Prove that each of the properties in 21–29 is an invariant for graph isomorphism. Assume that n, m, and k are all nonnegative integers.

21. Has n vertices

22. Has m edges

23. Has a circuit of length k

24. Has a simple circuit of length k

H **25.** Has m vertices of degree k

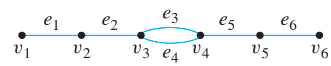
26. Has m simple circuits of length k

H **27.** Is connected

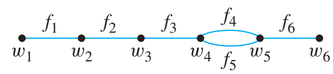
28. Has an Euler circuit

29. Has a Hamiltonian circuit

30. Show that the following two graphs are not isomorphic by supposing they are isomorphic and deriving a contradiction.



G



G'

Answers for Test Yourself

1. $g(v)$ is an endpoint of $h(e)$ 2. G' has property P 3. has n vertices; has m edges; has a vertex of degree k; has m vertices of degree k; has a circuit of length k; has a simple circuit of length k; has m simple circuits of length k; is connected; has an Euler circuit; has a Hamiltonian circuit

10.5 Trees

We are not very pleased when we are forced to accept a mathematical truth by virtue of a complicated chain of formal conclusions and computations, which we traverse blindly, link by link, feeling our way by touch. We want first an overview of the aim and of the road; we want to understand the idea of the proof, the deeper context.
— Hermann Weyl, 1885–1955

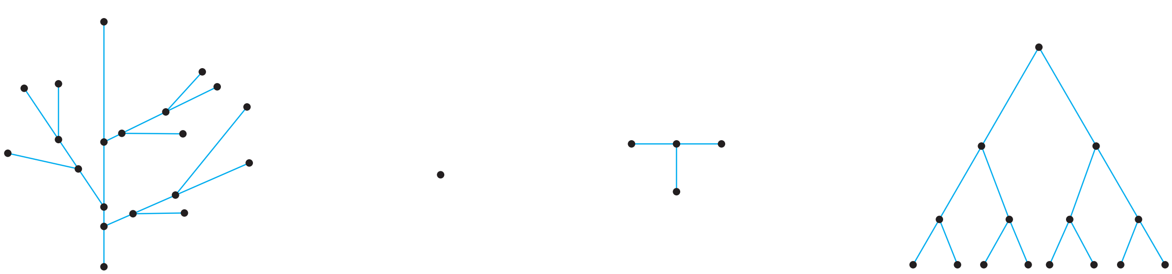
If a friend asks what you are studying and you answer "trees," your friend is likely to infer you are taking a course in botany. But trees are also a subject for mathematical investigation. In mathematics, a tree is a connected graph that does not contain any circuits. Mathematical trees are similar in certain ways to their botanical namesakes.

• Definition

A graph is said to be **circuit-free** if, and only if, it has no circuits. A graph is called a **tree** if, and only if, it is circuit-free and connected. A **trivial tree** is a graph that consists of a single vertex. A graph is called a **forest** if, and only if, it is circuit-free and not connected.

Example 10.5.1 Trees and Non-Trees

All the graphs shown in Figure 10.5.1 are trees, whereas those in Figure 10.5.2 are not.



(a) (b) (c) (d)

Figure 10.5.1 Trees. All the graphs in (a)–(d) are connected and circuit-free.

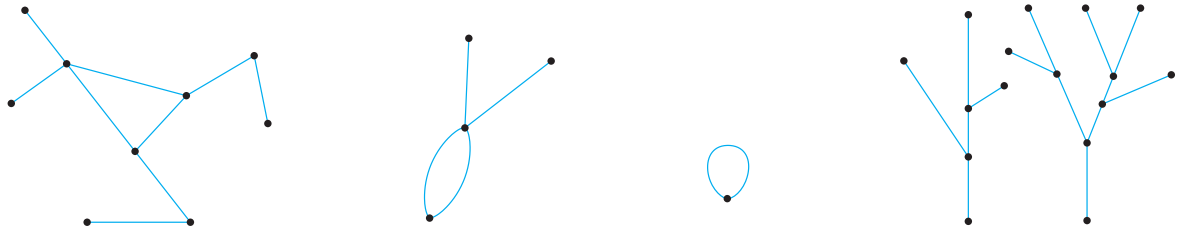


Figure 10.5.2 **Non-Trees. The graphs in (a), (b), and (c) all have circuits, and the graph in (d) is not connected.** ■

Examples of Trees

The following examples illustrate just a few of the many and varied situations in which mathematical trees arise.

