travel all around the city crossing each bridge exactly once.

Definitions

Travel in a graph is accomplished by moving from one vertex to another along a sequence of adjacent edges. In the graph below, for instance, you can go from u_1 to u_4 by taking f_1 to u_2 and then f_7 to u_4. This is represented by writing

$$u_1 f_1 u_2 f_7 u_4.$$



Or you could take the roundabout route

$$u_1 f_1 u_2 f_3 u_3 f_4 u_2 f_3 u_3 f_5 u_4 f_6 u_4 f_7 u_2 f_3 u_3 f_5 u_4.$$

Certain types of sequences of adjacent vertices and edges are of special importance in graph theory: those that do not have a repeated edge, those that do not have a repeated vertex, and those that start and end at the same vertex.

• Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from *v* to *w*** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from *v* to *v*** consists of the single vertex v.

A **trail from *v* to *w*** is a walk from v to w that does not contain a repeated edge.

A **path from *v* to *w*** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

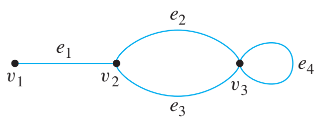
For ease of reference, these definitions are summarized in the following table:

	Repeated Edge?	Repeated Vertex?	Starts and Ends at Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

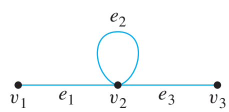
Often a walk can be specified unambiguously by giving either a sequence of edges or a sequence of vertices. The next two examples show how this is done.

Example 10.2.1 Notation for Walks

a. In the graph below, the notation $e_1e_2e_4e_3$ refers unambiguously to the following walk: $v_1e_1v_2e_2v_3e_4v_3e_3v_2$. On the other hand, the notation e_1 is ambiguous if used to refer to a walk. It could mean either $v_1e_1v_2$ or $v_2e_1v_1$.



b. In the graph of part (a), the notation v_2v_3 is ambiguous if used to refer to a walk. It could mean $v_2e_2v_3$ or $v_2e_3v_3$. On the other hand, in the graph below, the notation $v_1v_2v_2v_3$ refers unambiguously to the walk $v_1e_1v_2e_2v_2e_3v_3$.



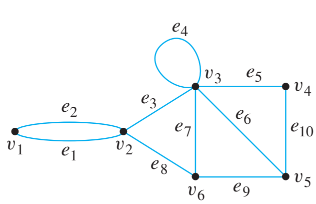
■

Note that if a graph G does not have any parallel edges, then any walk in G is uniquely determined by its sequence of vertices.

Example 10.2.2 Walks, Trails Paths, and Circuits

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1e_1v_2e_3v_3e_4v_3e_5v_4$ b. $e_1e_3e_5e_5e_6$ c. $v_2v_3v_4v_5v_3v_6v_2$
d. $v_2v_3v_4v_5v_6v_2$ e. $v_1e_1v_2e_1v_1$ f. v_1



Solution

a. This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 to v_4 but not a path.

b. This is just a walk from v_1 to v_5. It is not a trail because it has a repeated edge.

c. This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.

d. This walk starts and ends at v_2, contains at least one edge, does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e. This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f. The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit. It is a closed walk from v_1 to v_1. (It is also a trail from v_1 to v_1.) ■

Because most of the major developments in graph theory have happened relatively recently and in a variety of different contexts, the terms used in the subject have not been standardized. For example, what this book calls a *graph* is sometimes called a *multigraph*, what this book calls a *simple graph* is sometimes called a *graph*, what this book calls a *vertex* is sometimes called a *node*, and what this book calls an *edge* is sometimes called an *arc*. Similarly, instead of the word *trail*, the word *path* is sometimes used; instead of the word *path*, the words *simple path* are sometimes used; and instead of the words *simple circuit*, the word *cycle* is sometimes used. The terminology in this book is among the most common, but if you consult other sources, be sure to check their definitions.

Connectedness

It is easy to understand the concept of connectedness on an intuitive level. Roughly speaking, a graph is connected if it is possible to travel from any vertex to any other vertex along a sequence of adjacent edges of the graph. The formal definition of connectedness is stated in terms of walks.

• Definition

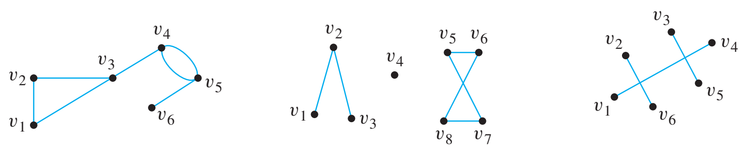
Let G be a graph. Two **vertices *v* and *w* of *G* are connected** if, and only if, there is a walk from v to w. The **graph *G* is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically,

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v, w \in V(G), \exists \text{ a walk from } v \text{ to } w.$$

If you take the negation of this definition, you will see that a graph G is *not connected* if, and only if, there are two vertices of G that are not connected by any walk.

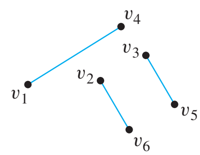
Example 10.2.3 Connected and Disconnected Graphs

Which of the following graphs are connected?



