5.6 Defining Sequences Recursively

So, Nat'ralists observe, a Flea/Hath smaller Fleas that on him prey,/And these have smaller Fleas to bite 'em,/And so proceed ad infinitum. — Jonathan Swift, 1733

A sequence can be defined in a variety of different ways. One informal way is to write the first few terms with the expectation that the general pattern will be obvious. We might say, for instance, "consider the sequence 3, 5, 7, ..." Unfortunately, misunderstandings can occur when this approach is used. The next term of the sequence could be 9 if we mean a sequence of odd integers, or it could be 11 if we mean the sequence of odd prime numbers.

The second way to define a sequence is to give an explicit formula for its nth term. For example, a sequence $a_0, a_1, a_2 \ldots$ can be specified by writing

$$a_n = \frac{(-1)^n}{n+1} \quad \text{for all integers } n \geq 0.$$

The advantage of defining a sequence by such an explicit formula is that each term of the sequence is uniquely determined and can be computed in a fixed, finite number of steps, by substitution.

The third way to define a sequence is to use recursion, as was done in Examples 5.3.3 and 5.4.2. This requires giving both an equation, called a *recurrence relation*, that defines each later term in the sequence by reference to earlier terms and also one or more initial values for the sequence.

Sometimes it is very difficult or impossible to find an explicit formula for a sequence, but it *is* possible to define the sequence using recursion. Note that defining sequences recursively is similar to proving theorems by mathematical induction. The recurrence relation is like the inductive step and the initial conditions are like the basis step. Indeed, the fact that sequences can be defined recursively is equivalent to the fact that mathematical induction works as a method of proof.

• Definition

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$. The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on k.

Example 5.6.1 Computing Terms of a Recursively Defined Sequence

Define a sequence $c_0, c_1, c_2, \ldots$ recursively as follows: For all integers $k \geq 2$,

(1) $c_k = c_{k-1} + kc_{k-2} + 1$ recurrence relation

(2) $c_0 = 1$ and $c_1 = 2$ initial conditions.

Find c_2, c_3, and c_4.

Solution

$$\begin{aligned} c_2 &= c_1 + 2c_0 + 1 && \text{by substituting } k = 2 \text{ into (1)} \\ &= 2 + 2\cdot 1 + 1 && \text{since } c_1 = 2 \text{ and } c_0 = 1 \text{ by (2)} \end{aligned}$$

$$\begin{aligned}
&(3)\ \therefore c_2 && = 5 \\
&\qquad\quad c_3 && = c_2 + 3c_1 + 1 && \text{by substituting } k = 3 \text{ into (1)} \\
& && = 5 + 3\cdot 2 + 1 && \text{since } c_2 = 5 \text{ by (3) and } c_1 = 2 \text{ by (2)} \\
&(4)\ \therefore c_3 && = 12 \\
&\qquad\quad c_4 && = c_3 + 4c_2 + 1 && \text{by substituting } k = 4 \text{ into (1)} \\
& && = 12 + 4\cdot 5 + 1 && \text{since } c_3 = 12 \text{ by (4) and } c_2 = 5 \text{ by (3)} \\
&(5)\ \therefore c_4 && = 33
\end{aligned}$$

■

A given recurrence relation may be expressed in several different ways.

Example 5.6.2 Writing a Recurrence Relation in More Than One Way

Note Think of the recurrence relation as $s_{\square} = 3s_{\square - 1} - 1$, where any positive integer expression may be placed in the box.

Let $s_0, s_1, s_2, \ldots$ be a sequence that satisfies the following recurrence relation:

$$\text{for all integers } k \geq 1, \quad s_k = 3s_{k-1} - 1.$$

Explain why the following statement is true:

$$\text{for all integers } k \geq 0, \quad s_{k+1} = 3s_k - 1.$$

Solution In informal language, the recurrence relation says that any term of the sequence equals 3 times the previous term minus 1. Now for any integer $k \geq 0$, the term previous to s_{k+1} is s_k. Thus for any integer $k \geq 0$, $s_{k+1} = 3s_k - 1$. ■

A sequence defined recursively need not start with a subscript of zero. Also, a given recurrence relation may be satisfied by many different sequences; the actual values of the sequence are determined by the initial conditions.