Example 10.5.2 A Decision Tree

During orientation week, a college administers an exam to all entering students to determine placement in the mathematics curriculum. The exam consists of two parts, and placement recommendations are made as indicated by the tree shown in Figure 10.5.3. Read the tree from left to right to decide what course should be recommended for a student who scored 9 on part I and 7 on part II.

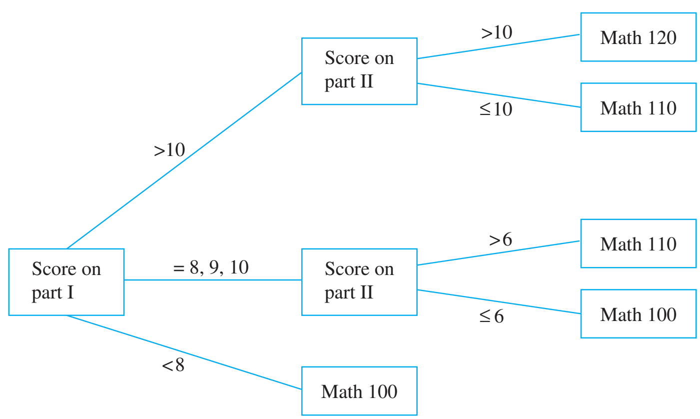


Figure 10.5.3

Solution Since the student scored 9 on part I, the score on part II is checked. Since it is greater than 6, the student should be advised to take Math 110. ■

Example 10.5.3 A Parse Tree

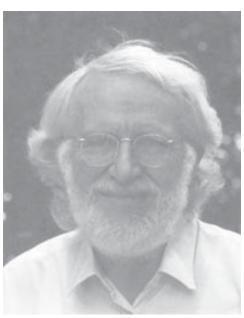
In the last 30 years, Noam Chomsky and others have developed new ways to describe the syntax (or grammatical structure) of natural languages such as English. As is discussed briefly in Chapter 12, this work has proved useful in constructing compilers for high-level computer languages. In the study of grammars, trees are often used to show the derivation of grammatically correct sentences from certain basic rules. Such trees are called **syntactic derivation trees** or **parse trees.**

A very small subset of English grammar, for example, specifies that

1. a sentence can be produced by writing first a noun phrase and then a verb phrase;
2. a noun phrase can be produced by writing an article and then a noun;
3. a noun phrase can also be produced by writing an article, then an adjective, and then a noun;
4. a verb phrase can be produced by writing a verb and then a noun phrase;
5. one article is “the”;
6. one adjective is “young”;
7. one verb is “caught”;
8. one noun is “man”;
9. one (other) noun is “ball.”

Courtesy of IBM Corporation

John Backus (1924–1998)



Courtesy of Peter Naur

Peter Naur (born 1928)

The rules of a grammar are called **productions.** It is customary to express them using the shorthand notation illustrated below. This notation, introduced by John Backus in 1959 and modified by Peter Naur in 1960, was used to describe the computer language Algol and is called the **Backus-Naur notation.** In the notation, the symbol | represents the word *or,* and angle brackets ⟨ ⟩ are used to enclose terms to be defined (such as a sentence or noun phrase).

