Find the mistakes in the proof fragments in 33–35.

H **33.** **Theorem:** For any integer $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

"**Proof (by mathematical induction):** Certainly the theorem is true for $n = 1$ because $1^2 = 1$ and $\frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$. So the basis step is true.

For the inductive step, suppose that for some integer $k \geq 1$, $k^2 = \frac{k(k+1)(2k+1)}{6}$. We must show that $(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$."

H **34.** **Theorem:** For any integer $n \geq 0$,

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

"**Proof (by mathematical induction): Let the property** $P(n)$ be $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Show that P(0) is true:

The left-hand side of $P(0)$ is $1 + 2 + 2^2 + \cdots + 2^0 = 1$ and the right-hand side is $2^{0+1} - 1 = 2 - 1 = 1$ also. So $P(0)$ is true."

H **35.** **Theorem:** For any integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

"**Proof (by mathematical induction): Let the property** $P(n)$ be $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$.

Show that P(1) is true: When $n = 1$

$$\sum_{i=1}^{1} i(i!) = (1+1)! - 1$$

So $\quad 1(1!) = 2! - 1$

and $\quad 1 = 1$

Thus $P(1)$ is true."

✶ 36. Use Theorem 5.2.2 to prove that if m and n are any positive integers and m is odd, then $\sum_{k=0}^{m-1}(n+k)$ is divisible by m. Does the conclusion hold if m is even? Justify your answer.

H ✶ 37. Use Theorem 5.2.2 and the result of exercise 10 to prove that if p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p.

Answers for Test Yourself

1. greater than or equal to some initial value 2. (a) $P(a)$ is true (b) $P(k)$ is true; inductive hypothesis; $P(k+1)$ is true

5.3 Mathematical Induction II

A good proof is one which makes us wiser. — I. Manin, *A Course in Mathematical Logic,* 1977

In natural science courses, deduction and induction are presented as alternative modes of thought—deduction being to infer a conclusion from general principles using the laws of logical reasoning, and induction being to enunciate a general principle after observing it to hold in a large number of specific instances. In this sense, then, *mathematical* induction is not inductive but deductive. Once proved by mathematical induction, a theorem is known just as certainly as if it were proved by any other mathematical method. Inductive reasoning, in the natural sciences sense, *is* used in mathematics, but only to make conjectures, not to prove them. For example, observe that

$$1 - \frac{1}{2} = \frac{1}{2}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

This pattern seems so unlikely to occur by pure chance that it is reasonable to conjecture (though it is by no means certain) that the pattern holds true in general. In a case like this, a proof by mathematical induction (which you are asked to write in exercise 1 at the end of this section) gets to the essence of why the pattern holds in general. It reveals the mathematical mechanism that necessitates the truth of each successive case from the previous one. For instance, in this example observe that if

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k},$$

then by substitution

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right)$$
$$=\frac{1}{k}\left(1-\frac{1}{k+1}\right)=\frac{1}{k}\left(\frac{k+1-1}{k+1}\right)=\frac{1}{k}\left(\frac{k}{k+1}\right)=\frac{1}{k+1}.$$

Thus mathematical induction makes knowledge of the general pattern a matter of mathematical certainty rather than vague conjecture.

In the remainder of this section we show how to use mathematical induction to prove additional kinds of statements such as divisibility properties of the integers and inequalities. The basic outlines of the proofs are the same in all cases, but the details of the basis and inductive steps differ from one to another.

Example 5.3.1 Proving a Divisibility Property

Use mathematical induction to prove that for all integers $n \geq 0$, $2^{2n}-1$ is divisible by 3.

Solution As in the previous proofs by mathematical induction, you need to identify the property $P(n)$. In this example, $P(n)$ is the sentence

$2^{2n}-1$ is divisible by 3. ← the property ($P(n)$)

By substitution, the statement for the basis step, $P(0)$, is

$2^{2\cdot 0}-1$ is divisible by 3. ← basis ($P(0)$)

The supposition for the inductive step, $P(k)$, is

$2^{2k}-1$ is divisible by 3, ← inductive hypothesis ($P(k)$)

and the conclusion to be shown, $P(k+1)$, is

$2^{2(k+1)}-1$ is divisible by 3. ← to show ($P(k+1)$)

Recall that an integer m is divisible by 3 if, and only if, $m=3r$ for some integer r. Now the statement $P(0)$ is true because $2^{2\cdot 0}-1=2^0-1=1-1=0$, which is divisible by 3 because $0=3\cdot 0$.

