37. "**Proof:** Suppose r and s are rational numbers. By definition of rational, $r = a/b$ for some integers a and b with $b \neq 0$, and $s = a/b$ for some integers a and b with $b \neq 0$. Then

$$r + s = \frac{a}{b} + \frac{a}{b} = \frac{2a}{b}.$$

Let $p = 2a$. Then p is an integer since it is a product of integers. Hence $r + s = p/b$, where p and b are integers and $b \neq 0$. Thus $r + s$ is a rational number by definition of rational. This is what was to be shown."

38. "**Proof:** Suppose r and s are rational numbers. Then $r = a/b$ and $s = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ (by definition of rational). Then

$$r + s = \frac{a}{b} + \frac{c}{d}.$$

But this is a sum of two fractions, which is a fraction. So $r + s$ is a rational number since a rational number is a fraction."

39. "**Proof:** Suppose r and s are rational numbers. If $r + s$ is rational, then by definition of rational $r + s = a/b$ for some integers a and b with $b \neq 0$. Also since r and s are rational, $r = i/j$ and $s = m/n$ for some integers i, j, m, and n with $j \neq 0$ and $n \neq 0$. It follows that

$$r + s = \frac{i}{j} + \frac{m}{n} = \frac{a}{b},$$

which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown."

Answers for Test Yourself

1. a ratio of integers with a nonzero denominator 2. real number; not rational 3. $0 = \frac{0}{1}$

4.3 Direct Proof and Counterexample III: Divisibility

The essential quality of a proof is to compel belief. — Pierre de Fermat

When you were first introduced to the concept of division in elementary school, you were probably taught that 12 divided by 3 is 4 because if you separate 12 objects into groups of 3, you get 4 groups with nothing left over.

xxx	xxx	xxx	xxx

You may also have been taught to describe this fact by saying that "12 is evenly divisible by 3" or "3 divides 12 evenly."

The notion of divisibility is the central concept of one of the most beautiful subjects in advanced mathematics: **number theory,** the study of properties of integers.

• Definition

If n and d are integers and $d \neq 0$ then

n is **divisible by** d if, and only if, n equals d times some integer.

Instead of "n is divisible by d," we can say that

n **is a multiple of** d, or
d **is a factor of** n, or
d **is a divisor of** n, or
d **divides** n.

The notation $\mathbf{d \mid n}$ is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$:

$$d \mid n \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = dk.$$

Example 4.3.1 Divisibility

a. Is 21 divisible by 3? b. Does 5 divide 40? c. Does $7 \mid 42$?

d. Is 32 a multiple of -16? e. Is 6 a factor of 54? f. Is 7 a factor of -7?

Solution

a. Yes, $21 = 3 \cdot 7$. b. Yes, $40 = 5 \cdot 8$. c. Yes, $42 = 7 \cdot 6$.

d. Yes, $32 = (-16) \cdot (-2)$. e. Yes, $54 = 6 \cdot 9$. f. Yes, $-7 = 7 \cdot (-1)$. ■

Example 4.3.2 Divisors of Zero

If k is any nonzero integer, does k divide 0?

Solution Yes, because $0 = k \cdot 0$. ■

Two useful properties of divisibility are (1) that if one positive integer divides a second positive integer, then the first is less than or equal to the second, and (2) that the only divisors of 1 are 1 and -1.

Theorem 4.3.1 A Positive Divisor of a Positive Integer

For all integers a and b, if a and b are positive and a divides b, then $a \leq b$.

Proof:

Suppose a and b are positive integers and a divides b. *[We must show that $a \leq b$.]* Then there exists an integer k so that $b = ak$. By property T25 of Appendix A, k must be positive because both a and b are positive. It follows that

$$1 \leq k$$

because every positive integer is greater than or equal to 1. Multiplying both sides by a gives

$$a \leq ka = b$$

because multiplying both sides of an inequality by a positive number preserves the inequality by property T20 of Appendix A. Thus $a \leq b$ *[as was to be shown]*.