Example 5.6.3 Sequences That Satisfy the Same Recurrence Relation

Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ satisfy the recurrence relation that the kth term equals 3 times the $(k-1)$st term for all integers $k \geq 2$:

$$(1) \quad a_k = 3a_{k-1} \quad \text{and} \quad b_k = 3b_{k-1}.$$

But suppose that the initial conditions for the sequences are different:

$$(2) \quad a_1 = 2 \quad \text{and} \quad b_1 = 1.$$

Find (a) a_2, a_3, a_4 and (b) b_2, b_3, b_4.

Solution

$$\begin{aligned}
&\text{a.}\quad a_2 = 3a_1 = 3\cdot 2 = 6 &&\qquad \text{b.}\quad b_2 = 3b_1 = 3\cdot 1 = 3 \\
&\quad\;\;\; a_3 = 3a_2 = 3\cdot 6 = 18 &&\qquad\quad\;\;\; b_3 = 3b_2 = 3\cdot 3 = 9 \\
&\quad\;\;\; a_4 = 3a_3 = 3\cdot 18 = 54 &&\qquad\quad\;\;\; b_4 = 3b_3 = 3\cdot 9 = 27
\end{aligned}$$

Thus $a_1, a_2, a_3, \ldots$ begins $2, 6, 18, 54, \ldots$ and

$b_1, b_2, b_3, \ldots$ begins $1, 3, 9, 27, \ldots$.

■

Example 5.6.4 Showing That a Sequence Given by an Explicit Formula Satisfies a Certain Recurrence Relation

The sequence of **Catalan numbers**, named after the Belgian mathematician Eugène Catalan (1814–1894), arises in a remarkable variety of different contexts in discrete mathematics. It can be defined as follows: For each integer $n \geq 1$,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

a. Find C_1, C_2, and C_3.

b. Show that this sequence satisfies the recurrence relation $C_k = \dfrac{4k-2}{k+1}C_{k-1}$ for all integers $k \geq 2$

Academie Royale de Belgique

Eugène Catalan (1814–1894)

Solution

a. $C_1 = \dfrac{1}{2}\dbinom{2}{1} = \dfrac{1}{2}\cdot 2 = 1,\quad C_2 = \dfrac{1}{3}\dbinom{4}{2} = \dfrac{1}{3}\cdot 6 = 2,\quad C_3 = \dfrac{1}{4}\dbinom{6}{3} = \dfrac{1}{4}\cdot 20 = 5$

b. To obtain the kth and $(k-1)$st terms of the sequence, just substitute k and $k-1$ in place of n in the explicit formula for $C_1, C_2, C_3, \ldots$.

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

$$C_{k+1} = \frac{1}{(k-1)+1}\binom{2(k-1)}{k-1} = \frac{1}{k}\binom{2k-2}{k-1}.$$

Then start with the right-hand side of the recurrence relation and transform it into the left-hand side: For each integer $k \geq 2$,

$$\begin{aligned}
\frac{4k-2}{k+1}C_{k-1} &= \frac{4k-2}{k+1}\left[\frac{1}{k}\binom{2k-2}{k-1}\right] && \text{by substituting}\\
&= \frac{2(2k-1)}{k+1}\cdot\frac{1}{k}\cdot\frac{(2k-2)!}{(k-1)!(2k-2-(k-1))!} && \text{by the formula for } n \text{ choose } r\\
&= \frac{1}{k+1}\cdot(2(2k-1))\cdot\frac{(2k-2)!}{(k(k-1)!)(k-1)!} && \text{by rearranging the factors}\\
&= \frac{1}{k+1}\cdot(2(2k-1))\cdot\frac{1}{k!(k-1)!}\cdot(2k-2)!\cdot\frac{1}{2}\cdot\frac{1}{k}\cdot 2k. && \text{because } \tfrac{1}{2}\cdot\tfrac{1}{k}\cdot 2k = 1\\
&= \frac{1}{k+1}\cdot\frac{2}{2}\cdot\frac{1}{k!}\cdot\frac{1}{(k-1)!}\cdot\frac{1}{k}\cdot(2k)\cdot(2k-1)\cdot(2k-2)! && \text{by rearranging the factors}\\
&= \frac{1}{k+1}\cdot\frac{(2k)!}{k!k!} && \text{because } k(k-1)! = k!,\ \tfrac{2}{2} = 1, \text{ and } 2k\cdot(2k-1)\cdot(2k-2)! = (2k)!\\
&= \frac{1}{k+1}\binom{2k}{k} && \text{by the formula for } n \text{ choose } r\\
&= C_k && \text{by defintion of } C_1, C_2, C_3, \ldots.
\end{aligned}$$