To prove the inductive step, you suppose that k is any integer greater than or equal to 0 such that $P(k)$ is true. This means that $2^{2k}-1$ is divisible by 3. You must then prove the truth of $P(k+1)$. Or, in other words, you must show that $2^{2(k+1)}-1$ is divisible by 3. But

$$\begin{aligned} 2^{2(k+1)}-1 &= 2^{2k+2}-1 \\ &= 2^{2k}\cdot 2^2-1 && \text{by the laws of exponents} \\ &= 2^{2k}\cdot 4-1. \end{aligned}$$

The aim is to show that this quantity, $2^{2k}\cdot 4 - 1$, is divisible by 3. Why should that be so? By the inductive hypothesis, $2^{2k} - 1$ is divisible by 3, and $2^{2k}\cdot 4 - 1$ resembles $2^{2k} - 1$. Observe what happens, if you subtract $2^{2k} - 1$ from $2^{2k}\cdot 4 - 1$:

$$\underbrace{2^{2k}\cdot 4 - 1}_{\text{divisible by 3?}} - \underbrace{(2^{2k} - 1)}_{\text{divisible by 3}} = \underbrace{2^{2k}\cdot 3.}_{\text{divisible by 3}}$$

Adding $2^{2k} - 1$ to both sides gives

$$\underbrace{2^{2k}\cdot 4 - 1}_{\text{divisible by 3?}} = \underbrace{2^{2k}\cdot 3}_{\text{divisible by 3}} + \underbrace{2^{2k} - 1.}_{\text{divisible by 3}}$$

Both terms of the sum on the right-hand side of this equation are divisible by 3; hence the sum is divisible by 3. (See exercise 15 of Section 4.3.) Therefore, the left-hand side of the equation is also divisible by 3, which is what was to be shown.

This discussion is summarized as follows:

Proposition 5.3.1

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Proof (by mathematical induction):

Let the property $P(n)$ be the sentence "$2^{2n} - 1$ is divisible by 3."

$$2^{2n} - 1 \text{ is divisible by 3.} \qquad \leftarrow P(n)$$

***Show that P(0) is true*:**
To establish $P(0)$, we must show that

$$2^{2\cdot 0} - 1 \text{ is divisible by 3.} \qquad \leftarrow P(0)$$

But

$$2^{2\cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 3 because $0 = 3\cdot 0$. Hence $P(0)$ is true.

***Show that for all integers $k \geq 0$, if P(k) is true then P(k + 1) is also true*:**
[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]
Let k be any integer with $k \geq 0$, and suppose that

$$2^{2k} - 1 \text{ is divisible by 3.} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

By definition of divisibility, this means that

$$2^{2k} - 1 = 3r \quad \text{for some integer } r.$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$2^{2(k+1)} - 1 \text{ is divisible by 3.} \qquad \leftarrow P(k+1)$$

But

$$
\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\
&= 2^{2k} \cdot 2^2 - 1 && \text{by the laws of exponents} \\
&= 2^{2k} \cdot 4 - 1 \\
&= 2^{2k}(3+1) - 1 \\
&= 2^{2k} \cdot 3 + (2^{2k} - 1) && \text{by the laws of algebra} \\
&= 2^{2k} \cdot 3 + 3r && \text{by inductive hypothesis} \\
&= 3(2^{2k} + r) && \text{by factoring out the 3.}
\end{aligned}
$$

But $2^{2k} + r$ is an integer because it is a sum of products of integers, and so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3 *[as was to be shown]*.
[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]

■

The next example illustrates the use of mathematical induction to prove an inequality.