■

Theorem 4.3.2 Divisors of 1

The only divisors of 1 are 1 and -1.

Proof:

Since $1 \cdot 1 = 1$ and $(-1)(-1) = 1$, both 1 and -1 are divisors of 1. Now suppose m is any integer that divides 1. Then there exists an integer n such that $1 = mn$. By Theorem T25 in Appendix A, either both m and n are positive or both m and n are negative. If both m and n are positive, then m is a positive integer divisor of 1. By Theorem 4.3.1, $m \leq 1$, and, since the only positive integer that is less than or equal

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to 1 is 1 itself, it follows that $m = 1$. On the other hand, if both m and n are negative, then, by Theorem T12 in Appendix A, $(-m)(-n) = mn = 1$. In this case $-m$ is a positive integer divisor of 1, and so, by the same reasoning, $-m = 1$ and thus $m = -1$. Therefore there are only two possibilities: either $m = 1$ or $m = -1$. So the only divisors of 1 are 1 and -1.

Example 4.3.3 Divisibility of Algebraic Expressions

a. If a and b are integers, is $3a + 3b$ divisible by 3?

b. If k and m are integers, is $10km$ divisible by 5?

Solution

a. Yes. By the distributive law of algebra, $3a + 3b = 3(a + b)$ and $a + b$ is an integer because it is a sum of two integers.

b. Yes. By the associative law of algebra, $10km = 5 \cdot (2km)$ and $2km$ is an integer because it is a product of three integers. ■

When the definition of divides is rewritten formally using the existential quantifier, the result is

$$d \mid n \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = dk.$$

Since the negation of an existential statement is universal, it follows that d does not divide n (denoted $d \nmid n$) if, and only if, $\forall$ integers k, $n \neq dk$, or, in other words, the quotient n/d is not an integer.

$$\text{For all integers } n \text{ and } d, \quad d \nmid n \quad \Leftrightarrow \quad \frac{n}{d} \text{ is not an integer.}$$

Example 4.3.4 Checking Nondivisibility

Does $4 \mid 15$?

Solution No, $\frac{15}{4} = 3.75$, which is not an integer. ■

Caution!
$a \mid b$ denotes the *sentence* "a divides b," whereas a/b denotes the *number* a divided by b.

Be careful to distinguish between the notation $a \mid b$ and the notation a/b. The notation $a \mid b$ stands for the sentence "a divides b," which means that there is an integer k such that $b = ak$. Dividing both sides by a gives $b/a = k$, an integer. Thus, when $a \neq 0$, $a \mid b$ if, and only if, b/a is an integer. On the other hand, the notation a/b stands for the number a/b which is the result of dividing a by b and which may or may not be an integer. In particular, be sure to avoid writing things like

$$\cancel{4 \mid (3 + 5) = 4 \mid 8}.$$

If read out loud, this becomes, "4 divides the quantity 3 plus 5 equals 4 divides 8," which is nonsense.

Example 4.3.5 Prime Numbers and Divisibility

An alternative way to define a prime number is to say that an integer $n > 1$ is prime if, and only if, its only positive integer divisors are 1 and itself. ■

Proving Properties of Divisibility

One of the most useful properties of divisibility is that it is transitive. If one number divides a second and the second number divides a third, then the first number divides the third.

Example 4.3.6 Transitivity of Divisibility

Prove that for all integers a, b, and c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

Solution Since the statement to be proved is already written formally, you can immediately pick out the starting point, or first sentence of the proof, and the conclusion that must be shown.

Starting Point: Suppose a, b, and c are particular but arbitrarily chosen integers such that $a \mid b$ and $b \mid c$.

To Show: $a \mid c$.