(a) (b) (c)

Solution The graph represented in (a) is connected, whereas those of (b) and (c) are not. To understand why (c) is not connected, recall that in a drawing of a graph, two edges may cross at a point that is not a vertex. Thus the graph in (c) can be redrawn as follows:



Some useful facts relating circuits and connectedness are collected in the following lemma. Proofs of (a) and (b) are left for the exercises. The proof of (c) is in Section 10.5.

Lemma 10.2.1

Let G be a graph.

a. If G is connected, then any two distinct vertices of G can be connected by a path.

b. If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

c. If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

Look back at Example 10.2.3. The graphs in (b) and (c) are both made up of three pieces, each of which is itself a connected graph. A *connected component* of a graph is a connected subgraph of largest possible size.

• Definition

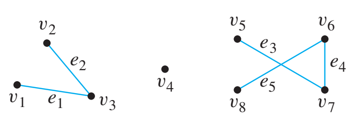
A graph H is a **connected component** of a graph G if, and only if,

1. H is subgraph of G;
2. H is connected; and
3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

The fact is that any graph is a kind of union of its connected components.

Example 10.2.4 Connected Components

Find all connected components of the following graph G.



Solution G has three connected components: H_1, H_2, and H_3 with vertex sets V_1, V_2, and V_3 and edge sets E_1, E_2, and E_3, where

$$\begin{aligned} V_1 &= \{v_1, v_2, v_3\}, & E_1 &= \{e_1, e_2\}, \\ V_2 &= \{v_4\}, & E_2 &= \emptyset, \\ V_3 &= \{v_5, v_6, v_7, v_8\}, & E_3 &= \{e_3, e_4, e_5\}. \end{aligned}$$

Euler Circuits

Now we return to consider general problems similar to the puzzle of the Königsberg bridges. The following definition is made in honor of Euler.

• Definition

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

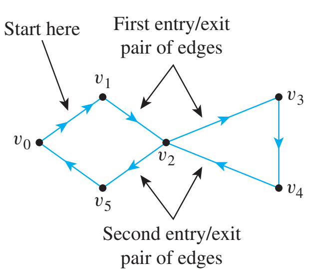
The analysis used earlier to solve the puzzle of the Königsberg bridges generalizes to prove the following theorem:

Theorem 10.2.2

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Proof:

Suppose G is a graph that has an Euler circuit. *[We must show that given any vertex v of G, the degree of v is even.]* Let v be any particular but arbitrarily chosen vertex of G. Since the Euler circuit contains every edge of G, it contains all edges incident on v. Now imagine taking a journey that begins in the middle of one of the edges adjacent to the start of the Euler circuit and continues around the Euler circuit to end in the middle of the starting edge. (See Figure 10.2.3. There is such a starting edge because the Euler circuit has at least one edge.) Each time v is entered by traveling along one edge, it is immediately exited by traveling along another edge (since the journey ends in the *middle* of an edge).



In this example, the Euler circuit is $v_0v_1v_2v_3v_4v_5v_0$, and v is v_2. Each time v_2 is entered by one edge, it is exited by another edge.

Figure 10.2.3 Example for the Proof of Theorem 10.2.2

Because the Euler circuit uses every edge of G exactly once, every edge incident on v is traversed exactly once in this process. Hence the edges incident on v occur in entry/exit pairs, and consequently the degree of v must be a positive multiple of 2. But that means that v has positive even degree *[as was to be shown].*

Recall that the contrapositive of a statement is logically equivalent to the statement. The contrapositive of Theorem 10.2.2 is as follows:

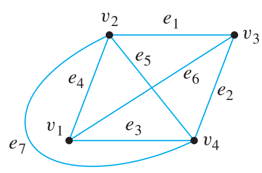
Contrapositive Version of Theorem 10.2.2

If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

This version of Theorem 10.2.2 is useful for showing that a given graph does *not* have an Euler circuit.

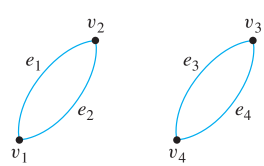
Example 10.2.5 Showing That a Graph Does Not Have an Euler Circuit

Show that the graph below does not have an Euler circuit.



Solution Vertices v_1 and v_3 both have degree 3, which is odd. Hence by (the contrapositive form of) Theorem 10.2.2, this graph does not have an Euler circuit. ■

Now consider the converse of Theorem 10.2.2: If every vertex of a graph has even degree, then the graph has an Euler circuit. Is this true? The answer is no. There is a graph G such that every vertex of G has even degree but G does not have an Euler circuit. In fact, there are many such graphs. The illustration below shows one example.



Every vertex has even degree, but the graph does not have an Euler circuit.

Note that the graph in the preceding drawing is not connected. It turns out that although the converse of Theorem 10.2.2 is false, a modified converse is true: If every vertex of a graph has positive even degree *and* if the graph is connected, then the graph has an Euler circuit. The proof of this fact is constructive: It contains an algorithm to find an Euler circuit for any connected graph in which every vertex has even degree.

