40. Trace Algorithm 11.3.4 for the input $n = 3, a[0] = 2, a[1] = 1, a[2] = -1, a[3] = 3$, and $x = 2$.

41. Trace Algorithm 11.3.4 for the input $n = 2, a[0] = 5, a[1] = -1, a[2] = 2$, and $x = 3$.

H **42.** Let t_n = the number of additions and multiplications that must be performed when Algorithm 11.3.4 is executed for a polynomial of degree n. Express t_n as a function of n.

43. Use the theorem on polynomial orders to find an order for Algorithm 11.3.4. How does this order compare with that of Algorithm 11.3.3?

Answers for Test Yourself

1. one iteration of the innermost loop 2. n 3. n^2; n^2

11.4 Exponential and Logarithmic Functions: Graphs and Orders

We ought never to allow ourselves to be persuaded of the truth of anything unless on the evidence of our own reason. — René Descartes, 1596–1650

Exponential and logarithmic functions are of great importance in mathematics in general and in computer science in particular. Several important computer algorithms have execution times that involve logarithmic functions of the size of the input data (which means they are relatively efficient for large data sets), and some have execution times that are exponential functions of the size of the input data (which means they are quite inefficient for large data sets). In addition, since exponential and logarithmic functions arise naturally in the descriptions of many growth and decay processes and in the computation of many kinds of probabilities, these functions are used in the analysis of computer operating systems, in queuing theory, and in the theory of information.

Graphs of Exponential Functions

As defined in Section 7.2, the exponential function with base $b > 0$ is the function that sends each real number x to b^x. The graph of the exponential function with base 2 (together with a partial table of its values) is shown in Figure 11.4.1. Note that the values of this function increase with extraordinary rapidity. If we tried to continue drawing the graph using the scale shown in Figure 11.4.1, we would have to plot the point $(10, 2^{10})$ more than 21 feet above the horizontal axis. And the point $(30, 2^{30})$ would be located more than 610,080 miles above the axis—well beyond the moon!

x	2^x
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
–1	$2^{-1} = 0.5$
–2	$2^{-2} = 0.25$
–3	$2^{-3} = 0.125$
0.5	$2^{0.5} \cong 1.414$
–0.5	$2^{-0.5} \cong 0.707$

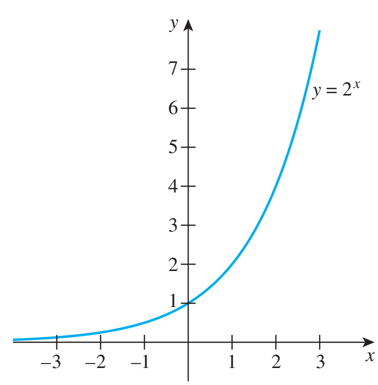
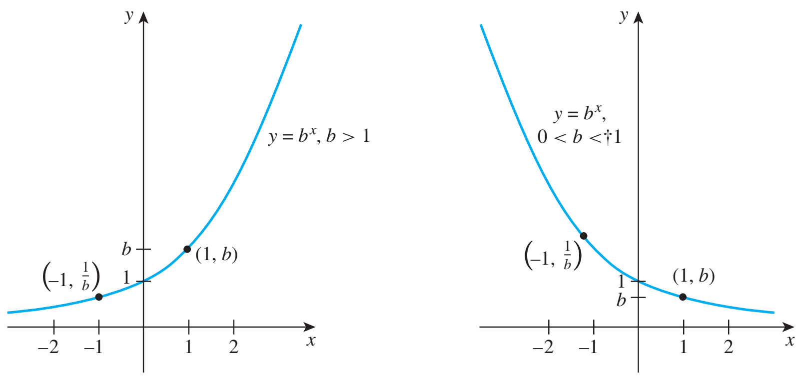


Figure 11.4.1 **The Exponential Function with Base 2**

The graph of any exponential function with base $b > 1$ has a shape that is similar to the graph of the exponential function with base 2. If $0 < b < 1$, then $1/b > 0$ and the graph of the exponential function with base b is the reflection across the vertical axis of the exponential function with base $1/b$. These facts are illustrated in Figure 11.4.2.



Graph of the exponential function with base $b > 1$

(a)

Graph of the exponential function with base b where $0 < b < 1$

(b)

Figure 11.4.2 Graphs of Exponential Functions

Graphs of Logarithmic Functions

Bettmann/CORBIS

John Napier (1550–1617)

Logarithms were first introduced by the Scotsman John Napier. Astronomers and navigators found them so useful for reducing the time needed to do multiplication and division that they quickly gained wide acceptance and played a crucial role in the remarkable development of those areas in the seventeenth century. Nowadays, however, electronic calculators and computers are available to handle most computations quickly and conveniently, and logarithms and logarithmic functions are used primarily as conceptual tools.

Recall the definition of the logarithmic function with base b from Section 7.1. We state it formally below.

• Definition

If b is a positive real number not equal to 1, then the **logarithmic function with base b, $\log_b: \mathbf{R}^+ \to \mathbf{R}$,** is the function that sends each positive real number x to the number $\log_b x$, which is the exponent to which b must be raised to obtain x.

The logarithmic function with base b is, in fact, the inverse of the exponential function with base b. (See exercise 10 at the end of this section.) It follows that the graphs of the two functions are symmetric with respect to the line $y = x$. The graph of the logarithmic function with base $b > 1$ is shown in Figure 11.4.3 on the next page.