You need to show that $a \mid c$, or, in other words, that

$$c = a \cdot (\text{some integer}).$$

But since $a \mid b$,

$$b = ar \quad \text{for some integer } r. \qquad 4.3.1$$

And since $b \mid c$,

$$c = bs \quad \text{for some integer } s. \qquad 4.3.2$$

Equation 4.3.2 expresses c in terms of b, and equation 4.3.1 expresses b in terms of a. Thus if you substitute 4.3.1 into 4.3.2, you will have an equation that expresses c in terms of a.

$$\begin{aligned} c &= bs && \text{by equation 4.3.2} \\ &= (ar)s && \text{by equation 4.3.1.} \end{aligned}$$

But $(ar)s = a(rs)$ by the associative law for multiplication. Hence

$$c = a(rs).$$

Now you are almost finished. You have expressed c as $a \cdot (\text{something})$. It remains only to verify that that something is an integer. But of course it is, because it is a product of two integers.

This discussion is summarized as follows:

Theorem 4.3.3 Transitivity of Divisibility

For all integers a, b, and c, if a divides b and b divides c, then a divides c.

Proof:

Suppose a, b, and c are *[particular but arbitrarily chosen]* integers such that a divides b and b divides c. *[We must show that a divides c.]* By definition of divisibility,

$$b = ar \quad \text{and} \quad c = bs \quad \text{for some integers } r \text{ and } s.$$

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By substitution

$$c = bs = (ar)s = a(rs) \quad \text{by basic algebra.}$$

Let $k = rs$. Then k is an integer since it is a product of integers, and therefore

$$c = ak \quad \text{where } k \text{ is an integer.}$$

Thus a divides c by definition of divisibility. *[This is what was to be shown.]*

■

It would appear from the definition of prime that to show that an integer is prime you would need to show that it is not divisible by any integer greater than 1 and less than itself. In fact, you need only check whether it is divisible by a prime number less than or equal to itself. This follows from Theorems 4.3.1, 4.3.3, and the following theorem, which says that any integer greater than 1 is divisible by a prime number. The idea of the proof is quite simple. You start with a positive integer. If it is prime, you are done; if not, it is a product of two smaller positive factors. If one of these is prime, you are done; if not, you can pick one of the factors and write it as a product of still smaller positive factors. You can continue in this way, factoring the factors of the number you started with, until one of them turns out to be prime. This must happen eventually because all the factors can be chosen to be positive and each is smaller than the preceding one.

Theorem 4.3.4 Divisibility by a Prime

Any integer $n > 1$ is divisible by a prime number.

Proof:

Suppose n is a *[particular but arbitrarily chosen]* integer that is greater than 1. *[We must show that there is a prime number that divides n.]* If n is prime, then n is divisible by a prime number (namely itself), and we are done. If n is not prime, then, as discussed in Example 4.1.2b,

$$n = r_0 s_0 \quad \text{where } r_0 \text{ and } s_0 \text{ are integers and } 1 < r_0 < n \text{ and } 1 < s_0 < n.$$

It follows by definition of divisibility that $r_0 \mid n$.

If r_0 is prime, then r_0 is a prime number that divides n, and we are done. If r_0 is not prime, then

$$r_0 = r_1 s_1 \quad \text{where } r_1 \text{ and } s_1 \text{ are integers and } 1 < r_1 < r_0 \text{ and } 1 < s_1 < r_0.$$

It follows by the definition of divisibility that $r_1 \mid r_0$. But we already know that $r_0 \mid n$. Consequently, by transitivity of divisibility, $r_1 \mid n$.

If r_1 is prime, then r_1 is a prime number that divides n, and we are done. If r_1 is not prime, then

$$r_1 = r_2 s_2 \quad \text{where } r_2 \text{ and } s_2 \text{ are integers and } 1 < r_2 < r_1 \text{ and } 1 < s_2 < r_1.$$

It follows by definition of divisibility that $r_2 \mid r_1$. But we already know that $r_1 \mid n$. Consequently, by transitivity of divisibility, $r_2 \mid n$.

If r_2 is prime, then r_2 is a prime number that divides n, and we are done. If r_2 is not prime, then we may repeat the previous process by factoring r_2 as $r_3 s_3$.