Theorem 10.2.3

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Proof:

Suppose that G is any connected graph and suppose that every vertex of G is a positive even integer. *[We must find an Euler circuit for G.]* Construct a circuit C by the following algorithm:

Step 1: Pick any vertex v of G at which to start.
[This step can be accomplished because the vertex set of G is nonempty by assumption.]

Step 2: Pick any sequence of adjacent vertices and edges, starting and ending at v and never repeating an edge. Call the resulting circuit C.
[This step can be performed for the following reasons: Since the degree of each vertex of G is a positive even integer, as each vertex of G is entered by traveling on one edge, either the vertex is v itself and there is no other unused edge adjacent to v, or the vertex can be exited by traveling on another previously unused edge. Since the number of edges of the graph is finite (by definition of graph), the sequence of distinct edges cannot go on forever. The sequence can eventually return to v because the degree of v is a positive even integer, and so if an edge connects v to another vertex, there must be a different edge that connects back to v.]

Step 3: Check whether C contains every edge and vertex of G. If so, C is an Euler circuit, and we are finished. If not, perform the following steps.

Step 3a: Remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Call the resulting subgraph G'.
[Note that G' may not be connected (as illustrated in Figure 10.2.4), but every vertex of G' has positive, even degree (since removing the edges of C removes an even number of edges from each vertex, the difference of two even integers is even, and isolated vertices with degree 0 were removed.)]

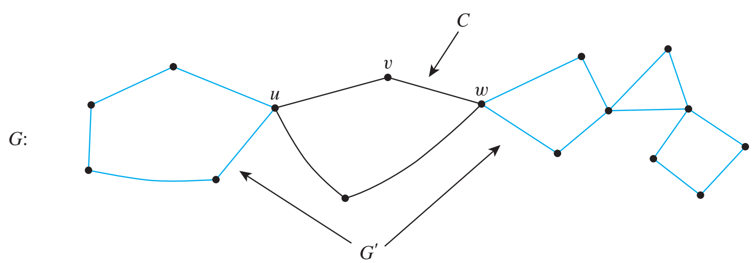


Figure 10.2.4

Step 3b: Pick any vertex w common to both C and G'.
[There must be at least one such vertex since G is connected. (See exercise 44.) (In Figure 10.2.4 there are two such vertices: u and w.)]

Step 3c: Pick any sequence of adjacent vertices and edges of G', starting and ending at w and never repeating an edge. Call the resulting circuit C'.
[This can be done since each vertex of G' has positive, even degree and G' is finite. See the justification for step 2.]

Step 3d: Patch C and C' together to create a new circuit C'' as follows: Start at v and follow C all the way to w. Then follow C' all the way back to w. After that, continue along the untraveled portion of C to return to v. *[The effect of executing steps 3c and 3d for the graph of Figure 10.2.4 is shown in Figure 10.2.5.]*

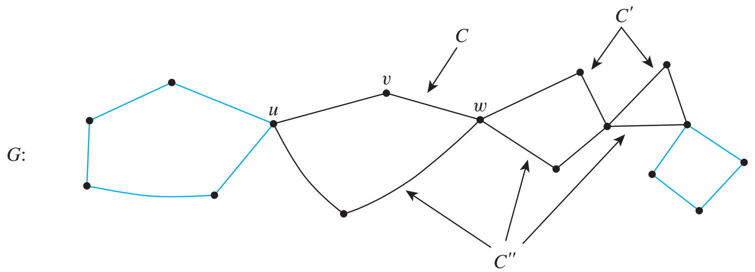


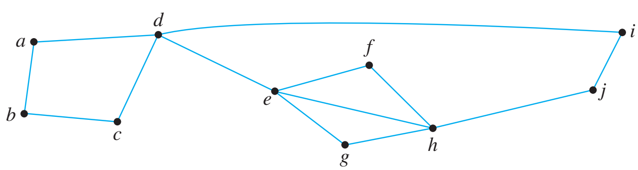
Figure 10.2.5

Step 3e: Let $C = C''$ and go back to step 3.

Since the graph G is finite, execution of the steps outlined in this algorithm must eventually terminate. At that point an Euler circuit for G will have been constructed. (Note that because of the element of choice in steps 1, 2, 3b, and 3c, a variety of different Euler circuits can be produced by using this algorithm.)

Example 10.2.6 Finding an Euler Circuit

Use Theorem 10.2.3 to check that the graph below has an Euler circuit. Then use the algorithm from the proof of the theorem to find an Euler circuit for the graph.



Solution Observe that

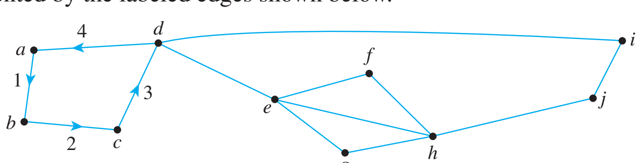
$$\deg(a) = \deg(b) = \deg(c) = \deg(f) = \deg(g) = \deg(i) = \deg(j) = 2$$

and that $\deg(d) = \deg(e) = \deg(h) = 4$. Hence all vertices have even degree. Also, the graph is connected. Thus, by Theorem 10.2.3, the graph has an Euler circuit.