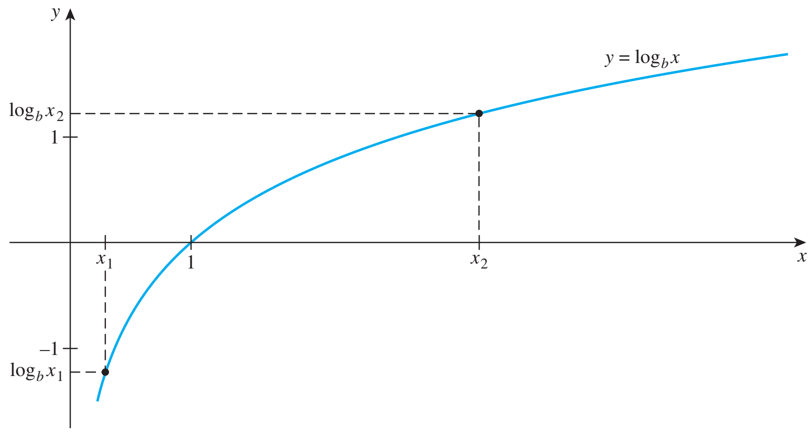


Figure 11.4.3 **The Graph of the Logarithmic Function with Base $b > 1$**

If its base b is greater than 1, the logarithmic function is increasing. Analytically, this means that

> if $b > 1$, then for all positive numbers x_1 and x_2,
>
> $$\text{if } x_1 < x_2, \text{ then } \log_b(x_1) < \log_b(x_2). \qquad 11.4.1$$

Note As examples, $\log_2(1{,}024)$ is only 10 and $\log_2(1{,}048{,}576)$ is just 20.

Corresponding to the rapid growth of the exponential function, however, is the very slow growth of the logarithmic function. Thus you must go very far out on the horizontal axis to find points whose logarithms are large numbers.

The following example shows how to make use of the increasing nature of the logarithmic function with base 2 to derive a remarkably useful property.

Example 11.4.1 Base 2 Logarithms of Numbers between Two Consecutive Powers of 2

Prove the following property:

a.
> If k is an integer and x is a real number with
>
> $$2^k \le x < 2^{k+1}, \text{ then } \lfloor \log_2 x \rfloor = k. \qquad 11.4.2$$

b. Describe property (11.4.2) in words and give a graphical interpretation of the property for $x > 1$.

Solution

a. Suppose that k is an integer and x is a real number with

$$2^k \le x < 2^{k+1}.$$

Because the logarithmic function with base 2 is increasing, this implies that

$$\log_2(2^k) \le \log_2 x < \log_2(2^{k+1}).$$

But $\log_2(2^k) = k$ *[the exponent to which you must raise* 2 *to get* 2^k *is* k*]* and $\log_2(2^{k+1}) = k + 1$ *[for a similar reason]*. Hence

$$k \le \log_2 x < k + 1.$$

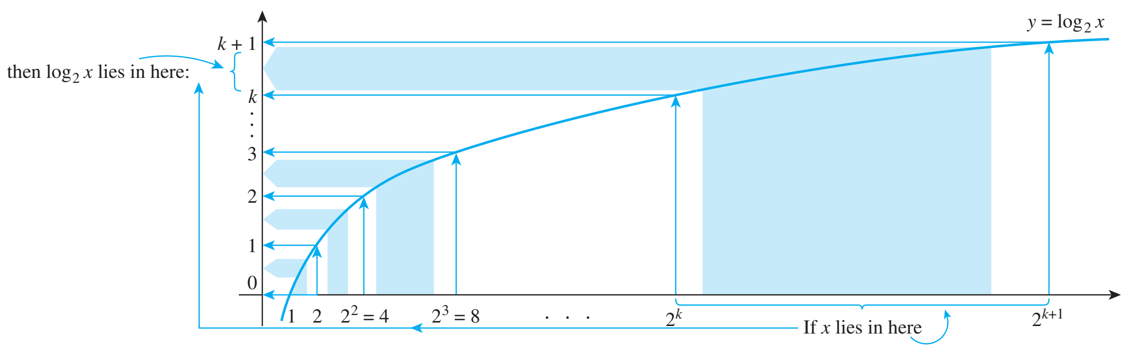
By definition of the floor function, then,

$$\lfloor \log_2 x \rfloor = k.$$

b. Recall that the floor of a positive number is its integer part. For instance, $\lfloor 2.82 \rfloor = 2$. Hence property (11.4.2) can be described in words as follows:

> If x is a positive number that lies between two consecutive integer powers of 2, the floor of the logarithm with base 2 of x is the exponent of the smaller power of 2.

A graphical interpretation follows:



■

One consequence of property (11.4.2) does not appear particularly interesting in its own right but is frequently needed as a step in the analysis of algorithm efficiency.

Example 11.4.2 When $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2 n \rfloor$

Prove the following property:

> For any odd integer $n > 1$, $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2 n \rfloor$. 11.4.3

Solution If n is an odd integer that is greater than 1, then n lies strictly between two successive powers of 2:

$$2^k < n < 2^{k+1} \quad \text{for some integer } k > 0. \qquad 11.4.4$$

It follows that $2^k \le n - 1$ because $2^k < n$ and both 2^k and n are integers. Consequently,

$$2^k \le n - 1 < 2^{k+1}. \qquad 11.4.5$$

Applying property (11.4.2) to both (11.4.4) and (11.4.5) gives

$$\lfloor \log_2 n \rfloor = k \quad \text{and also} \quad \lfloor \log_2(n-1) \rfloor = k.$$

Hence $\lfloor \log_2 n \rfloor = \lfloor \log_2(n-1) \rfloor$. ■

Application: Number of Bits Needed to Represent an Integer in Binary Notation

Given a positive integer n, how many binary digits are needed to represent n? To answer this question, recall from Section 5.4 that any positive integer n can be written in a unique way as

$$n = 2^k + c_{k-1}\cdot 2^{k-1} + \cdots + c_2\cdot 2^2 + c_1\cdot 2 + c_0,$$

where k is a nonnegative integer and each $c_0, c_1, c_2, \ldots c_{k-1}$ is either 0 or 1. Then the binary representation of n is

$$1c_{k-1}c_{k-2}\cdots c_2c_1c_0,$$

and so the number of binary digits needed to represent n is $k+1$.