We may continue in this way, factoring successive factors of n until we find a prime factor. We must succeed in a finite number of steps because each new factor is both less than the previous one (which is less than n) and greater than 1, and there are fewer than n integers strictly between 1 and n.* Thus we obtain a sequence

$$r_0, r_1, r_2, \ldots, r_k,$$

where $k \geq 0$, $1 < r_k < r_{k-1} < \cdots < r_2 < r_1 < r_0 < n$, and $r_i \mid n$ for each $i = 0, 1, 2, \ldots, k$. The condition for termination is that r_k should be prime. Hence r_k is a prime number that divides n. *[This is what we were to show.]*

Counterexamples and Divisibility

To show that a proposed divisibility property is not universally true, you need only find one pair of integers for which it is false.

Example 4.3.7 Checking a Proposed Divisibility Property

Is the following statement true or false? For all integers a and b, if $a \mid b$ and $b \mid a$ then $a = b$.

Solution This statement is false. Can you think of a counterexample just by concentrating for a minute or so?

The following discussion describes a mental process that may take just a few seconds. It is helpful to be able to use it consciously, however, to solve more difficult problems.

To discover the truth or falsity of a statement such as the one given above, start off much as you would if you were trying to prove it.

Starting Point: Suppose a and b are integers such that $a \mid b$ and $b \mid a$.

Ask yourself, "*Must* it follow that $a = b$, or *could* it happen that $a \neq b$ for some a and b?" Focus on the supposition. What does it mean? By definition of divisibility, the conditions $a \mid b$ and $b \mid a$ mean that

$$b = ka \quad \text{and} \quad a = lb \quad \text{for some integers } k \text{ and } l.$$

Must it follow that $a = b$, or can you find integers a and b that satisfy these equations for which $a \neq b$? The equations imply that

$$b = ka = k(lb) = (kl)b.$$

Since $b \mid a$, $b \neq 0$, and so you can cancel b from the extreme left and right sides to obtain

$$1 = kl.$$

In other words, k and l are divisors of 1. But, by Theorem 4.3.2, the only divisors of 1 are 1 and -1. Thus k and l are both 1 or are both -1. If $k = l = 1$, then $b = a$. But

*Strictly speaking, this statement is justified by an axiom for the integers called the well-ordering principle, which is discussed in Section 5.4. Theorem 4.3.4 can also be proved using strong mathematical induction, as shown in Example 5.4.1.

if $k = l = -1$, then $b = -a$ and so $a \neq b$. This analysis suggests that you can find a counterexample by taking $b = -a$. Here is a formal answer:

Proposed Divisibility Property: For all integers a and b, if $a \mid b$ and $b \mid a$ then $a = b$.

Counterexample: Let $a = 2$ and $b = -2$. Then

$$a \mid b \text{ since } 2 \mid (-2) \text{ and } b \mid a \text{ since } (-2) \mid 2, \text{ but } a \neq b \text{ since } 2 \neq -2.$$

Therefore, the statement is false.

■

The search for a proof will frequently help you discover a counterexample (provided the statement you are trying to prove is, in fact, false). Conversely, in trying to find a counterexample for a statement, you may come to realize the reason why it is true (if it is, in fact, true). The important thing is to keep an open mind until you are convinced by the evidence of your own careful reasoning.

The Unique Factorization of Integers Theorem

The most comprehensive statement about divisibility of integers is contained in the *unique factorization of integers theorem*. Because of its importance, this theorem is also called the *fundamental theorem of arithmetic.* Although Euclid, who lived about 300 B.C., seems to have been acquainted with the theorem, it was first stated precisely by the great German mathematician Carl Friedrich Gauss (rhymes with *house*) in 1801.