To construct an Euler circuit using the algorithm of Theorem 10.2.3, let $v = a$ and let C be

$$C\colon abcda.$$

C is represented by the labeled edges shown below.



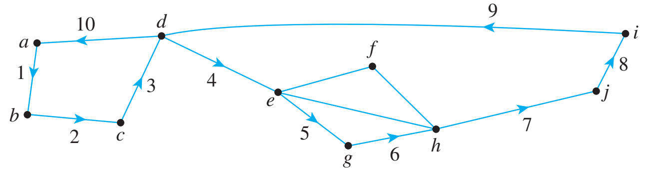
Observe that C is not an Euler circuit for the graph but that C intersects the rest of the graph at d. Let C' be

$$C'\colon deghjid.$$

Patch C' into C to obtain

$$C''\colon abcdeghjida.$$

Set $C = C''$. Then C is represented by the labeled edges shown below.



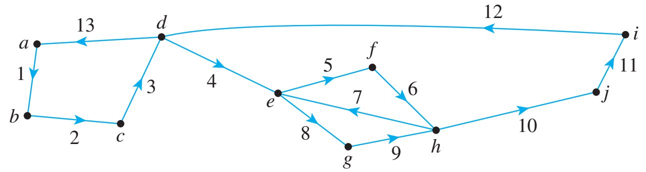
Observe that C is not an Euler circuit for the graph but that it intersects the rest of the graph at e. Let C' be

$$C'\colon efhe.$$

Patch C' into C to obtain

$$C''\colon abcdefheghjida.$$

Set $C = C''$. Then C is represented by the labeled edges shown below.



Since C includes every edge of the graph exactly once, C is an Euler circuit for the graph. ■

In exercise 45 at the end of this section you are asked to show that any graph with an Euler circuit is connected. This result can be combined with Theorems 10.2.2 and 10.2.3 to give a complete characterization of graphs that have Euler circuits, as stated in Theorem 10.2.4.

Theorem 10.2.4

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.

A corollary to Theorem 10.2.4 gives a criterion for determining when it is possible to find a walk from one vertex of a graph to another, passing through every vertex of the graph at least once and every edge of the graph exactly once.

• Definition

Let G be a graph, and let v and w be two distinct vertices of G. An **Euler trail from *v* to *w*** is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

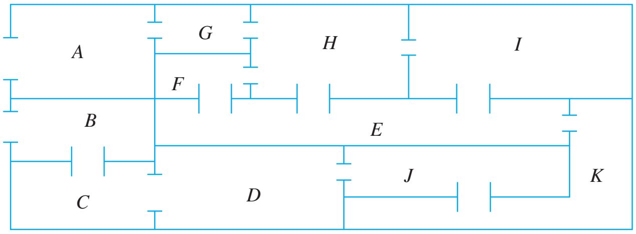
Corollary 10.2.5

Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler path from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

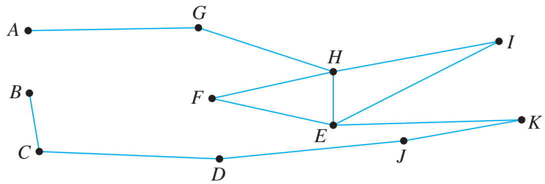
The proof of this corollary is left as an exercise.

Example 10.2.7 Finding an Euler Trail

The floor plan shown below is for a house that is open for public viewing. Is it possible to find a trail that starts in room A, ends in room B, and passes through every interior doorway of the house exactly once? If so, find such a trail.



Solution Let the floor plan of the house be represented by the graph below.



Each vertex of this graph has even degree except for A and B, each of which has degree 1. Hence by Corollary 10.2.5, there is an Euler path from A to B. One such trail is

$$AGHFEIHEKJDCB.$$ ■

Hamiltonian Circuits

Sir Wm. Hamilton (1805–1865)

Theorem 10.2.4 completely answers the following question: Given a graph G, is it possible to find a circuit for G in which all the *edges* of G appear exactly once? A related question is this: Given a graph G, is it possible to find a circuit for G in which all the *vertices* of G (except the first and the last) appear exactly once?

In 1859 the Irish mathematician Sir William Rowan Hamilton introduced a puzzle in the shape of a dodecahedron (DOH-dek-a-HEE-dron). (Figure 10.2.6 contains a drawing of a dodecahedron, which is a solid figure with 12 identical pentagonal faces.)

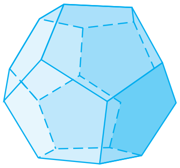
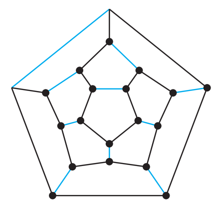


Figure 10.2.6 Dodecahedron

Each vertex was labeled with the name of a city—London, Paris, Hong Kong, New York, and so on. The problem Hamilton posed was to start at one city and tour the world by visiting each other city exactly once and returning to the starting city. One way to solve the puzzle is to imagine the surface of the dodecahedron stretched out and laid flat in the plane, as follows:



The circuit denoted with black lines is one solution. Note that although every city is visited, many edges are omitted from the circuit. (More difficult versions of the puzzle required that certain cities be visited in a certain order.)