What is $k+1$ as a function of n? Observe that since each $c_i \le 1$,

$$n = 2^k + c_{k-1}\cdot 2^{k-1} + \cdots + c_2\cdot 2^2 + c_1\cdot 2 + c_0 \le 2^k + 2^{k-1} + \cdots + 2^2 + 2 + 1.$$

But by the formula for the sum of a geometric sequence (Theorem 5.2.3),

$$2^k + 2^{k-1} + \cdots + 2^2 + 2 + 1 = \frac{2^{k+1}-1}{2-1} = 2^{k+1} - 1.$$

Hence, by transitivity of order,

$$n \le 2^{k+1} - 1 < 2^{k+1} \qquad 11.4.6$$

In addition, because each $c_i \ge 0$,

$$2^k \le 2^k + c_{k-1}\cdot 2^{k-1} + \cdots + c_2\cdot 2^2 + c_1\cdot 2 + c_0 = n. \qquad 11.4.7$$

Putting inequalities (11.4.6) and (11.4.7) together gives the double inequality

$$2^k \le n < 2^{k+1}.$$

But then, by property (11.4.2),

$$k = \lfloor \log_2 n \rfloor.$$

Thus the number of binary digits needed to represent n is $\lfloor \log_2 n \rfloor + 1$.

Example 11.4.3 Number of Bits in a Binary Representation

How many binary digits are needed to represent 52,837 in binary notation?

Solution If you compute the logarithm with base 2 using the formula in part (a) of Theorem 7.2.1 and a calculator that gives you approximate values of logarithms with base 10, you find that

$$\log_2(52{,}837) \cong \frac{\log_{10}(52{,}837)}{\log_{10}(2)} \cong \frac{4.722938151}{0.3010299957} \cong 15.7.$$

Thus the binary representation of 52,837 has $\lfloor 15.7 \rfloor + 1 = 15 + 1 = 16$ binary digits. ■

Application: Using Logarithms to Solve Recurrence Relations

In Chapter 5 we discussed methods for solving recurrence relations. One class of recurrence relations that is very important in computer science has solutions that can be expressed in terms of logarithms. One such recurrence relation is discussed in the next example.

Example 11.4.4 A Recurrence Relation with a Logarithmic Solution

Define a sequence $a_1, a_2, a_3, \ldots$ recursively as follows:

$$\begin{aligned} a_1 &= 1, \\ a_k &= 2a_{\lfloor k/2 \rfloor} \quad \text{for all integers } k \geq 2. \end{aligned}$$

a. Use iteration to guess an explicit formula for this sequence.

b. Use strong mathematical induction to confirm the correctness of the formula obtained in part (a).

Solution

a. Begin by iterating to find the values of the first few terms of the sequence.

$$\begin{array}{ll}
a_1 = 1 & 1 = 2^0 \\
\left.\begin{array}{l} a_2 = 2a_{\lfloor 2/2 \rfloor} = 2a_1 = 2 \cdot 1 = 2 \\ a_3 = 2a_{\lfloor 3/2 \rfloor} = 2a_1 = 2 \cdot 1 = 2 \end{array}\right\} & 2 = 2^1 \\
\left.\begin{array}{l} a_4 = 2a_{\lfloor 4/2 \rfloor} = 2a_2 = 2 \cdot 2 = 4 \\ a_5 = 2a_{\lfloor 5/2 \rfloor} = 2a_2 = 2 \cdot 2 = 4 \\ a_6 = 2a_{\lfloor 6/2 \rfloor} = 2a_3 = 2 \cdot 2 = 4 \\ a_7 = 2a_{\lfloor 7/2 \rfloor} = 2a_3 = 2 \cdot 2 = 4 \end{array}\right\} & 4 = 2^2 \\
\left.\begin{array}{l} a_8 = 2a_{\lfloor 8/2 \rfloor} = 2a_4 = 2 \cdot 4 = 8 \\ a_9 = 2a_{\lfloor 9/2 \rfloor} = 2a_4 = 2 \cdot 4 = 8 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{15} = 2a_{\lfloor 15/2 \rfloor} = 2a_7 = 2 \cdot 4 = 8 \end{array}\right\} & 8 = 2^3 \\
\left.\begin{array}{l} a_{16} = 2a_{\lfloor 16/2 \rfloor} = 2a_8 = 2 \cdot 8 = 16 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \end{array}\right| & 16 = 2^4
\end{array}$$