The unique factorization of integers theorem says that any integer greater than 1 either is prime or can be written as a product of prime numbers in a way that is unique except, perhaps, for the order in which the primes are written. For example,

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2$$

and so forth. The three 2's and two 3's may be written in any order, but any factorization of 72 as a product of primes must contain exactly three 2's and two 3's—no other collection of prime numbers besides three 2's and two 3's multiplies out to 72.

Note This theorem is the reason the number 1 is not allowed to be prime. If 1 were prime, then factorizations would not be unique. For example, $6 = 2 \cdot 3 = 1 \cdot 2 \cdot 3$, and so forth.

Theorem 4.3.5 Unique Factorization of Integers Theorem (Fundamental Theorem of Arithmetic)

Given any integer $n > 1$, there exist a positive integer k, distinct prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_k^{e_k},$$

and any other expression for n as a product of prime numbers is identical to this except, perhaps, for the order in which the factors are written.

The proof of the unique factorization theorem is outlined in the exercises for Sections 5.4 and 8.4.

Because of the unique factorization theorem, any integer $n > 1$ can be put into a *standard factored form* in which the prime factors are written in ascending order from left to right.

• Definition

Given any integer $n > 1$, the **standard factored form** of n is an expression of the form

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k},$$

where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; $e_1, e_2, \ldots, e_k$ are positive integers; and $p_1 < p_2 < \cdots < p_k$.

Example 4.3.8 Writing Integers in Standard Factored Form

Write 3,300 in standard factored form.

Solution First find all the factors of 3,300. Then write them in ascending order:

$$\begin{aligned} 3{,}300 &= 100 \cdot 33 = 4 \cdot 25 \cdot 3 \cdot 11 \\ &= 2 \cdot 2 \cdot 5 \cdot 5 \cdot 3 \cdot 11 = 2^2 \cdot 3^1 \cdot 5^2 \cdot 11^1. \end{aligned}$$

■

Example 4.3.9 Using Unique Factorization to Solve a Problem

Suppose m is an integer such that

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot m = 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10.$$

Does $17 \mid m$?

Solution Since 17 is one of the prime factors of the right-hand side of the equation, it is also a prime factor of the left-hand side (by the unique factorization of integers theorem). But 17 does not equal any prime factor of 8, 7, 6, 5, 4, 3, or 2 (because it is too large). Hence 17 must occur as one of the prime factors of m, and so $17 \mid m$. ■

Test Yourself

1. To show that a nonzero integer d divides an integer n, we must show that _____.
2. To say that d divides n means the same as saying that _____ is divisible by _____.
3. If a and b are positive integers and $a \mid b$, then _____ is less than or equal to _____.
4. For all integers n and d, $d \nmid n$ if, and only if, _____.
5. If a and b are integers, the notation $a \mid b$ denotes _____ and the notation a/b denotes _____.
6. The transitivity of divisibility theorem says that for all integers a, b, and c, if _____ then _____.
7. The divisibility by a prime theorem says that every integer greater than 1 is _____.
8. The unique factorization of integers theorem says that any integer greater than 1 is either _____ or can be written as _____ in a way that is unique except possibly for the _____ in which the numbers are written.

Exercise Set 4.3

Give a reason for your answer in each of 1–13. Assume that all variables represent integers.

1. Is 52 divisible by 13?

2. Does $7 \mid 56$?

3. Does $5 \mid 0$?

4. Does 3 divide $(3k + 1)(3k + 2)(3k + 3)$?

5. Is $6m(2m + 10)$ divisible by 4?

6. Is 29 a multiple of 3?

7. Is -3 a factor of 66?

8. Is $6a(a + b)$ a multiple of $3a$?

9. Is 4 a factor of $2a \cdot 34b$?

10. Does $7 \mid 34$? 11. Does $13 \mid 73$?

12. If $n = 4k + 1$, does 8 divide $n^2 - 1$?

13. If $n = 4k + 3$, does 8 divide $n^2 - 1$?

14. Fill in the blanks in the following proof that for all integers a and b, if $a \mid b$ then $a \mid (-b)$.

Proof: Suppose a and b are any integers such that __(a)__. By definition of divisibility, there exists an integer r such that __(b)__. By substitution,

$$-b = -ar = a(-r).$$

Let $t =$ __(c)__. Then t is an integer because $t = (-1) \cdot r$, and both -1 and r are integers. Thus, by substitution, $-b = at$, where r is an integer, and so by definition of divisibility, __(d)__, as was to be shown.