■

Examples of Recursively Defined Sequences

Recursion is one of the central ideas of computer science. To solve a problem recursively means to find a way to break it down into smaller subproblems each having the same form as the original problem—and to do this in such a way that when the process is repeated many times, the last of the subproblems are small and easy to solve and the solutions of the subproblems can be woven together to form a solution to the original problem.

Probably the most difficult part of solving problems recursively is to figure out how knowing the solution to smaller subproblems of the same type as the original problem will give you a solution to the problem as a whole. You *suppose* you know the solutions to smaller subproblems and ask yourself how you would best make use of that knowledge to solve the larger problem. The supposition that the smaller subproblems have already been solved has been called the *recursive paradigm* or the *recursive leap of faith*. Once you take this leap, you are right in the middle of the most difficult part of the problem, but generally, the path to a solution from this point, though difficult, is short. The recursive leap of faith is similar to the inductive hypothesis in a proof by mathematical induction.

Example 5.6.5 The Tower of Hanoi


Courtesy of Francis Lucas

Édouard Lucas (1842–1891)

In 1883 a French mathematician, Édouard Lucas, invented a puzzle that he called The Tower of Hanoi (La Tour D'Hanoï). The puzzle consisted of eight disks of wood with holes in their centers, which were piled in order of decreasing size on one pole in a row of three. A facsimile of the cover of the box is shown in Figure 5.6.1. Those who played the game were supposed to move all the disks one by one from one pole to another, never placing a larger disk on top of a smaller one. The directions to the puzzle claimed it was based on an old Indian legend:

> *On the steps of the altar in the temple of Benares, for many, many years Brahmins have been moving a tower of 64 golden disks from one pole to another; one by one, never placing a larger on top of a smaller. When all the disks have been transferred the Tower and the Brahmins will fall, and it will be the end of the world.*

Courtesy of Paul Stockmeyer

Figure 5.6.1

The puzzle offered a prize of ten thousand francs (about $34,000 US today) to anyone who could move a tower of 64 disks by hand while following the rules of the game. (See Figure 5.6.2 on the following page.) Assuming that you transferred the disks as efficiently as possible, how many moves would be required to win the prize?

Note that the numbers m_n are independent of the labeling of the poles; it takes the same minimum number of moves to transfer n disks from pole A to pole C as to transfer n disks from pole A to pole B, for example. Also the values of m_n are independent of the number of larger disks that may lie below the top n, provided these remain stationary while the top n are moved. Because the disks on the bottom are all larger than the ones on the top, the top disks can be moved from pole to pole as though the bottom disks were not present.

Going from position (a) to position (b) requires m_{k-1} moves, going from position (b) to position (c) requires just one move, and going from position (c) to position (d) requires m_{k-1} moves. By substitution into equation (5.6.1), therefore,

$$\begin{aligned} m_k &= m_{k-1} + 1 + m_{k-1} \\ &= 2m_{k-1} + 1 \qquad \text{for all integers } k \geq 2. \end{aligned}$$

The initial condition, or base, of this recursion is found by using the definition of the sequence. Because just one move is needed to move one disk from one pole to another,

$$m_1 = \left[\begin{array}{l}\text{the minimum number of moves needed to move}\\ \text{a tower of one disk from one pole to another}\end{array}\right] = 1.$$

Hence the complete recursive specification of the sequence $m_1, m_2, m_3, \ldots$ is as follows: For all integers $k \geq 2$,

$$\begin{aligned} &(1) \quad m_k = 2m_{k-1} + 1 && \text{recurrence relation} \\ &(2) \quad m_1 = 1 && \text{initial conditions} \end{aligned}$$