Note that in each case when the subscript n is between two powers of 2, a_n equals the smaller power of 2. More precisely:

$$\text{If } 2^i \leq n < 2^{i+1}, \text{ then } a_n = 2^i. \tag{11.4.8}$$

But since n satisfies the inequality

$$2^i \leq n < 2^{i+1},$$

then (by property 11.4.2)

$$i = \lfloor \log_2 n \rfloor.$$

Substituting into statement (11.4.8) gives

$$a_n = 2^{\lfloor \log_2 n \rfloor}.$$

b. The following proof shows that if $a_1, a_2, a_3, \ldots$ is a sequence of numbers that satisfies

$$a_1 = 1, \quad \text{and} \quad a_k = 2a_{\lfloor k/2 \rfloor} \quad \text{for all integers } k \geq 2,$$

then the sequence satisfies the formula

$$a_n = 2^{\lfloor \log_2 n \rfloor} \quad \text{for all integers } n \geq 1.$$

Proof:

Let $a_1, a_2, a_3, \ldots$ be the sequence defined by specifying that $a_1 = 1$ and $a_k = 2_{\lfloor a_{k/2} \rfloor}$ for all integers $k \geq 2$, and let the property $P(n)$ be the equation

$$a_n = 2^{\lfloor \log_2 n \rfloor}. \qquad \leftarrow P(n)$$

We will use strong mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.

Show that P (1) is true: By definition of $a_1, a_2, a_3, \ldots$, we have that $a_1 = 1$. But it is also the case that $2^{\lfloor \log_2 1 \rfloor} = 2^0 = 1$. Thus $a_1 = 2^{\lfloor \log_2 1 \rfloor}$ and $P(1)$ is true.

Show that for all integers k ≥ 1, if P(i) is true for all integers i from 1 through k, then P(k + 1) is also true: Let k be any integer with $k \geq 1$, and suppose that

$$a_i = 2^{\lfloor \log_2 i \rfloor} \text{ for all integers } i \text{ with } 1 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$a_{k+1} = 2^{\lfloor \log_2 (k+1) \rfloor} \qquad \leftarrow P(k+1)$$

Consider the two cases: k is even and k is odd.

Case 1 (k is even): In this case, $k + 1$ is odd, and

$$\begin{aligned}
a_{k+1} &= 2a_{\lfloor (k+1)/2 \rfloor} && \text{by definition of } a_1, a_2, a_3, \ldots \\
&= 2a_{k/2} && \text{because } \lfloor (k+1)/2 \rfloor = k/2 \text{ since } k+1 \text{ is odd} \\
&= 2 \cdot 2^{\lfloor \log_2 (k/2) \rfloor} && \text{by inductive hypothesis because, since } k \text{ is even,} \\
& && k \geq 2, \text{ and so } k/2 \geq 1 \\
&= 2^{\lfloor \log_2 (k/2) \rfloor + 1} && \text{by the laws of exponents from algebra (7.2.1)} \\
&= 2^{\lfloor \log_2 k - \log_2 2 \rfloor + 1} && \text{by the identity } \log_b (x/y) = \log_b x - \log_b y \\
& && \text{from Theorem 7.2.1} \\
&= 2^{\lfloor \log_2 k - 1 \rfloor + 1} && \text{since } \log_2 2 = 1 \\
&= 2^{\lfloor \log_2 k \rfloor - 1 + 1} && \text{by substituting } x = \log_2 k \text{ into the identity} \\
& && \lfloor x - 1 \rfloor = \lfloor x \rfloor - 1 \text{ derived in exercise 15 of Section 4.5} \\
&= 2^{\lfloor \log_2 k \rfloor} \\
&= 2^{\lfloor \log_2 (k+1) \rfloor} && \text{by property (11.4.3)}
\end{aligned}$$

Case 2 (k is odd): The analysis of this case is very similar to that of case 1 and is left as exercise 56 at the end of the section.

Thus in either case, $a_n = 2^{\lfloor \log_2 (k+1) \rfloor}$, as was to be shown. ■

Exponential and Logarithmic Orders

Now consider the question "How do graphs of logarithmic and exponential functions compare with graphs of power functions?" It turns out that for large enough values of x, the graph of the logarithmic function with any base $b > 1$ lies *below* the graph of any positive power function, and the graph of the exponential function with any base $b > 1$ lies *above* the graph of any positive power function. In analytic terms, this says the following:

For all real numbers b and r with $b > 1$ and $r > 0$,

$$\log_b x \leq x^r \quad \text{for all sufficiently large real numbers } x. \qquad 11.4.9$$

and

$$x^r \leq b^x \quad \text{for all sufficiently large real numbers } x. \qquad 11.4.10$$

These statements have the following implications for O-notation.

For all real numbers b and r with $b > 1$ and $r > 0$,

$$\log_b x \quad \text{is} \quad O(x^r) \qquad \text{11.4.11}$$

and

$$x^r \quad \text{is} \quad O(b^x) \qquad \text{11.4.12}$$

Another important function in the analysis of algorithms is the function f defined by the formula

$$f(x) = x \log_b x \quad \text{for all real numbers } x > 0.$$

For large values of x, the graph of this function fits in between the graph of the identity function and the graph of the squaring function. More precisely:

For all real numbers b with $b > 1$ and for all sufficiently large real numbers x,

$$x \le x \log_b x \le x^2. \qquad \text{11.4.13}$$

The O-notation versions of these facts are as follows:

For all real numbers $b > 1$,

$$x \quad \text{is} \quad O(x \log_b x) \quad \text{and} \quad x \log_b x \quad \text{is} \quad O(x^2). \qquad \text{11.4.14}$$

Although proofs of some of these facts require calculus, proofs of some cases can be obtained using the algebra of inequalities. (See the exercises at the end of this section.) Figure 11.4.4 illustrates the relationships among some power functions, the logarithmic function with base 2, the exponential function with base 2, and the function defined by the formula $x \to x \log_2 x$. Note that different scales are used on the horizontal and vertical axes.