Prove statements 15 and 16 directly from the definition of divisibility.

15. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

H **16.** For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

17. Consider the following statement: The negative of any multiple of 3 is a multiple of 3.
a. Write the statement formally using a quantifier and a variable.
b. Determine whether the statement is true or false and justify your answer.

18. Show that the following statement is false: For all integers a and b, if $3 \mid (a + b)$ then $3 \mid (a - b)$.

For each statement in 19–31, determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

H **19.** For all integers a, b, and c, if a divides b then a divides bc.

20. The sum of any three consecutive integers is divisible by 3. (Two integers are **consecutive** if, and only if, one is one more than the other.)

21. The product of any two even integers is a multiple of 4.

H **22.** A necessary condition for an integer to be divisible by 6 is that it be divisible by 2.

23. A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.

24. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (2b - 3c)$.

25. For all integers a, b, and c, if a is a factor of c then ab is a factor of c.

H **26.** For all integers a, b, and c, if $ab \mid c$ then $a \mid c$ and $b \mid c$.

H **27.** For all integers a, b, and c, if $a \mid (b + c)$ then $a \mid b$ or $a \mid c$.

28. For all integers a, b, and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$.

29. For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

30. For all integers a and n, if $a \mid n^2$ and $a \leq n$ then $a \mid n$.

31. For all integers a and b, if $a \mid 10b$ then $a \mid 10$ or $a \mid b$.

32. A fast-food chain has a contest in which a card with numbers on it is given to each customer who makes a purchase. If some of the numbers on the card add up to 100, then the customer wins \$100. A certain customer receives a card containing the numbers

72, 21, 15, 36, 69, 81, 9, 27, 42, and 63.

Will the customer win \$100? Why or why not?

33. Is it possible to have a combination of nickels, dimes, and quarters that add up to \$4.72? Explain.

34. Is it possible to have 50 coins, made up of pennies, dimes, and quarters, that add up to \$3? Explain.

35. Two athletes run a circular track at a steady pace so that the first completes one round in 8 minutes and the second in 10 minutes. If they both start from the same spot at 4 P.M., when will be the first time they return to the start together?

36. It can be shown (see exercises 44–48) that an integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3. An integer is divisible by 9 if, and only if, the sum of its digits is divisible by 9. An integer is divisible by 5 if, and only if, its right-most digit is a 5 or a 0. And an integer is divisible by 4 if, and only if, the number formed by its right-most two digits is divisible by 4. Check the following integers for divisibility by 3, 4, 5 and 9.
a. 637,425,403,705,125 b. 12,858,306,120,312
c. 517,924,440,926,512 d. 14,328,083,360,232

37. Use the unique factorization theorem to write the following integers in standard factored form.
a. 1,176 b. 5,733 c. 3,675

38. Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; and $e_1, e_2, \ldots, e_k$ are positive integers.
a. What is the standard factored form for a^2?
b. Find the least positive integer n such that $2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot n$ is a perfect square. Write the resulting product as a perfect square.
c. Find the least positive integer m such that $2^2 \cdot 3^5 \cdot 7 \cdot 11 \cdot m$ is a perfect square. Write the resulting product as a perfect square.

39. Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; and $e_1, e_2, \ldots, e_k$ are positive integers.
a. What is the standard factored form for a^3?
b. Find the least positive integer k such that $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$ is a perfect cube (i.e., equals an integer to the third power). Write the resulting product as a perfect cube.

40. **a.** If a and b are integers and $12a = 25b$, does $12 \mid b$? does $25 \mid a$? Explain.
    b. If x and y are integers and $10x = 9y$, does $10 \mid y$? does $9 \mid x$? Explain.