1. ⟨sentence⟩ → ⟨noun phrase⟩⟨verb phrase⟩

2., 3. ⟨noun phrase⟩ → ⟨article⟩⟨noun⟩ | ⟨article⟩⟨adjective⟩⟨noun⟩

4. ⟨verb phrase⟩ → ⟨verb⟩⟨noun phrase⟩

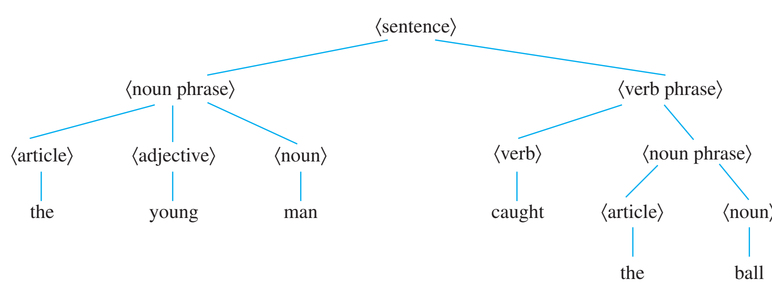
5. ⟨article⟩ → the

6. ⟨adjective⟩ → young

7, 8. ⟨noun⟩ → man | ball

9. ⟨verb⟩ → caught

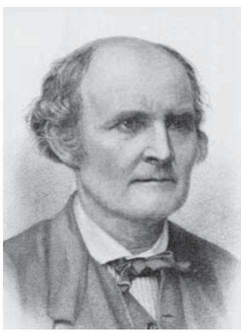
The derivation of the sentence “The young man caught the ball” from the above rules is described by the tree shown below.



In the study of linguistics, **syntax** refers to the grammatical structure of sentences, and **semantics** refers to the meanings of words and their interrelations. A sentence can be syntactically correct but semantically incorrect, as in the nonsensical sentence “The young ball caught the man,” which can be derived from the rules given above. Or a sentence can contain syntactic errors but not semantic ones, as, for instance, when a two-year-old child says, “Me hungry!” ■

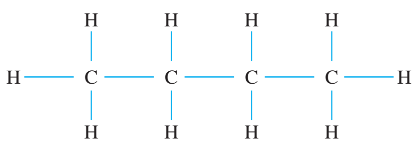
Example 10.5.4 Structure of Hydrocarbon Molecules

The German physicist Gustav Kirchhoff (1824–1887) was the first to analyze the behavior of mathematical trees in connection with the investigation of electrical circuits. Soon after (and independently), the English mathematician Arthur Cayley used the mathematics of trees to enumerate all isomers for certain hydrocarbons. Hydrocarbon molecules are composed of carbon and hydrogen; each carbon atom can form up to four chemical bonds with other atoms, and each hydrogen atom can form one bond with another atom. Thus the structure of hydrocarbon molecules can be represented by graphs such as those shown following, in which the vertices represent atoms of hydrogen and carbon, denoted H and C, and the edges represent the chemical bonds between them.

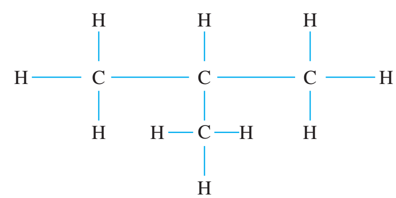


Bettmann/CORBIS

Arthur Cayley (1821–1895)



Butane

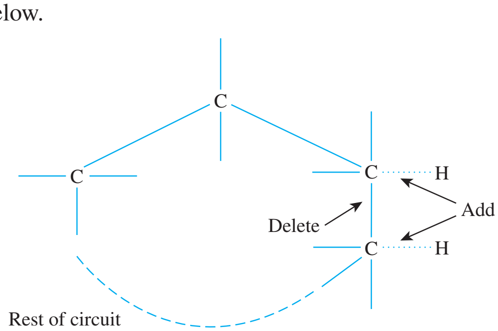


Isobutane

Note that each of these graphs has four carbon atoms and ten hydrogen atoms, but the two graphs show different configurations of atoms. When two molecules have the same chemical formulae (in this case C_4H_{10}) but different chemical bonds, they are called *isomers*.

Certain *saturated hydrocarbon* molecules contain the maximum number of hydrogen atoms for a given number of carbon atoms. Cayley showed that if such a saturated hydrocarbon molecule has k carbon atoms, then it has $2k + 2$ hydrogen atoms. The first step in doing so is to prove that the graph of such a saturated hydrocarbon molecule is a tree. Prove this using proof by contradiction. (You are asked to finish the derivation of Cayley's result in exercise 4 at the end of this section.)

Solution Suppose there is a hydrocarbon molecule that contains the maximum number of hydrogen atoms for the number of its carbon atoms and whose graph G is not a tree. *[We must derive a contradiction.]* Since G is not a tree, G is not connected or G has a circuit. But the graph of any molecule is connected (all the atoms in a molecule must be connected to each other), and so G must have a nontrivial circuit. Now the edges of the circuit can link only carbon atoms because every vertex of a circuit has degree at least 2 and a hydrogen atom vertex has degree 1. Delete one edge of the circuit and add two new edges to join each of the newly disconnected carbon atom vertices to a hydrogen atom vertex as shown below.