Example 11.4.5 shows how to use inequalities such as (11.4.9), (11.4.10), and (11.4.13) to derive additional orders involving the logarithmic function.

Example 11.4.5 Deriving an Order from Logarithmic Inequalities

Show that $x + x \log_2 x$ is $\Theta(x \log_2 x)$.

Solution First observe that $x + x \log_2 x$ is $\Omega(x \log_2 x)$ because for all real numbers $x > 1$,

$$x \log_2 x \le x + x \log_2 x,$$

and since all quantities are positive,

$$|x \log_2 x| \le |x + x \log_2 x|.$$

Let $A = 1$ and $a = 1$. Then

$$A|x \log_2 x| \le |x + x \log_2 x| \quad \text{for all } x > a.$$

Hence, by definition of Ω-notation,

$$x + x \log_2 x \quad \text{is} \quad \Omega(x \log_2 x).$$

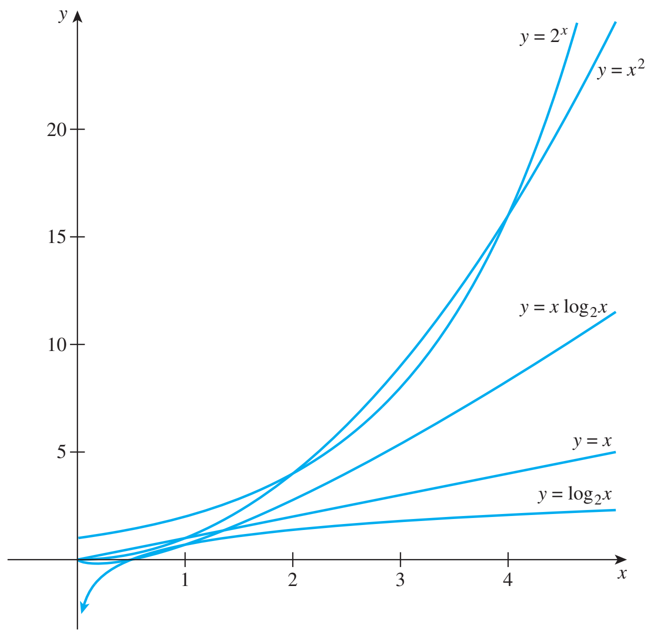


Figure 11.4.4 **Graphs of Some Logarithmic, Exponential, and Power Functions**

To show that $x + x\log_2 x$ is $O(x\log_2 x)$, note that according to property (11.4.13) with $b = 2$, there is a number b such that for all $x > b$,

$$x < x\log_2 x$$

$$\Rightarrow \quad x + x\log_2 x < 2x\log_2 x \qquad \text{by adding } x\log_2 x \text{ to both sides}$$

Thus, if b is taken to be greater than 2, then

$$|x + x\log_2 x| \; < \; 2|x\log_2 x|$$

because when $x > 2$, $x\log_2 x > 0$, and so $|x + x\log_2 x| = x + x\log_2 x$ and $\log_2 x = |x\log_2 x|$.

Let $B = 2$. Then

$$|x + x\log_2 x| \le B|x\log_2 x| \quad \text{for all } x > b.$$

Hence, by definition of O-notation

$$x + x\log_2 x \quad \text{is} \quad O(x\log_2 x).$$

Therefore, since $x + x\log_2 x$ is $\Omega(x\log_2 x)$ and $x + x\log_2 x$ is $O(x\log_2 x)$, by Theorem 11.2.1,

$$x + x\log_2 x \quad \text{is} \quad \Theta(x\log_2 x).$$

■

Example 11.4.5 illustrates a special case of a useful general fact about O-notation: *If one function "dominates" another (in the sense of being larger for large values of the variable), then the sum of the two is big-O of the dominating function.* (See exercise 49a in Section 11.2.)

Example 11.4.6 shows that any two logarithmic functions with bases greater than 1 have the same order.

Example 11.4.6 Logarithm with Base b Is Big-Theta of Logarithm with Base c

Show that if b and c are real numbers such that $b > 1$ and $c > 1$, then $\log_b x$ is $\Theta(\log_c x)$.

Solution Suppose b and c are real numbers and $b > 1$ and $c > 1$. To show that $\log_b x$ is $\Theta(\log_c x)$, positive real numbers A, B, and k must be found such that

$$A|\log_c x| \le |\log_b x| \le B|\log_c x| \quad \text{for all real numbers } x > k.$$

By part (d) of Theorem 7.2.1,

$$\log_b x = \frac{\log_c x}{\log_c b} = \left(\frac{1}{\log_c b}\right)\log_c x. \qquad (*)$$

Since $b > 1$ and the logarithmic function with base c is strictly increasing, then $\log_c b > \log_c 1 = 0$, and so $\frac{1}{\log_c b} > 0$ also. Furthermore, if $x > 1$, then $\log_b x > 0$ and $\log_c x > 0$. It follows from equation $(*)$, therefore, that

$$\left(\frac{1}{\log_c b}\right)\log_c x \le \log_b x \le \left(\frac{1}{\log_c b}\right)\log_c x \qquad (**)$$

for all real numbers $x > 1$. Accordingly, let $A = \frac{1}{\log_c b}$, $B = \frac{1}{\log_c b}$, and $k = 1$. Then, since all quantities in $(**)$ are positive,

$$A|\log_c x| \le |\log_b x| \le B|\log_c x| \quad \text{for all real numbers } x > k.$$

Hence, by definition of Θ-notation,

$$\log_b x \quad \text{is} \quad \Theta(\log_c x).$$ ■

Example 11.4.7 shows how a logarithmic order can arise from the computation of a certain kind of sum. It requires the following fact from calculus:

The area underneath the graph of $y = 1/x$ between $x = 1$ and $x = n$ equals $\ln n$, where $\ln n = \log_e n$. This fact is illustrated in Figure 11.4.5.