The following definition is made in honor of Hamilton.

• Definition

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

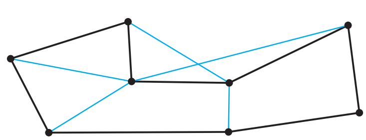
Note that although an Euler circuit for a graph G must include every vertex of G, it may visit some vertices more than once and hence may not be a Hamiltonian circuit. On the other hand, a Hamiltonian circuit for G does not need to include all the edges of G and hence may not be an Euler circuit.

Despite the analogous-sounding definitions of Euler and Hamiltonian circuits, the mathematics of the two are very different. Theorem 10.2.4 gives a simple criterion for determining whether a given graph has an Euler circuit. Unfortunately, there is no analogous criterion for determining whether a given graph has a Hamiltonian circuit, nor is there even an efficient algorithm for finding such a circuit. There is, however, a simple technique that can be used in many cases to show that a graph does *not* have a Hamiltonian circuit. This follows from the following considerations:

Suppose a graph G with at least two vertices has a Hamiltonian circuit C given concretely as

$$C: v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n.$$

Since C is a simple circuit, all the e_i are distinct and all the v_j are distinct except that $v_0 = v_n$. Let H be the subgraph of G that is formed using the vertices and edges of C. An example of such an H is shown below.



H is indicated by the black lines.

Note that H has the same number of edges as it has vertices since all its n edges are distinct and so are its n vertices $v_1, v_2, \ldots, v_n$. Also, by definition of Hamiltonian circuit, every vertex of G is a vertex of H, and H is connected since any two of its vertices lie on a circuit. In addition, every vertex of H has degree 2. The reason for this is that there are exactly two edges incident on any vertex. These are e_i and e_{i+1} for any vertex v_i except $v_0 = v_n$, and they are e_1 and e_n for $v_0\ (= v_n)$. These observations have established the truth of the following proposition in all cases where G has at least two vertices.

Proposition 10.2.6

If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

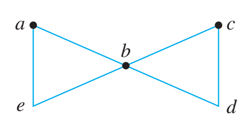
1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

Note that if G contains only one vertex and G has a Hamiltonian circuit, then the circuit has the form $v\ e\ v$, where v is the vertex of G and e is an edge incident on v. In this case, the subgraph H consisting of v and e satisfies conditions (1)–(4) of Proposition 10.2.6.

Recall that the contrapositive of a statement is logically equivalent to the statement. The contrapositive of Proposition 10.2.6 says that if a graph G does *not* have a subgraph H with properties (1)–(4), then G does *not* have a Hamiltonian circuit.

Example 10.2.8 Showing That a Graph Does Not Have a Hamiltonian Circuit

Prove that the graph G shown below does not have a Hamiltonian circuit.

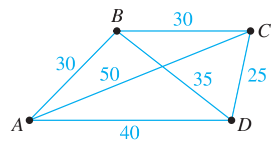


Solution If G has a Hamiltonian circuit, then by Proposition 10.2.6, G has a subgraph H that (1) contains every vertex of G, (2) is connected, (3) has the same number of edges as vertices, and (4) is such that every vertex has degree 2. Suppose such a subgraph H exists. In other words, suppose there is a connected subgraph H of G such that H has five vertices (a, b, c, d, e) and five edges and such that every vertex of H has degree 2. Since the degree of b in G is 4 and every vertex of H has degree 2, two edges incident on b must be removed from G to create H. Edge $\{a, b\}$ cannot be removed because if it were, vertex a would have degree less than 2 in H. Similar reasoning shows that edges $\{e, b\}$, $\{b, a\}$, and $\{b, d\}$ cannot be removed either. It follows that the degree of b in H must be 4, which contradicts the condition that every vertex in H has degree 2 in H. Hence no such subgraph H exists, and so G does not have a Hamiltonian circuit. ■

The next example illustrates a type of problem known as a **traveling salesman problem.** It is a variation of the problem of finding a Hamiltonian circuit for a graph.

Example 10.2.9 A Traveling Salesman Problem

Imagine that the drawing below is a map showing four cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city A. Which route from city to city will minimize the total distance that must be traveled?



Solution This problem can be solved by writing all possible Hamiltonian circuits starting and ending at A and calculating the total distance traveled for each.