The resulting molecule has two more hydrogen atoms than the given molecule, but the number of carbon atoms is unchanged. This contradicts the supposition that the given molecule has the maximum number of hydrogen atoms for the given number of carbon atoms. Hence the supposition is false, and so G is a tree. ■

Characterizing Trees

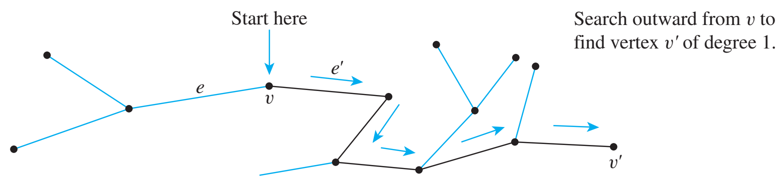
There is a somewhat surprising relation between the number of vertices and the number of edges of a tree. It turns out that if n is a positive integer, then any tree with n vertices (no matter what its shape) has $n - 1$ edges. Perhaps even more surprisingly, a partial converse to this fact is also true—namely, any *connected* graph with n vertices and $n - 1$ edges is a tree. It follows from these facts that if even one new edge (but no new vertex) is added to a tree, the resulting graph must contain a circuit. Also, from the fact that removing an edge from a circuit does not disconnect a graph, it can be shown that every connected graph has a subgraph that is a tree. It follows that if n is a positive integer, any graph with n vertices and *fewer* than $n - 1$ edges is not connected.

A small but very important fact necessary to derive the first main theorem about trees is that any nontrivial tree must have at least one vertex of degree 1.

Lemma 10.5.1

Any tree that has more than one vertex has at least one vertex of degree 1.

A constructive way to understand this lemma is to imagine being given a tree T with more than one vertex. You pick a vertex v at random and then search outward along a path from v looking for a vertex of degree 1. As you reach each new vertex, you check whether it has degree 1. If it does, you are finished. If it does not, you exit from the vertex along a different edge from the one you entered on. Because T is circuit-free, the vertices included in the path never repeat. And since the number of vertices of T is finite, the process of building a path must eventually terminate. When that happens, the final vertex v' of the path must have degree 1. This process is illustrated below.



This discussion is made precise in the following proof.

Proof:

Let T be a particular but arbitrarily chosen tree that has more than one vertex, and consider the following algorithm:

Step 1: Pick a vertex v of T and let e be an edge incident on v.
[If there were no edge incident on v, then v would be an isolated vertex. But this would contradict the assumption that T is connected (since it is a tree) and has at least two vertices.]

Step 2: While $\deg(v) > 1$, repeat steps 2a, 2b, and 2c:

continued on page 688

Step 2a: Choose e' to be an edge incident on v such that $e' \neq e$. *[Such an edge exists because* $\deg(v) > 1$ *and so there are at least two edges incident on* v.*]*

Step 2b: Let v' be the vertex at the other end of e' from v. *[Since T is a tree, e' cannot be a loop and therefore e' has two distinct endpoints.]*

Step 2c: Let $e = e'$ and $v = v'$. *[This is just a renaming process in preparation for a repetition of step 2.]*

The algorithm just described must eventually terminate because the set of vertices of the tree T is finite and T is circuit-free. When it does, a vertex v of degree 1 will have been found.

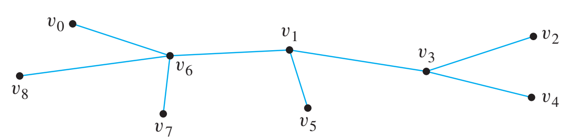
Using Lemma 10.5.1 it is not difficult to show that, in fact, any tree that has more than one vertex has at least *two* vertices of degree 1. This extension of Lemma 10.5.1 is left to the exercises at the end of this section.

• Definition

Let T be a tree. If T has only one or two vertices, then each is called a **terminal vertex.** If T has at least three vertices, then a vertex of degree 1 in T is called a **terminal vertex** (or a **leaf**), and a vertex of degree greater than 1 in T is called an **internal vertex** (or a **branch vertex**).