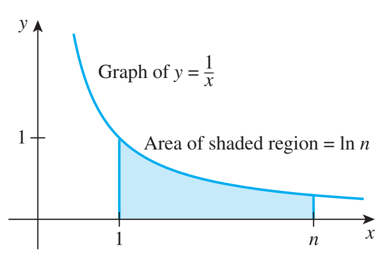


Figure 11.4.5 **Area Under Graph of $y = \frac{1}{x}$ Between $x = 1$ and $x = n$**

Example 11.4.7 Order of a Harmonic Sum

Sums of the form $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ are called *harmonic sums*. They occur in the analysis of various computer algorithms such as quick sort. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is $\Omega(\ln n)$ by performing the steps on the next page:

a. Interpret Figure 11.4.6 to show that

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \;\le\; \ln n.$$

and

$$\ln n \;\le\; 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

b. Show that if n is an integer that is at least 3, then $1 \le \ln n$.

c. Deduce from (a) and (b) that if the integer n is greater than or equal to 3, then

$$\ln n \;\le\; 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \;\le\; 2\ln n.$$

d. Deduce from (c) that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad\text{is}\quad \Theta(\ln n).$$

Solution

a. Figure 11.4.6(a) shows rectangles whose bases are the intervals between each pair of integers from 1 to n and whose heights are the heights of the graph of $y = 1/x$ above the right-hand endpoints of the intervals. Figure 11.4.6(b) shows rectangles with the same bases but whose heights are the heights of the graph above the left-hand endpoints of the intervals.

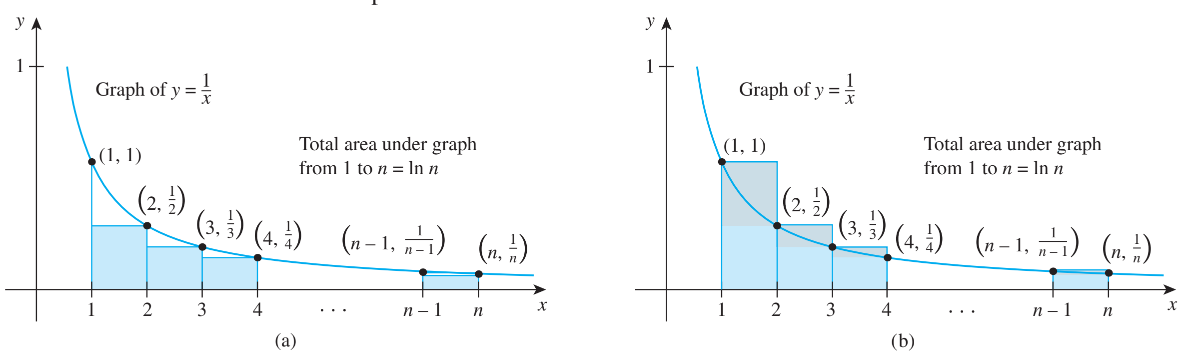


Figure 11.4.6

Now the area of each rectangle is its base times its height. Since all the rectangles have base 1, the area of each rectangle equals its height. Thus in Figure 11.4.6(a),

$$\text{the area of the rectangle from 1 to 2 is } \frac{1}{2};$$

$$\text{the area of the rectangle from 2 to 3 is } \frac{1}{3};$$

$$\vdots$$

$$\text{the area of the rectangle from } n-1 \text{ to } n \text{ is } \frac{1}{n}.$$

So the sum of the areas of all the rectangles is $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. From the picture it is clear that this sum is less than the area underneath the graph of f between $x = 1$ and $x = n$, which is known to equal $\ln n$. Hence

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le \ln n.$$

A similar analysis of the areas of the combined blue and gray rectangles in Figure 11.4.6(b) shows that

$$\ln n \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

b. Suppose n is an integer and $n \ge 3$. Since $e \cong 2.718$, then $n \ge e$. Now the logarithmic function with base e is strictly increasing. Thus since $e \le n$, then $1 = \ln e \le \ln n$.

c. By part (a),

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le \ln n,$$

and by part (b),

$$1 \le \ln n.$$

Adding these two inequalities together gives

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le 2\ln n \quad \text{for any integer } n \ge 3.$$

d. Putting together the results of parts (a) and (c) leads to the conclusion that for all integers $n \ge 3$,

$$\ln n \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le 2\ln n.$$

And because all the quantities are positive for $n \ge 3$,

$$|\ln n| \le \left|1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right| \le 2|\ln n|.$$

Let $A = 1$, $B = 2$, and $k = 3$. Then

$$A|\ln n| \le \left|1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right| \le B|\ln n| \quad \text{for all } n > k.$$

Hence by definition of Θ-notation,

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \text{ is } \Theta(\ln n).$$ ■

Test Yourself

1. The domain of any exponential function is _____, and its range is _____.
2. The domain of any logarithmic function is _____, and its range is _____.
3. If k is an integer and $2^k \le x < 2^{k+1}$, then $\lfloor \log_2 x \rfloor =$ _____.
4. If b is a real number with $b > 1$ and if x is a sufficiently large real number, then when the quantities x, x^2, $\log_b x$, and $x \log_b x$ are arranged in order of increasing size, the result is _____.
5. If n is a positive integer, then $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ has order _____.