Route	Total Distance (In Kilometers)
$ABCDA$	$30 + 30 + 25 + 40 = 125$
$ABDCA$	$30 + 35 + 25 + 50 = 140$
$ACBDA$	$50 + 30 + 35 + 40 = 155$
$ACDBA$	140 [$ABDCA$ backwards]
$ADBCA$	155 [$ACBDA$ backwards]
$ADCBA$	125 [$ABCDA$ backwards]

Thus either route $ABCDA$ or $ADCBA$ gives a minimum total distance of 125 kilometers. ■

The general traveling salesman problem involves finding a Hamiltonian circuit to minimize the total distance traveled for an arbitrary graph with n vertices in which each edge is marked with a distance. One way to solve the general problem is to use the method of Example 10.2.9: Write down all Hamiltonian circuits starting and ending at a particular vertex, compute the total distance for each, and pick one for which this total is minimal. However, even for medium-sized values of n this method is impractical. For a complete graph with 30 vertices, there would be $(29!)/2 \cong 4.42 \times 10^{30}$ Hamiltonian circuits starting and ending at a particular vertex to check. Even if each circuit could be found and its total distance computed in just one nanosecond, it would require approximately 1.4×10^{14} years to finish the computation. At present, there is no known algorithm for solving the general traveling salesman problem that is more efficient. However, there are efficient algorithms that find "pretty good" solutions—that is, circuits that, while not necessarily having the least possible total distances, have smaller total distances than most other Hamiltonian circuits.

Test Yourself

1. Let G be a graph and let v and w be vertices in G.
   (a) A walk from v to w is _____.
   (b) A trail from v to w is _____.
   (c) A path from v to w is _____.
   (d) A closed walk is _____.
   (e) A circuit is _____.
   (f) A simple circuit is _____.
   (g) A trivial walk is _____.
   (h) Vertices v and w are connected if, and only if, _____.

2. A graph is connected if, and only if, _____.

3. Removing an edge from a circuit in a graph does not _____.

4. An Euler circuit in a graph is _____.

5. A graph has an Euler circuit if, and only if, _____.

6. Given vertices v and w in a graph, there is an Euler path from v to w if, and only if, _____.

7. A Hamiltonian circuit in a graph is _____.

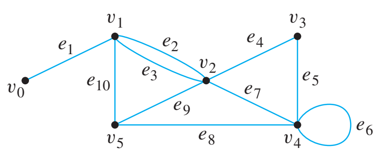
8. If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties: _____, _____, _____, and _____.

9. A traveling salesman problem involves finding a _____ that minimizes the total distance traveled for a graph in which each edge is marked with a distance.

Exercise Set 10.2

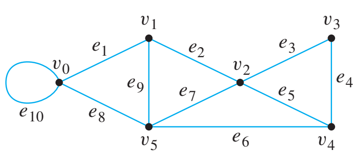
1. In the graph below, determine whether the following walks are trails, paths, closed walks, circuits, simple circuits, or just walks.

a. $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ b. $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$
c. v_2 d. $v_5v_2v_3v_4v_4v_5$
e. $v_2v_3v_4v_5v_2v_4v_3v_2$ f. $e_5e_8e_{10}e_3$

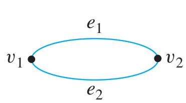


2. In the graph below, determine whether the following walks are trails, paths, closed walks, circuits, simple circuits, or just walks.

a. $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$ b. $v_2v_3v_4v_5v_2$
c. $v_4v_2v_3v_4v_5v_2v_4$ d. $v_2v_1v_5v_2v_3v_4v_2$
e. $v_0v_5v_2v_3v_4v_2v_1$ f. $v_5v_4v_2v_1$



3. Let G be the graph

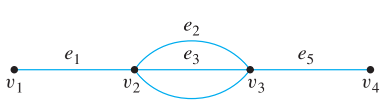


and consider the walk $v_1e_1v_2e_2v_1$.

a. Can this walk be written unambiguously as $v_1v_2v_1$? Why?

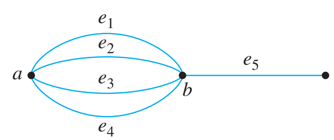
b. Can this walk be written unambiguously as e_1e_2? Why?

4. Consider the following graph.



a. How many paths are there from v_1 to v_4?
b. How many trails are there from v_1 to v_4?
c. How many walks are there from v_1 to v_4?

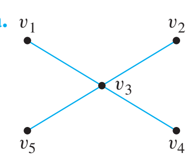
5. Consider the following graph.



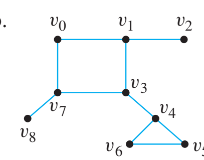
a. How many paths are there from a to c?
b. How many trails are there from a to c?
c. How many walks are there from a to c?

6. An edge whose removal disconnects the graph of which it is a part is called a **bridge.** Find all bridges for each of the following graphs.

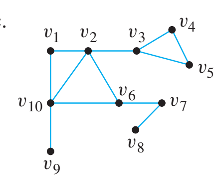
a.



b.



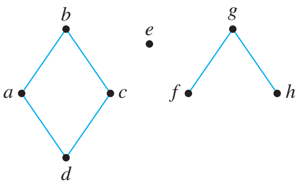
c.



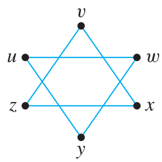
7. Given any positive integer n, (a) find a connected graph with n edges such that removal of just one edge disconnects the graph; (b) find a connected graph with n edges that cannot be disconnected by the removal of any single edge.