Example 5.3.2 Proving an Inequality

Use mathematical induction to prove that for all integers $n \geq 3$,

$$2n + 1 < 2^n.$$

Solution In this example the property $P(n)$ is the inequality

$$\boxed{2n + 1 < 2^n.} \quad \leftarrow \text{the property } (P(n))$$

By substitution, the statement for the basis step, $P(3)$, is

$$\boxed{2 \cdot 3 + 1 < 2^3.} \quad \leftarrow \text{basis } (P(3))$$

The supposition for the inductive step, $P(k)$, is

$$\boxed{2k + 1 < 2^k,} \quad \leftarrow \text{inductive hypothesis } (P(k))$$

and the conclusion to be shown is

$$\boxed{2(k+1) + 1 < 2^{k+1}.} \quad \leftarrow \text{to show } (P(k+1))$$

To prove the basis step, observe that the statement $P(3)$ is true because $2 \cdot 3 + 1 = 7$, $2^3 = 8$, and $7 < 8$.

To prove the inductive step, suppose the inductive hypothesis, that $P(k)$ is true for an integer $k \geq 3$. This means that $2k + 1 < 2^k$ is assumed to be true for a particular but arbitrarily chosen integer $k \geq 3$. Then derive the truth of $P(k+1)$. Or, in other words, show that the inequality $2(k+1) + 1 < 2^{k+1}$ is true. But by multiplying out and regrouping,

$$2(k+1) + 1 = 2k + 3 = (2k+1) + 2, \qquad 5.3.1$$

and by substitution from the inductive hypothesis,

$$(2k+1)+2 < 2^k + 2. \qquad 5.3.2$$

Hence

$$2(k+1)+1 < 2^k + 2$$ The left-most part of equation (5.3.1) is less than the right-most part of inequality (5.3.2).

Note Properties of order are listed in Appendix A.

If it can be shown that $2^k + 2$ is less than 2^{k+1}, then the desired inequality will have been proved. But since the quantity 2^k can be added to or subtracted from an inequality without changing its direction,

$$2^k + 2 < 2^{k+1} \quad \Leftrightarrow \quad 2 < 2^{k+1} - 2^k = 2^k(2-1) = 2^k.$$

And since multiplying or dividing an inequality by 2 does not change its direction,

$$2 < 2^k \quad \Leftrightarrow \quad 1 = \frac{2}{2} < \frac{2^k}{2} = 2^{k-1}$$ by the laws of exponents.

This last inequality is clearly true for all $k \geq 2$. Hence it is true that $2(k+1)+1 < 2^{k+1}$.

This discussion is made more flowing (but less intuitive) in the following formal proof:

Proposition 5.3.2

For all integers $n \geq 3$, $2n+1 < 2^n$.

Proof (by mathematical induction):

Let the property $P(n)$ be the inequality

$$2n+1 < 2^n. \qquad \leftarrow P(n)$$

Show that P(3) is true:
To establish $P(3)$, we must show that

$$2 \cdot 3 + 1 < 2^3. \qquad \leftarrow P(3)$$

But

$$2 \cdot 3 + 1 = 7 \quad \text{and} \quad 2^3 = 8 \quad \text{and} \quad 7 < 8.$$

Hence $P(3)$ is true.

Show that for all integers k ≥ 3, if P(k) is true then P(k + 1) is also true:
[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 3. That is:]
Suppose that k is any integer with $k \geq 3$ such that

$$2k+1 < 2^k. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that P(k + 1) is true. That is:] We must show that

$$2(k+1)+1 < 2^{(k+1)},$$

or, equivalently,

$$2k+3 < 2^{(k+1)}. \qquad \leftarrow P(k+1)$$

But

$$2k+3 = (2k+1)+2 \quad \text{by algebra}$$
$$< 2^k + 2^k \quad \text{because } 2k+1 < 2^k \text{ by the inductive hypothesis and because } 2 < 2^k \text{ for all integers } k \geq 2$$
$$\therefore 2k+3 < 2 \cdot 2^k = 2^{k+1} \quad \text{by the laws of exponents.}$$

[This is what we needed to show.]
[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]

■

The next example demonstrates how to use mathematical induction to show that the terms of a sequence satisfy a certain explicit formula.

Example 5.3.3 Proving a Property of a Sequence

Define a sequence $a_1, a_2, a_3, \ldots$ as follows.*

$$a_1 = 2$$
$$a_k = 5a_{k-1} \quad \text{for all integers } k \geq 2.$$

a. Write the first four terms of the sequence.

b. It is claimed that for each integer $n \geq 0$, the nth term of the sequence has the same value as that given by the formula $2 \cdot 5^{n-1}$. In other words, the claim is that the terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$. Prove that this is true.