Example 10.5.5 Terminal and Internal Vertices

Find all terminal vertices and all internal vertices in the following tree:



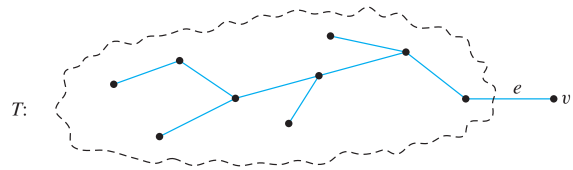
Solution The terminal vertices are v_0, v_2, v_4, v_5, v_7, and v_8. The internal vertices are v_6, v_1, and v_3. ■

The following is the first of the two main theorems about trees:

Theorem 10.5.2

For any positive integer n, any tree with n vertices has $n - 1$ edges.

The proof is by mathematical induction. To do the inductive step, you assume the theorem is true for a positive integer k and then show it is true for $k + 1$. Thus you assume you have a tree T with $k + 1$ vertices, and you must show that T has $(k + 1) - 1 = k$ edges. As you do this, you are free to use the inductive hypothesis that *any* tree with k vertices has $k - 1$ edges. To make use of the inductive hypothesis, you need to reduce the tree T with $k + 1$ vertices to a tree with just k vertices. But by Lemma 10.5.1, T has a vertex v of degree 1, and since T is connected, v is attached to the rest of T by a single edge e as sketched on the next page.



Rest of T

Now if e and v are removed from T, what remains is a tree T' with $(k+1)-1=k$ vertices. By inductive hypothesis, then, T' has $k-1$ edges. But the original tree T has one more vertex and one more edge than T'. Hence T must have $(k-1)+1=k$ edges, as was to be shown. A formal version of this argument is given below.

Proof (by mathematical induction):

Let the property $P(n)$ be the sentence

Any tree with n vertices has $n-1$ edges. $\leftarrow P(n)$

We use mathematical induction to show that this property is true for all integers $n \geq 1$.

Show that $P(1)$ is true: Let T be any tree with one vertex. Then T has zero edges (since it contains no loops). But $0 = 1 - 1$, so $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:
Suppose k is any positive integer for which $P(k)$ is true. In other words, suppose that

Any tree with k vertices has k - 1 edges. $\leftarrow P(k)$ inductive hypothesis

We must show that $P(k+1)$ is true. In other words, we must show that

Any tree with $k+1$ vertices has $(k+1)-1=k$ edges. $\leftarrow P(k+1)$

Let T be a particular but arbitrarily chosen tree with $k+1$ vertices. *[We must show that T has k edges.]* Since k is a positive integer, $(k+1) \geq 2$, and so T has more than one vertex. Hence by Lemma 10.5.1, T has a vertex v of degree 1. Also, since T has more than one vertex, there is at least one other vertex in T besides v. Thus there is an edge e connecting v to the rest of T. Define a subgraph T' of T so that

$$V(T') = V(T) - \{v\}$$

Then

$$E(T') = E(T) - \{e\}.$$

1. The number of vertices of T' is $(k+1)-1=k$.
2. T' is circuit-free (since T is circuit-free, and removing an edge and a vertex cannot create a circuit).
3. T' is connected (see exercise 24 at the end of this section).

Hence, by the definition of tree, T' is a tree. Since T' has k vertices, by inductive hypothesis

$$\begin{aligned}\text{the number of edges of } T' &= (\text{the number of vertices of } T') - 1 \\ &= k - 1.\end{aligned}$$

continued on page 690

But then

$$\begin{aligned}\text{the number of edges of } T &= (\text{the number of edges of } T') + 1\\ &= (k-1)+1\\ &= k.\end{aligned}$$

[This is what was to be shown.]

Example 10.5.6 Determining Whether a Graph Is a Tree

A graph G has ten vertices and twelve edges. Is it a tree?

Solution No. By Theorem 10.5.2, any tree with ten vertices has nine edges, not twelve. ■

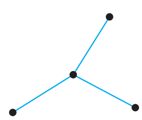
Example 10.5.7 Finding Trees Satisfying Given Conditions

Find all nonisomorphic trees with four vertices.