8. Find the number of connected components for each of the following graphs.

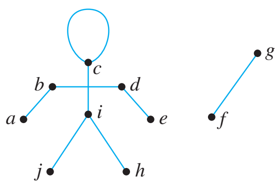
a.



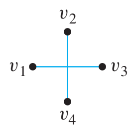
b.



c.



d.



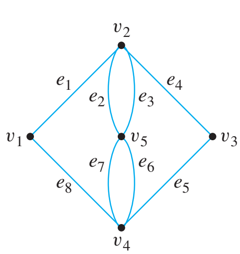
9. Each of (a)–(c) describes a graph. In each case answer *yes, no,* or *not necessarily* to this question: Does the graph have an Euler circuit? Justify your answers.
   a. G is a connected graph with five vertices of degrees 2, 2, 3, 3, and 4.
   b. G is a connected graph with five vertices of degrees 2, 2, 4, 4, and 6.
   c. G is a graph with five vertices of degrees 2, 2, 4, 4, and 6.

10. The solution for Example 10.2.5 shows a graph for which every vertex has even degree but which does not have an Euler circuit. Give another example of a graph satisfying these properties.

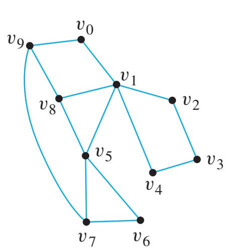
11. Is it possible for a citizen of Königsberg to make a tour of the city and cross each bridge exactly twice? (See Figure 10.2.1.) Why?

Determine which of the graphs in 12–17 have Euler circuits. If the graph does not have an Euler circuit, explain why not. If it does have an Euler circuit, describe one.

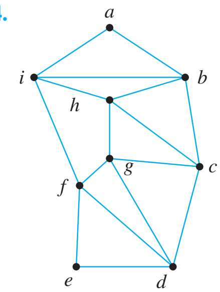
12.



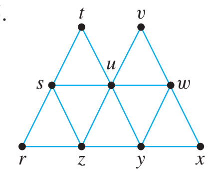
13.



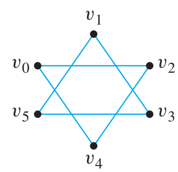
14.



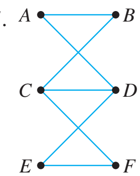
15.



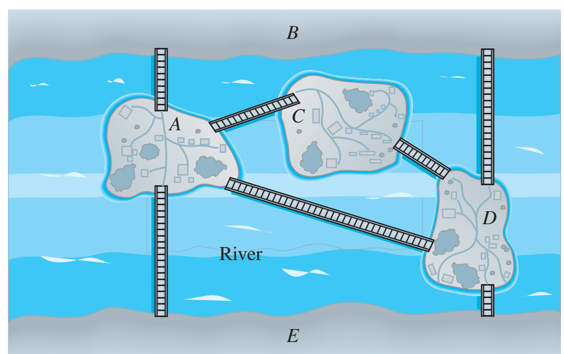
16.



17.

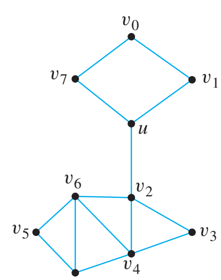


18. Is it possible to take a walk around the city whose map is shown below, starting and ending at the same point and crossing each bridge exactly once? If so, how can this be done?

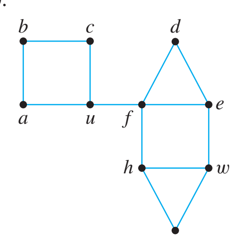


For each of the graphs in 19–21, determine whether there is an Euler path from u to w. If there is, find such a path.

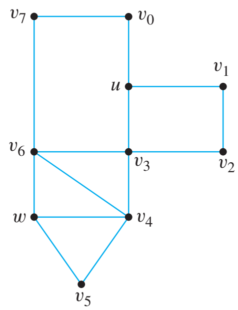
19.



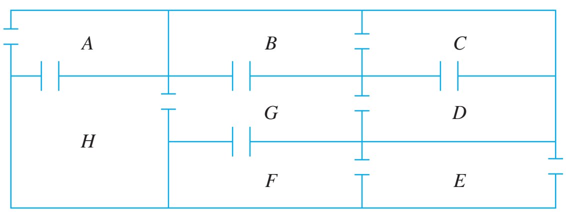
20.



21.

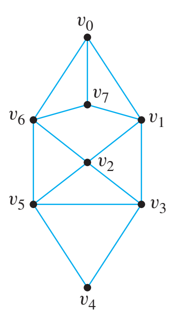


22. The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E? If so, how can this be done?

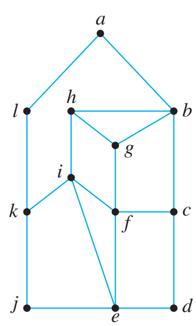


Find Hamiltonian circuits for each of the graphs in 23 and 24.