Exercise Set 11.4

Graph each function defined in 1–8.

1. $f(x) = 3^x$ for all real numbers x

2. $g(x) = \left(\frac{1}{3}\right)^x$ for all real numbers x

3. $h(x) = \log_{10} x$ for all positive real numbers x

4. $k(x) = \log_2 x$ for all positive real numbers x

5. $F(x) = \lfloor \log_2 x \rfloor$ for all positive real numbers x

6. $G(x) = \lceil \log_2 x \rceil$ for all positive real numbers x

7. $H(x) = x \log_2 x$ for all positive real numbers x

8. $K(x) = x \log_{10} x$ for all positive real numbers x

9. The scale of the graph shown in Figure 11.4.1 is one-fourth inch to each unit. If the point $(2, 2^{64})$ is plotted on the graph of $y = 2^x$, how many miles will it lie above the horizontal axis? What is the ratio of the height of the point to the distance of the earth from the sun? (There are 12 inches per foot and 5,280 feet per mile. The earth is approximately 93,000,000 miles from the sun on average.)
($\frac{1}{4}$ inch $\cong$ 0.635 cm, 1 mile $\cong$ 0.62 km)

10. a. Use the definition of logarithm to show that $\log_b b^x = x$ for all real numbers x.
b. Use the definition of logarithm to show that $b^{\log_b x} = x$ for all positive real numbers x.
c. By the result of exercise 25 in Section 7.3, if $f: X \to Y$ and $g: Y \to X$ are functions and $g \circ f = I_X$ and $f \circ g = I_Y$, then f and g are inverse functions. Use this result to show that $\log_b$ and $\exp_b$ (the exponential function with base b) are inverse functions.

11. Let $b > 1$.
a. Use the fact that $u = \log_b v \Leftrightarrow v = b^u$ to show that a point (u, v) lies on the graph of the logarithmic function with base b if, and only if, (v, u) lies on the graph of the exponential function with base b.
b. Plot several pairs of points of the form (u, v) and (v, u) on a coordinate system. Describe the geometric relationship between the locations of the points in each pair.
c. Draw the graphs of $y = \log_2 x$ and $y = 2^x$. Describe the geometric relationship between these graphs.

12. Give a graphical interpretation for property (11.4.2) in Example 11.4.1(a) for $0 < x < 1$.

H **13.** Suppose a positive real number x satisfies the inequality $10^m \le x < 10^{m+1}$ where m is an integer. What can be inferred about $\lfloor \log_{10} x \rfloor$? Justify your answer.

14. a. Prove that if x is a positive real number and k is a nonnegative integer such that $2^{k-1} < x \le 2^k$, then $\lceil \log_2 x \rceil = k$.
b. Describe in words the statement proved in part (a).

15. If n is an odd integer and $n > 1$, is $\lceil \log_2(n-1) \rceil = \lceil \log_2(n) \rceil$? Justify your answer.

H **16.** If n is an odd integer and $n > 1$, is $\lceil \log_2(n+1) \rceil = \lceil \log_2(n) \rceil$? Justify your answer.

17. If n is an odd integer and $n > 1$, is $\lfloor \log_2(n+1) \rfloor = \lfloor \log_2(n) \rfloor$? Justify your answer.

In 18 and 19, indicate how many binary digits are needed to represent the numbers in binary notation. Use the method shown in Example 11.4.3.

18. 148,206 19. 5,067,329

20. It was shown in the text that the number of binary digits needed to represent a positive integer n is $\lfloor \log_2 n \rfloor + 1$. Can this also be given as $\lceil \log_2 n \rceil$? Why or why not?

In each of 21 and 22, a sequence is specified by a recurrence relation and initial conditions. In each case, (a) use iteration to guess an explicit formula for the sequence; (b) use strong mathematical induction to confirm the correctness of the formula you obtained in part (a).

21. $a_k = a_{\lfloor k/2 \rfloor} + 2$, for all integers $k \ge 2$
$a_1 = 1$

22. $b_k = b_{\lceil k/2 \rceil} + 1$, for all integers $k \ge 2$
$b_1 = 1$.

H **23.** Define a sequence $c_1, c_2, c_3, \ldots$, recursively as follows:

$$c_1 = 0,$$
$$c_k = 2c_{\lfloor k/2 \rfloor} + k, \quad \text{for all integers } k \ge 2.$$

Use strong mathematical induction to show that $c_n \le n^2$ for all integers $n \ge 1$.

✶*H* **24.** Use strong mathematical induction to show that for the sequence of exercise 23, $c_n \le n \log_2 n$, for all integers $n \ge 4$.

Exercises 25–28 refer to properties 11.4.9 and 11.4.10. To solve them, think big!

25. Find a real number $x > 3$ such that $\log_2 x < x^{1/10}$.

26. Find a real number $x > 1$ such that $x^{50} < 2^x$.

27. Find a real number $x > 2$ such that $x < 1.0001^x$.

28. Use a graphing calculator or computer graphing program to find two distinct approximate values of x such that $x = 1.0001^x$. On what approximate intervals is $x > 1.0001^x$? On what approximate intervals is $x < 1.0001^x$?

29. Use Θ-notation to express the following statement:

$$|x^2| \le |7x^2 + 3x \log_2 x| \le 10|x^2|,$$

for all real numbers $x > 2$.

Derive each statement in 30–33.