Solution By Theorem 10.5.2, any tree with four vertices has three edges. Thus the total degree of a tree with four vertices must be 6. Also, every tree with more than one vertex has at least two vertices of degree 1 (see the comment following Lemma 10.5.1 and exercises 5 and 29 at the end of this section). Thus the following combinations of degrees for the vertices are the only ones possible:

$$1, 1, 1, 3 \quad \text{and} \quad 1, 1, 2, 2.$$

There are two nonisomorphic trees corresponding to both of these possibilities, as shown below.



and

■

To prove the second major theorem about trees, we need another lemma.

Lemma 10.5.3

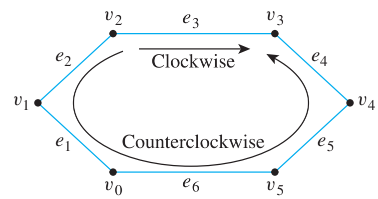
If G is any connected graph, C is any circuit in G, and any one of the edges of C is removed from G, then the graph that remains is connected.

Essentially, the reason why Lemma 10.5.3 is true is that any two vertices in a circuit are connected by two distinct paths. It is possible to draw the graph so that one of these goes "clockwise" and the other goes "counterclockwise" around the circuit. For example, in the circuit shown on the next page, the clockwise path from v_2 to v_3 is

$$v_2 e_3 v_3$$

and the counterclockwise path from v_2 to v_3 is

$$v_2 e_2 v_1 e_1 v_0 e_6 v_5 e_5 v_4 e_4 v_3.$$



Proof:

Suppose G is a connected graph, C is a circuit in G, and e is an edge of C. Form a subgraph G' of G by removing e from G. Thus

$$V(G') = V(G)$$
$$E(G') = E(G) - \{e\}.$$

We must show that G' is connected. *[To show a graph is connected, we must show that if u and w are any vertices of the graph, then there exists a walk in G' from u to w.]* Suppose u and w are any two vertices of G'. *[We must find a walk from u to w.]* Since the vertex sets of G and G' are the same, u and w are both vertices of G, and since G is connected, there is a walk W in G from u to w.

Case 1 (e is not an edge of W): The only edge in G that is not in G' is e, so in this case W is also a walk in G'. Hence u is connected to w by a walk in G'.

Case 2 (e is an edge of W): In this case the walk W from u to w includes a section of the circuit C that contains e. Let C be denoted as follows:

$$C\colon v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n \; (= v_0).$$

Now e is one of the edges of C, so, to be specific, let $e = e_k$. Then the walk W contains either the sequence

$$v_{k-1} e_k v_k \quad \text{or} \quad v_k e_k v_{k-1}.$$

If W contains $v_{k-1} e_k v_k$, connect v_{k-1} to v_k by taking the "counterclockwise" walk W' defined as follows:

$$W'\colon v_{k-1} e_{k-1} v_{k-2} \cdots v_0 e_n v_{n-1} \cdots e_{k+1} v_k.$$

An example showing how to go from u to w while avoiding e_k is given in Figure 10.5.4.

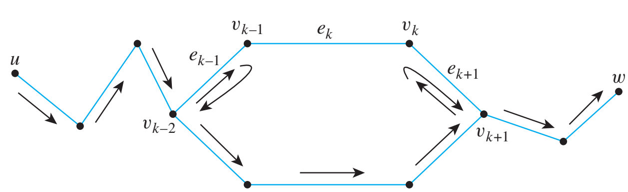


Figure 10.5.4 **An Example of a Walk from u to w That Does Not Include Edge e_k**

If W contains $v_k e_k v_{k-1}$, connect v_k to v_{k-1} by taking the "clockwise" walk W'' defined as follows:

$$W''\colon v_k e_{k+1} v_{k+1} \cdots v_n e_1 v_1 e_2 \cdots e_{k-1} v_{k-1}.$$

continued on page 692

Now patch either W' or W'' into W to form a new walk from u to w. For instance, to patch W' into W, start with the section of W from u to v_{k-1}, then take W' from v_{k-1} to v_k, and finally take the section of W from v_k to w. If this new walk still contains an occurrence of e, just repeat the process described previously until all occurrences are eliminated. *[This must happen eventually since the number of occurrences of e in C is finite.]* The result is a walk from u to w that does not contain e and hence is a walk in G'.