Here is a computation of the next five terms of the sequence:

$$\begin{aligned} &(3) \quad m_2 = 2m_1 + 1 = 2 \cdot 1 + 1 = 3 && \text{by (1) and (2)} \\ &(4) \quad m_3 = 2m_2 + 1 = 2 \cdot 3 + 1 = 7 && \text{by (1) and (3)} \\ &(5) \quad m_4 = 2m_3 + 1 = 2 \cdot 7 + 1 = 15 && \text{by (1) and (4)} \\ &(6) \quad m_5 = 2m_4 + 1 = 2 \cdot 15 + 1 = 31 && \text{by (1) and (5)} \\ &(7) \quad m_6 = 2m_5 + 1 = 2 \cdot 31 + 1 = 63 && \text{by (1) and (6)} \end{aligned}$$

Going back to the legend, suppose the priests work rapidly and move one disk every second. Then the time from the beginning of creation to the end of the world would be m_{64} seconds. In the next section we derive an explicit formula for m_n. Meanwhile, we can compute m_{64} on a calculator or a computer by continuing the process started above (Try it!). The approximate result is

$$\begin{aligned} 1.844674 \times 10^{19} \text{ seconds} &\cong 5.84542 \times 10^{11} \text{ years} \\ &\cong 584.5 \text{ billion years,} \end{aligned}$$

which is obtained by the estimate of

$$\underset{\substack{\uparrow\\ \text{seconds per}\\ \text{minute}}}{60} \cdot \underset{\substack{\uparrow\\ \text{minutes}\\ \text{per}\\ \text{hour}}}{60} \cdot \underset{\substack{\nwarrow\\ \text{hours}\\ \text{per}\\ \text{day}}}{24} \cdot \underset{\substack{\nwarrow\\ \text{days}\\ \text{per}\\ \text{year}}}{(365.25)} = \underset{\substack{\uparrow\\ \text{seconds}\\ \text{per}\\ \text{year}}}{31,557,600}$$

seconds in a year (figuring 365.25 days in a year to take leap years into account). Surprisingly, this figure is close to some scientific estimates of the life of the universe! ■

Example 5.6.6 The Fibonacci Numbers


Bettmann/CORBIS

Fibonacci (Leonardo of Pisa) (ca. 1175–1250)

One of the earliest examples of a recursively defined sequence arises in the writings of Leonardo of Pisa, commonly known as Fibonacci, who was the greatest European mathematician of the Middle Ages. In 1202 Fibonacci posed the following problem:

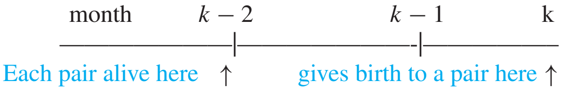
A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

1. Rabbit pairs are not fertile during their first month of life but thereafter give birth to one new male/female pair at the end of every month.
2. No rabbits die.

How many rabbits will there be at the end of the year?

Solution One way to solve this problem is to plunge right into the middle of it using recursion. Suppose you know how many rabbit pairs there were at the ends of previous months. How many will there be at the end of the current month?

The crucial observation is that the number of rabbit pairs born at the end of month k is the same as the number of pairs alive at the end of month $k-2$. Why? Because it is exactly the rabbit pairs that were alive at the end of month $k-2$ that were fertile during month k. The rabbits born at the end of month $k-1$ were not.



Now the number of rabbit pairs alive at the end of month k equals the ones alive at the end of month $k-1$ plus the pairs newly born at the end of the month. Thus

Note It is essential to rephrase this observation in terms of a sequence.

$$\left[\begin{array}{l}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k\end{array}\right] = \left[\begin{array}{l}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-1\end{array}\right] + \left[\begin{array}{l}\text{the number}\\ \text{of rabbit}\\ \text{pairs born}\\ \text{at the end}\\ \text{of month } k\end{array}\right]$$

$$= \left[\begin{array}{l}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-1\end{array}\right] + \left[\begin{array}{l}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-2\end{array}\right] \tag{5.6.2}$$