H **41.** How many zeros are at the end of $45^8 \cdot 88^5$? Explain how you can answer this question without actually computing the number. (*Hint:* $10 = 2 \cdot 5$.)

42. If n is an integer and $n > 1$, then $n!$ is the product of n and every other positive integer that is less than n. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.
    a. Write 6! in standard factored form.
    b. Write 20! in standard factored form.
    c. Without computing the value of $(20!)^2$ determine how many zeros are at the end of this number when it is written in decimal form. Justify your answer.

✶43. In a certain town 2/3 of the adult men are married to 3/5 of the adult women. Assume that all marriages are monogamous (no one is married to more than one other person). Also assume that there are at least 100 adult men in the town. What is the least possible number of adult men in the town? of adult women in the town?

Definition: Given any nonnegative integer n, the **decimal representation** of n is an expression of the form

$$d_k d_{k-1} \cdots d_2 d_1 d_0,$$

where k is a nonnegative integer; $d_0, d_1, d_2, \ldots, d_k$ (called the **decimal digits** of n) are integers from 0 to 9 inclusive; $d_k \neq 0$ unless $n = 0$ and $k = 0$; and

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0.$$

(For example, $2{,}503 = 2 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 3$.)

44. Prove that if n is any nonnegative integer whose decimal representation ends in 0, then $5 \mid n$. (*Hint:* If the decimal representation of a nonnegative integer n ends in d_0, then $n = 10m + d_0$ for some integer m.)

45. Prove that if n is any nonnegative integer whose decimal representation ends in 5, then $5 \mid n$.

46. Prove that if the decimal representation of a nonnegative integer n ends in $d_1 d_0$ and if $4 \mid (10d_1 + d_0)$, then $4 \mid n$. (*Hint:* If the decimal representation of a nonnegative integer n ends in $d_1 d_0$, then there is an integer s such that $n = 100s + 10d_1 + d_0$.)

H ✶**47.** Observe that

$$\begin{aligned}
7524 &= 7 \cdot 1000 + 5 \cdot 100 + 2 \cdot 10 + 4 \\
&= 7(999 + 1) + 5(99 + 1) + 2(9 + 1) + 4 \\
&= (7 \cdot 999 + 7) + (5 \cdot 99 + 5) + (2 \cdot 9 + 2) + 4 \\
&= (7 \cdot 999 + 5 \cdot 99 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 \cdot 9 + 5 \cdot 11 \cdot 9 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 + 5 \cdot 11 + 2) \cdot 9 + (7 + 5 + 2 + 4) \\
&= \text{(an integer divisible by 9)} \\
&\qquad + \text{(the sum of the digits of 7524)}.
\end{aligned}$$

Since the sum of the digits of 7524 is divisible by 9, 7524 can be written as a sum of two integers each of which is divisible by 9. It follows from exercise 15 that 7524 is divisible by 9.

Generalize the argument given in this example to any nonnegative integer n. In other words, prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 9, then n is divisible by 9.

✶48. Prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 3, then n is divisible by 3.

✶49. Given a positive integer n written in decimal form, the alternating sum of the digits of n is obtained by starting with the right-most digit, subtracting the digit immediately to its left, adding the next digit to the left, subtracting the next digit, and so forth. For example, the alternating sum of the digits of 180,928 is $8 - 2 + 9 - 0 + 8 - 1 = 22$. Justify the fact that for any nonnegative integer n, if the alternating sum of the digits of n is divisible by 11, then n is divisible by 11.

Answers for Test Yourself

1. n equals d times some integer (Or: there is an integer r such that $n = dr$) 2. n; d 3. a; b 4. $\frac{n}{d}$ is not an integer 5. the sentence "a divides b"; the number obtained when a is divided by b 6. a divides b and b divides c; a divides c 7. divisible by some prime number 8. prime; a product of prime numbers; order