The previous arguments show that both in case 1 and in case 2 there is a walk in G' from u to w. Since the choice of u and w was arbitrary, G' is connected.

The second major theorem about trees is a modified converse to Theorem 10.5.2.

Theorem 10.5.4

For any positive integer n, if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Proof:

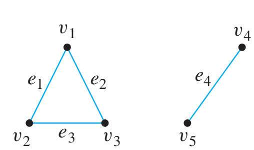
Let n be a positive integer and suppose G is a particular but arbitrarily chosen graph that is connected and has n vertices and $n - 1$ edges. *[We must show that G is a tree. Now a tree is a connected, circuit-free graph. Since we already know G is connected, it suffices to show that G is circuit-free.]* Suppose G is not circuit-free. That is, suppose G has a circuit C. *[We must derive a contradiction.]* By Lemma 10.5.3, an edge of C can be removed from G to obtain a graph G' that is connected. If G' has a circuit, then repeat this process: Remove an edge of the circuit from G' to form a new connected graph. Continue repeating the process of removing edges from circuits until eventually a graph G'' is obtained that is connected and is circuit-free. By definition, G'' is a tree. Since no vertices were removed from G to form G'', G'' has n vertices just as G does. Thus, by Theorem 10.5.2, G'' has $n - 1$ edges. But the supposition that G has a circuit implies that at least one edge of G is removed to form G''. Hence G'' has no more than $(n - 1) - 1 = n - 2$ edges, which contradicts its having $n - 1$ edges. So the supposition is false. Hence G is circuit-free, and therefore G is a tree *[as was to be shown]*.

Theorem 10.5.4 is not a full converse of Theorem 10.5.2. Although it is true that every *connected* graph with n vertices and $n - 1$ edges (where n is a positive integer) is a tree, it is not true that *every* graph with n vertices and $n - 1$ edges is a tree.

Example 10.5.8 A Graph with n Vertices and $n - 1$ Edges That Is Not a Tree

Give an example of a graph with five vertices and four edges that is not a tree.

Solution By Theorem 10.5.4, such a graph cannot be connected. One example of such an unconnected graph is shown below.



■

Test Yourself

1. A circuit-free graph is a graph with _____.
2. A forest is a graph that is _____, and a tree is a graph that is _____.
3. A trivial tree is a graph that consists of _____.
4. Any tree with at least two vertices has at least one vertex of degree _____.
5. If a tree T has at least two vertices, then a terminal vertex (or leaf) in T is a vertex of degree _____ and an internal vertex (or branch vertex) in T is a vertex of degree _____.
6. For any positive integer n, any tree with n vertices has _____.
7. For any positive integer n, if G is a connected graph with n vertices and $n - 1$ edges then _____.

Exercise Set 10.5

1. Read the tree in Example 10.5.2 from left to right to answer the following questions:
   a. What course should a student who scored 12 on part I and 4 on part II take?
   b. What course should a student who scored 8 on part I and 9 on part II take?

2. Draw trees to show the derivations of the following sentences from the rules given in Example 10.5.3.
   a. The young ball caught the man.
   b. The man caught the young ball.

H **3.** What is the total degree of a tree with n vertices? Why?

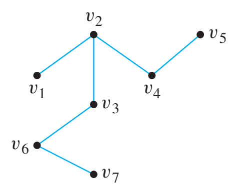
4. Let G be the graph of a hydrocarbon molecule with the maximum number of hydrogen atoms for the number of its carbon atoms.
   a. Draw the graph of G if G has three carbon atoms and eight hydrogen atoms.
   b. Draw the graphs of three isomers of C_5H_{12}.
   c. Use Example 10.5.4 and exercise 3 to prove that if the vertices of G consist of k carbon atoms and m hydrogen atoms, then G has a total degree of $2k + 2m - 2$.
   H **d.** Prove that if the vertices of G consist of k carbon atoms and m hydrogen atoms, then G has a total degree of $4k + m$.
   e. Equate the results of (c) and (d) to prove Cayley's result that a saturated hydrocarbon molecule with k carbon atoms and a maximum number of hydrogen atoms has $2k + 2$ hydrogen atoms.