Solution

a. $a_1 = 2$.
$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = 10$
$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$
$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = 250$.

b. To use mathematical induction to show that every term of the sequence satisfies the equation, begin by showing that the first term of the sequence satisfies the equation. Then suppose that an arbitrarily chosen term a_k satisfies the equation and prove that the next term a_{k+1} also satisfies the equation.

Proof:

Let $a_1, a_2, a_3, \ldots$ be the sequence defined by specifying that $a_1 = 2$ and $a_k = 5a_{k-1}$ for all integers $k \geq 2$, and let the property $P(n)$ be the equation

$$a_n = 2 \cdot 5^{n-1}. \qquad \leftarrow P(n)$$

We will use mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.

Show that P(1) is true:
To establish $P(1)$, we must show that

$$a_1 = 2 \cdot 5^{1-1}. \qquad \leftarrow P(1)$$

*This is another example of a recursive definition. The general subject of recursion is discussed in Section 5.6.

But the left-hand side of $P(1)$ is

$$a_1 = 2 \qquad \text{by definition of } a_1, a_2, a_3, \ldots,$$

and the right-hand side of $P(1)$ is

$$2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2 \cdot 1 = 2.$$

Thus the two sides of $P(1)$ are equal to the same quantity, and hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true: *[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:]* Let k be any integer with $k \geq 0$, and suppose that

$$a_k = 2 \cdot 5^{k-1}. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

By definition of divisibility, this means that

$$a_k = 2 \cdot 5^{k-1}.$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$a_{k+1} = 2 \cdot 5^{(k+1)-1},$$

or, equivalently,

$$a_{k+1} = 2 \cdot 5^k. \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

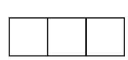
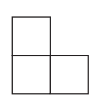
$$\begin{aligned} a_{k+1} &= 5a_{(k+1)-1} && \text{by definition of } a_1, a_2, a_3, \ldots \\ &= 5a_k && \text{since } (k+1) - 1 = k \\ &= 5 \cdot (2 \cdot 5^{k-1}) && \text{by inductive hypothesis} \\ &= 2 \cdot (5 \cdot 5^{k-1}) && \text{by regrouping} \\ &= 2 \cdot 5^k && \text{by the laws of exponents} \end{aligned}$$

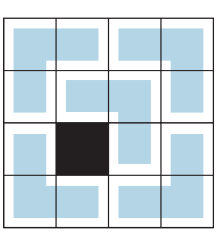
which is the right-hand side of the equation *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the formula holds for all terms of the sequence.] ■

A Problem with Trominoes

The word *polyomino*, a generalization of *domino*, was introduced by Solomon Golomb in 1954 when he was a 22-year-old student at Harvard. Subsequently, he and others proved many interesting properties about them, and they became the basis for the popular computer game Tetris. A particular type of polyomino, called a *tromino*, is made up of three attached squares, which can be of two types:

straight  and L-shaped 



Call a checkerboard that is formed using m squares on a side an $m \times m$ ("m by m") checkerboard. Observe that if one square is removed from a 4×4 checkerboard, the remaining squares can be completely covered by L-shaped trominoes. For instance, a covering for one such board is illustrated in the figure to the left.

In his first article about polyominoes, Golomb included a proof of the following theorem. It is a beautiful example of an argument by mathematical induction.

Theorem Covering a Board with Trominoes

For any integer $n \geq 1$, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by L-shaped trominoes.

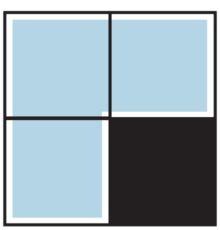
The main insight leading to a proof of this theorem is the observation that because $2^{k+1} = 2 \cdot 2^k$, when a $2^{k+1} \times 2^{k+1}$ board is split in half both vertically and horizontally, each half side will have length 2^k and so each resulting quadrant will be a $2^k \times 2^k$ checkerboard.