23.

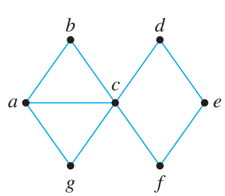


24.

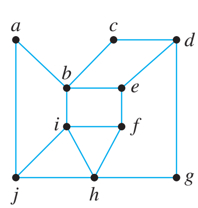


Show that none of the graphs in 25–27 has a Hamiltonian circuit.

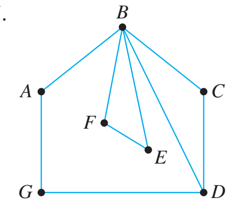
***H* 25.**



26.

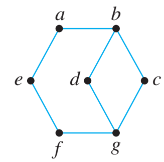


27.

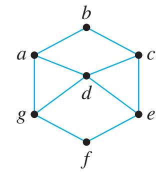


In 28–31 find Hamiltonian circuits for those graphs that have them. Explain why the other graphs do not.

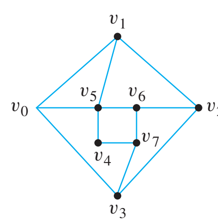
***H* 28.**



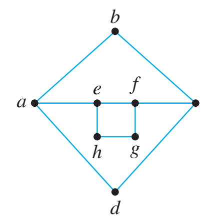
29.



30.



31.



***H* 32.** Give two examples of graphs that have Euler circuits but not Hamiltonian circuits.

***H* 33.** Give two examples of graphs that have Hamiltonian circuits but not Euler circuits.

***H* 34.** Give two examples of graphs that have circuits that are both Euler circuits and Hamiltonian circuits.

***H* 35.** Give two examples of graphs that have Euler circuits and Hamiltonian circuits that are not the same.

36. A traveler in Europe wants to visit each of the cities shown on the map exactly once, starting and ending in Brussels. The distance (in kilometers) between each pair of cities is given in the table. Find a Hamiltonian circuit that minimizes the total distance traveled. (Use the map to narrow the possible circuits down to just a few. Then use the table to find the total distance for each of those.)

	Berlin	Brussels	Düsseldorf	Luxembourg	Munich
Brussels	783				
Düsseldorf	564	223			
Luxembourg	764	219	224		
Munich	585	771	613	517	
Paris	1,057	308	497	375	832

37. **a.** Prove that if a walk in a graph contains a repeated edge, then the walk contains a repeated vertex.
b. Explain how it follows from part (a) that any walk with no repeated vertex has no repeated edge.

38. Prove Lemma 10.2.1(a): If G is a connected graph, then any two distinct vertices of G can be connected by a path.

39. Prove Lemma 10.2.1(b): If vertices v and w are part of a circuit in a graph G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

40. Draw a picture to illustrate Lemma 10.2.1(c): If a graph G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

41. Prove that if there is a trail in a graph G from a vertex v to a vertex w, then there is a trail from w to v.

H **42.** If a graph contains a circuit that starts and ends at a vertex v, does the graph contain a simple circuit that starts and ends at v? Why?

43. Prove that if there is a circuit in a graph that starts and ends at a vertex v and if w is another vertex in the circuit, then there is a circuit in the graph that starts and ends at w.

44. Let G be a connected graph, and let C be any circuit in G that does not contain every vertex of C. Let G' be the subgraph obtained by removing all the edges of C from G and also any vertices that become isolated when the edges of C are removed. Prove that there exists a vertex v such that v is in both C and G'.

45. Prove that any graph with an Euler circuit is connected.

46. Prove Corollary 10.2.5.

47. For what values of n does the complete graph K_n with n vertices have (a) an Euler circuit? (b) a Hamiltonian circuit? Justify your answers.

✶ 48. For what values of m and n does the complete bipartite graph on (m, n) vertices have (a) an Euler circuit? (b) a Hamiltonian circuit? Justify your answers.

✶ 49. What is the maximum number of edges a simple disconnected graph with n vertices can have? Prove your answer.

✶ 50. Show that a graph is bipartite if, and only if, it does not have a circuit with an odd number of edges. (See exercise 37 of Section 10.1 for the definition of bipartite graph.)

Answers for Test Yourself

1. (a) a finite alternating sequence of adjacent vertices and edges of G (b) a walk that does not contain a repeated edge (c) a trail that does not contain a repeated vertex (d) a walk that starts and ends at the same vertex (e) a closed walk that contains at least one edge and does not contain a repeated edge (f) a circuit that does not have any repeated vertex other than the first and the last (g) a walk consisting of a single vertex and no edge (h) there is a walk from v to w 2. given any two vertices in the graph, there is a walk from one to the other 3. disconnect the graph 4. a circuit that contains every vertex and every edge of the graph 5. the graph is connected, and every vertex has positive, even degree 6. the graph is connected, v and w have odd degree, and all other vertices have positive even degree 7. a simple circuit that includes every vertex of the graph 8. H contains every vertex of G; H is connected; H has the same number of edges as vertices; every vertex of H has degree 2 9. Hamiltonian circuit