H **5.** Extend the argument given in the proof of Lemma 10.5.1 to show that a tree with more than one vertex has at least two vertices of degree 1.

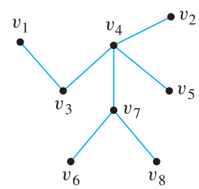
6. If graphs are allowed to have an infinite number of vertices and edges, then Lemma 10.5.1 is false. Give a counterexample that shows this. In other words, give an example of an "infinite tree" (a connected, circuit-free graph with an infinite number of vertices and edges) that has no vertex of degree 1.

7. Find all terminal vertices and all internal vertices for the following trees.

a.



b.



In each of 8–21, either draw a graph with the given specifications or explain why no such graph exists.

8. Tree, nine vertices, nine edges

9. Graph, connected, nine vertices, nine edges

10. Graph, circuit-free, nine vertices, six edges

11. Tree, six vertices, total degree 14

12. Tree, five vertices, total degree 8

13. Graph, connected, six vertices, five edges, has a nontrivial circuit

14. Graph, two vertices, one edge, not a tree

15. Graph, circuit-free, seven vertices, four edges

16. Tree, twelve vertices, fifteen edges

17. Graph, six vertices, five edges, not a tree

18. Tree, five vertices, total degree 10

19. Graph, connected, ten vertices, nine edges, has a nontrivial circuit

20. Simple graph, connected, six vertices, six edges

21. Tree, ten vertices, total degree 24

22. A connected graph has twelve vertices and eleven edges. Does it have a vertex of degree 1? Why?

23. A connected graph has nine vertices and twelve edges. Does it have a nontrivial circuit? Why?

24. Suppose that v is a vertex of degree 1 in a connected graph G and that e is the edge incident on v. Let G' be the subgraph of G obtained by removing v and e from G. Must G' be connected? Why?

25. A graph has eight vertices and six edges. Is it connected? Why?

H 26. If a graph has n vertices and $n - 2$ or fewer edges, can it be connected? Why?

27. A circuit-free graph has ten vertices and nine edges. Is it connected? Why?

H 28. Is a circuit-free graph with n vertices and at least $n - 1$ edges connected? Why?

29. Prove that every nontrivial tree has at least two vertices of degree 1 by filling in the details and completing the following argument: Let T be a nontrivial tree and let S be the set of all paths from one vertex to another of T. Among all the paths in S, choose a path P with the most edges. (Why is it possible to find such a P?) What can you say about the initial and final vertices of P? Why?

30. Find all nonisomorphic trees with five vertices.

31. a. Prove that the following is an invariant for graph isomorphism: A vertex of degree i is adjacent to a vertex of degree j.

H b. Find all nonisomorphic trees with six vertices.

Answers for Test Yourself

1. no circuits 2. circuit-free and not connected; connected and circuit-free 3. a single vertex (and no edges) 4. 1 5. 1; greater than 1 (*Or*: at least 2) 6. $n - 1$ edges 7. G is a tree

10.6 Rooted Trees

Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings. — Alfred North Whitehead, 1861–1947

An outdoor tree is rooted and so is the kind of family tree that shows all the descendants of one particular person. The terminology and notation of rooted trees blends the language of botanical trees and that of family trees. In mathematics, a rooted tree is a tree in which one vertex has been distinguished from the others and is designated the *root*. Given any other vertex v in the tree, there is a unique path from the root to v. (After all, if there were two distinct paths, a circuit could be constructed.) The number of edges in such a path is called the *level* of v, and the *height* of the tree is the length of the longest such path. It is traditional in drawing rooted trees to place the root at the top (as is done in family trees) and show the branches descending from it.

• Definition

A **rooted tree** is a tree in which there is one vertex that is distinguished from the others and is called the **root.** The **level** of a vertex is the number of edges along the unique path between it and the root. The **height** of a rooted tree is the maximum level of any vertex of the tree. Given the root or any internal vertex v of a rooted tree, the **children** of v are all those vertices that are adjacent to v and are one level farther away from the root than v. If w is a child of v, then v is called the **parent** of w, and two distinct vertices that are both children of the same parent are called **siblings.** Given two distinct vertices v and w, if v lies on the unique path between w and the root, then v is an **ancestor** of w and w is a **descendant** of v.