Proof (by mathematical induction):

Let the property $P(n)$ be the sentence

> If any square is removed from a $2^n \times 2^n$ checkerboard,
> then the remaining squares can be completely covered. $\leftarrow P(n)$
> by L-shaped trominoes

Show that P(1) is true:



A $2^1 \times 2^1$ checkerboard just consists of four squares. If one square is removed, the remaining squares form an L, which can be covered by a single L-shaped tromino, as illustrated in the figure to the left. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true:

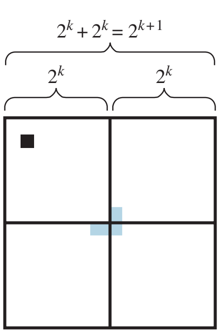
[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 3$. That is:] Let k be any integer such that $k \geq 1$, and suppose that

> If any square is removed from a $2^k \times 2^k$ checkerboard,
> then the remaining squares can be completely covered $\leftarrow P(k)$
> by L-shaped trominoes.

$P(k)$ is the inductive hypothesis.

[We must show that $P(k+1)$ is true. That is:] We must show that

> If any square is removed from a $2^{k+1} \times 2^{k+1}$ checkerboard,
> then the remaining squares can be completely covered $\leftarrow P(k+1)$
> by L-shaped trominoes.



Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Divide it into four equal quadrants: Each will consist of a $2^k \times 2^k$ checkerboard. In one of the quadrants, one square will have been removed, and so, by inductive hypothesis, all the remaining squares in this quadrant can be completely covered by L-shaped trominoes. The other three quadrants meet at the center of the checkerboard, and the center of the checkerboard serves as a corner of a square from each of those quadrants. An L-shaped tromino can, therefore, be placed on those three central squares. This situation is illustrated in the figure to the left. By inductive hypothesis, the remaining squares in each of the three quadrants can be completely covered by L-shaped trominoes. Thus every square in the $2^{k+1} \times 2^{k+1}$ checkerboard except the one that was removed can be completely covered by L-shaped trominoes *[as was to be shown]*.

Test Yourself

1. Mathematical induction differs from the kind of induction used in the natural sciences because it is actually a form of ____ reasoning.

2. Mathematical induction can be used to _____ conjectures that have been made using inductive reasoning.

Exercise Set 5.3

1. Based on the discussion of the product $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)$ at the beginning of this section, conjecture a formula for general n. Prove your conjecture by mathematical induction.

2. Experiment with computing values of the product $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)$ for small values of n to conjecture a formula for this product for general n. Prove your conjecture by mathematical induction.

3. Observe that

$$\frac{1}{1\cdot 3}=\frac{1}{3}$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}=\frac{2}{5}$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}=\frac{3}{7}$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}=\frac{4}{9}$$

Guess a general formula and prove it by mathematical induction.

H 4. Observe that

$$1=1,$$
$$1-4=-(1+2),$$
$$1-4+9=1+2+3,$$
$$1-4+9-16=-(1+2+3+4),$$
$$1-4+9-16+25=1+2+3+4+5.$$

Guess a general formula and prove it by mathematical induction.

5. Evaluate the sum $\sum_{k=1}^{n}\frac{k}{(k+1)!}$ for $n=1,2,3,4,$ and 5. Make a conjecture about a formula for this sum for general n, and prove your conjecture by mathematical induction.

6. For each positive integer n, let $P(n)$ be the property

$$5^n-1 \text{ is divisible by } 4.$$

   a. Write $P(0)$. Is $P(0)$ true?
   b. Write $P(k)$.
   c. Write $P(k+1)$.
   d. In a proof by mathematical induction that this divisibility property holds for all integers $n\geq 0$, what must be shown in the inductive step?

7. For each positive integer n, let $P(n)$ be the property

$$2^n<(n+1)!.$$

   a. Write $P(2)$. Is $P(2)$ true?
   b. Write $P(k)$.
   c. Write $P(k+1)$.
   d. In a proof by mathematical induction that this inequality holds for all integers $n\geq 2$, what must be shown in the inductive step?

Prove each statement in 8–23 by mathematical induction.

8. 5^n-1 is divisible by 4, for each integer $n\geq 0$.

9. 7^n-1 is divisible by 6, for each integer $n\geq 0$.