30. $2x + \log_2 x$ is $\Theta(x)$.

31. $x^2 + 5x \log_2 x$ is $\Theta(x^2)$.

32. $n^2 + 2^n$ is $\Theta(2^n)$.

H **33.** 2^{n+1} is $\Theta(2^n)$.

H **34.** Show that 4^n is not $O(2^n)$.

Prove each of the statements in 35–40, assuming n is an integer variable that takes positive integer values. Use identities from Section 5.2 as needed.

35. $1 + 2 + 2^2 + 2^3 + \cdots + 2^n$ is $\Theta(2^n)$.

H **36.** $4 + 4^2 + 4^3 + \cdots + 4^n$ is $\Theta(4^n)$.

37. $2 + 2 \cdot 3^2 + 2 \cdot 3^4 + \cdots + 2 \cdot 3^{2n}$ is $\Theta(3^{2n})$.

38. $\frac{1}{5} + \frac{4}{5^2} + \frac{4^2}{5^3} + \cdots + \frac{4^n}{5^{n+1}}$ is $\Theta(1)$.

39. $n + \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^n}$ is $\Theta(n)$.

40. $\frac{2n}{3} + \frac{2n}{3^2} + \frac{2n}{3^3} + \cdots + \frac{2n}{3^n}$ is $\Theta(n)$.

41. Quantities of the form

$$kn + kn \log_2 n \quad \text{for positive integers } k_1 \cdot k_2, \text{ and } n$$

arise in the analysis of the merge sort algorithm in computer science. Show that for any positive integer k,

$$k_1 n + k_2 n \log_2 n \quad \text{is} \quad \Theta(n \log_2 n).$$

42. Calculate the values of the harmonic sums

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{for } n = 2, 3, 4, \text{ and } 5.$$

43. Use part (d) of Example 11.4.7 to show that

$$n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n} \quad \text{is} \quad \Theta(n \ln n).$$

44. Use the fact that $\log_2 x = \left(\frac{1}{\log_e 2}\right) \log_e x$ and $\log_e x = \ln x$, for all positive numbers x, and part (c) of Example 11.4.7 to show that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{is} \quad \Theta(\log_2 n).$$

45. a. Show that $\lfloor \log_2 n \rfloor$ is $\Theta(\log_2 n)$.
 b. Show that $\lfloor \log_2 n \rfloor + 1$ is $\Theta(\log_2 n)$.

46. Prove by mathematical induction that $n \le 10^n$ for all integers $n \ge 1$.

H **47.** Prove by mathematical induction that $\log_2 n \le n$ for all integers $n \ge 1$.

H **48.** Show that if n is a variable that takes positive integer values, then 2^n is $O(n!)$.

49. Let n be a variable that takes positive integer values.
 a. Show that $n!$ is $O(n^n)$.
 b. Use part (a) to show that $\log_2(n!)$ is $O(n \log_2 n)$.
 H **c.** Show that $n^n \le (n!)^2$ for all integers $n \ge 2$.
 d. Use part (c) to show that $\log_2(n!)$ is $\Omega(n \log_2 n)$.
 e. Use parts (b) and (d) to find an order for $\log_2(n!)$.

✶ **50.** a. For all positive real numbers u, $\log_2 u < u$. Use this fact to show that for any positive integer n, $\log_2 x < n x^{1/n}$ for all real numbers $x > 0$.
 b. Interpret the statement of part (a) using O-notation.

51. a. For all real numbers x, $x < 2^x$. Use this fact to show that for any positive integer n, $x^n < n^n 2^x$ for all real numbers $x > 0$.
 b. Interpret the statement of part (a) using O-notation.

✶ **52.** For all positive real numbers u, $\log_2 u < u$. Use this fact and the result of exercise 21 in Section 11.1 to prove the following: For all integers $n \ge 1$, $\log_2 x < x^{1/n}$ for all real numbers $x > (2n)^{2n}$.

53. Use the result of exercise 52 above to prove the following: For all integers $n \ge 1$, $x^n < 2^x$ for all real numbers $x > (2n)^{2n}$.

Exercises 54 and 55 use L'Hôpital's rule from calculus.

54. a. Let b be any real number greater than 1. Use L'Hôpital's rule and mathematical induction to prove that for all integers $n \ge 1$,

$$\lim_{x \to \infty} \frac{x^n}{b^x} = 0.$$

 b. Use the result of part (a) and the definitions of limit and of O-notation to prove that x^n is $O(b^x)$ for any integer $n \ge 1$.

55. a. Let b be any real number greater than 1. Use L'Hôpital's rule to prove that for all integers $n \ge 1$,

$$\lim_{x \to \infty} \frac{\log_b x}{x^{1/n}} = 0.$$

 b. Use the result of part (a) and the definitions of limit and of O-notation to prove that $\log_b x$ is $O(x^{1/n})$ for any integer $n \ge 1$.

56. Complete the proof in Example 11.4.4.

Answers for Test Yourself

1. the set of all real numbers; the set of all positive real numbers 2. the set of all positive real numbers; the set of all real numbers 3. k 4. $\log_b x < x < x \log_b x < x^2$ 5. $\ln x$ (or, equivalently, $\log_2 x$)

11.5 Application: Analysis of Algorithm Efficiency II

Pick a Number, Any Number — Donal O'Shea, 2007

Have you ever played the "guess my number" game? A person thinks of a number between two other numbers, say 1 and 10 or 1 and 100 for example, and you try to figure out what it is, using the least possible number of guesses. Each time you guess a number, the person tells you whether you are correct, too low, or too high.