10. n^3-7n+3 is divisible by 3, for each integer $n\geq 0$.

11. $3^{2n}-1$ is divisible by 8, for each integer $n\geq 0$.

12. For any integer $n\geq 0$, 7^n-2^n is divisible by 5.

H 13. For any integer $n\geq 0$, x^n-y^n is divisible by $x-y$, where x and y are any integers with $x\neq y$.

H 14. n^3-n is divisible by 6, for each integer $n\geq 0$.

15. $n(n^2+5)$ is divisible by 6, for each integer $n\geq 0$.

16. $2^n<(n+1)!$, for all integers $n\geq 2$.

17. $1+3n\leq 4^n$, for every integer $n\geq 0$.

18. $5^n+9<6^n$, for all integers $n\geq 2$.

19. $n^2<2^n$, for all integers $n\geq 5$.

20. $2^n<(n+2)!$, for all integers $n\geq 0$.

21. $\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$, for all integers $n\geq 2$.

22. $1+nx\leq(1+x)^n$, for all real numbers $x>-1$ and integers $n\geq 2$.

23. a. $n^3>2n+1$, for all integers $n\geq 2$.
    b. $n!>n^2$, for all integers $n\geq 4$.

24. A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1=3$ and $a_k=7a_{k-1}$ for all integers $k\geq 2$. Show that $a_n=3\cdot 7^{n-1}$ for all integers $n\geq 1$.

25. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0=5$ and $b_k=4+b_{k-1}$ for all integers $k\geq 1$. Show that $b_n>4n$ for all integers $n\geq 0$.

26. A sequence $c_0, c_1, c_2, \ldots$ is defined by letting $c_0 = 3$ and $c_k = (c_{k-1})^2$ for all integers $k \geq 1$. Show that $c_n = 3^{2^n}$ for all integers $n \geq 0$.

27. A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$ for all integers $k \geq 2$. Show that for all integers $n \geq 1$, $d_n = \dfrac{2}{n!}$.

28. Prove that for all integers $n \geq 1$,

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+\cdots+(2n-1)}{(2n+1)+\cdots+(4n-1)}.$$

29. As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

In order for a proof by mathematical induction to be valid, the basis statement must be true for $n = a$ and the argument of the inductive step must be correct for every integer $k \geq a$. In 30 and 31 find the mistakes in the "proofs" by mathematical induction.

30. "**Theorem:**" For any integer $n \geq 1$, all the numbers in a set of n numbers are equal to each other.

"**Proof (by mathematical induction):** It is obviously true that all the numbers in a set consisting of just one number are equal to each other, so the basis step is true. For the inductive step, let $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ be any set of $k+1$ numbers. Form two subsets each of size k:

$$B = \{a_1, a_2, a_3, \ldots, a_k\} \quad \text{and}$$
$$C = \{a_1, a_3, a_4, \ldots, a_{k+1}\}.$$

(B consists of all the numbers in A except a_{k+1}, and C consists of all the numbers in A except a_2.) By inductive hypothesis, all the numbers in B equal a_1 and all the numbers in C equal a_1 (since both sets have only k numbers). But every number in A is in B or C, so all the numbers in A equal a_1; hence all are equal to each other."

H **31.** "**Theorem:**" For all integers $n \geq 1$, $3^n - 2$ is even.

"**Proof (by mathematical induction):** Suppose the theorem is true for an integer k, where $k \geq 1$. That is, suppose that $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is even. But

$$3^{k+1} - 2 = 3^k \cdot 3 - 2 = 3^k(1+2) - 2 = (3^k - 2) + 3^k \cdot 2.$$

Now $3^k - 2$ is even by inductive hypothesis and $3^k \cdot 2$ is even by inspection. Hence the sum of the two quantities is even (by Theorem 4.1.1). It follows that $3^{k+1} - 2$ is even, which is what we needed to show."

H **32.** Some 5×5 checkerboards with one square removed can be completely covered by L-shaped trominoes, whereas other 5×5 checkerboards cannot. Find examples of both kinds of checkerboards. Justify your answers.

33. Consider a 4×6 checkerboard. Draw a covering of the board by L-shaped trominoes.

H 34. **a.** Use mathematical induction to prove that any checkerboard with dimensions $3 \times 2n$ can be completely covered with L-shaped trominoes for any integer $n \geq 1$.

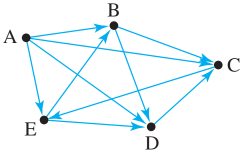
b. Let n be any integer greater than or equal to 1. Use the result of part (a) to prove by mathematical induction that for all integers m, any checkerboard with dimensions $2m \times 3n$ can be completely covered with L-shaped trominoes.

35. Let m and n be any integers that are greater than or equal to 1.

a. Prove that a necessary condition for an $m \times n$ checkerboard to be completely coverable by L-shaped trominoes is that mn be divisible by 3.

H b. Prove that having mn be divisible by 3 is not a sufficient condition for an $m \times n$ checkerboard to be completely coverable by L-shaped trominoes.

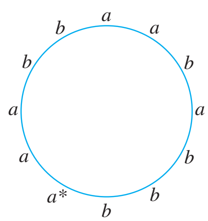
36. In a round-robin tournament each team plays every other team exactly once. If the teams are labeled $T_1, T_2, \ldots, T_n$, then the outcome of such a tournament can be represented by a drawing, called a *directed graph,* in which the teams are represented as dots and an arrow is drawn from one dot to another if, and only if, the team represented by the first dot beats the team represented by the second dot. For example, the directed graph below shows one outcome of a round-robin tournament involving five teams, A, B, C, D, and E.



Use mathematical induction to show that in any round-robin tournament involving n teams, where $n \geq 2$, it is possible to label the teams $T_1, T_2, \ldots, T_n$ so that T_i beats T_{i+1} for all $i = 1, 2, \ldots, n-1$. (For instance, one such labeling in the example above is $T_1 = A, T_2 = B, T_3 = C, T_4 = E, T_5 = D$.) (*Hint:* Given $k+1$ teams, pick one—say T'—and apply the inductive hypothesis to the remaining teams to obtain an ordering $T_1, T_2, \ldots, T_k$. Consider three cases: T' beats T_1, T' loses to the first m teams (where $1 \leq m \leq k-1$) and beats the $(m+1)$st team, and T' loses to all the other teams.)

H ✶ **37.** On the outside rim of a circular disk the integers from 1 through 30 are painted in random order. Show that no matter what this order is, there must be three successive integers whose sum is at least 45.

H 38. Suppose that n a's and n b's are distributed around the outside of a circle. Use mathematical induction to prove that for all integers $n \geq 1$, given any such arrangement, it is possible to find a starting point so that if one travels around the circle in a clockwise direction, the number of a's one has passed is never less than the number of b's one has passed. For example, in the diagram shown below, one could start at the a with an asterisk.



39. For a polygon to be **convex** means that all of its interior angles are less than 180 degrees. Use mathematical induction to prove that for all integers $n \geq 3$, the angles of any n-sided convex polygon add up to $180(n-2)$ degrees.

40. a. Prove that in an 8×8 checkerboard with alternating black and white squares, if the squares in the top right and bottom left corners are removed the remaining board cannot be covered with dominoes. (*Hint*: Mathematical induction is not needed for this proof.)
    b. Use mathematical induction to prove that for all integers n, if a $2n \times 2n$ checkerboard with alternating black and white squares has one white square and one black square removed anywhere on the board, the remaining squares can be covered with dominoes.

Answers for Test Yourself

1. deductive 2. prove

5.4 Strong Mathematical Induction and the Well-Ordering Principle for the Integers

Mathematics takes us still further from what is human into the region of absolute necessity, to which not only the actual world, but every possible world, must conform.
— Bertrand Russell, 1902

Strong mathematical induction is similar to ordinary mathematical induction in that it is a technique for establishing the truth of a sequence of statements about integers. Also, a proof by strong mathematical induction consists of a basis step and an inductive step. However, the basis step may contain proofs for several initial values, and in the inductive step the truth of the predicate $P(n)$ is assumed not just for one value of n but for *all* values through k, and then the truth of $P(k+1)$ is proved.

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots,$ and $P(b)$ are all true. **(basis step)**
2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. **(inductive step)**

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis.** Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)