For each integer $n \geq 1$, let

$$F_n = \left[\begin{array}{l}\text{the number of rabbit pairs}\\ \text{alive at the end of month } n\end{array}\right]$$

and let

$$\begin{aligned}F_0 &= \text{the initial number of rabbit pairs}\\ &= 1.\end{aligned}$$

Then by substitution into equation (5.6.2), for all integers $k \geq 2$,

$$F_k = F_{k-1} + F_{k-2}.$$

Now $F_0 = 1$, as already noted, and $F_1 = 1$ also, because the first pair of rabbits is not fertile until the second month. Hence the complete specification of the Fibonacci sequence is as follows: For all integers $k \geq 2$,

$$\begin{array}{lll} (1) & F_k = F_{k-1} + F_{k-2} & \text{recurrence relation} \\ (2) & F_0 = 1, \quad F_1 = 1 & \text{initial conditions.} \end{array}$$

To answer Fibonacci's question, compute F_2, F_3, and so forth through F_{12}:

$$\begin{array}{lllllll} (3) & F_2 & = F_1 & + F_0 & = 1 + 1 & = 2 & \text{by (1) and (2)} \\ (4) & F_3 & = F_2 & + F_1 & = 2 + 1 & = 3 & \text{by (1), (2) and (3)} \\ (5) & F_4 & = F_3 & + F_2 & = 3 + 2 & = 5 & \text{by (1), (3) and (4)} \\ (6) & F_5 & = F_4 & + F_3 & = 5 + 3 & = 8 & \text{by (1), (4) and (5)} \\ (7) & F_6 & = F_5 & + F_4 & = 8 + 5 & = 13 & \text{by (1), (5) and (6)} \\ (8) & F_7 & = F_6 & + F_5 & = 13 + 8 & = 21 & \text{by (1), (6) and (7)} \\ (9) & F_8 & = F_7 & + F_6 & = 21 + 13 & = 34 & \text{by (1), (7) and (8)} \\ (10) & F_9 & = F_8 & + F_7 & = 34 + 21 & = 55 & \text{by (1), (8) and (9)} \\ (11) & F_{10} & = F_9 & + F_8 & = 55 + 34 & = 89 & \text{by (1), (9) and (10)} \\ (12) & F_{11} & = F_{10} & + F_9 & = 89 + 55 & = 144 & \text{by (1), (10) and (11)} \\ (13) & F_{12} & = F_{11} & + F_{10} & = 144 + 89 & = 233 & \text{by (1), (11) and (12)} \end{array}$$

At the end of the twelfth month there are 233 rabbit pairs, or 466 rabbits in all. ■

Example 5.6.7 Compound Interest

On your twenty-first birthday you get a letter informing you that on the day you were born an eccentric rich aunt deposited \$100,000 in a bank account earning 4% interest compounded annually and she now intends to turn the account over to you, provided you can figure out how much it is worth. What is the amount currently in the account?

Solution To approach this problem recursively, observe that

$$\left[\begin{array}{l}\text{the amount in}\\ \text{the account at}\\ \text{the end of any}\\ \text{particular year}\end{array}\right] = \left[\begin{array}{l}\text{the amount in}\\ \text{the account at}\\ \text{the end of the}\\ \text{previous year}\end{array}\right] + \left[\begin{array}{l}\text{the interest}\\ \text{earned on the}\\ \text{account during}\\ \text{the year}\end{array}\right].$$

Now the interest earned during the year equals the interest rate, $4\% = 0.04$ times the amount in the account at the end of the previous year. Thus

$$\left[\begin{array}{l}\text{the amount in}\\ \text{the account at}\\ \text{the end of any}\\ \text{particular year}\end{array}\right] = \left[\begin{array}{l}\text{the amount in}\\ \text{the account at}\\ \text{the end of the}\\ \text{previous year}\end{array}\right] + (0.04) \cdot \left[\begin{array}{l}\text{the amount in}\\ \text{the account at}\\ \text{the end of the}\\ \text{previous year}\end{array}\right]. \qquad 5.6.3$$

For each positive integer n, let

Note Again, a crucial step is to define the sequence explicitly.

$$A_n = \left[\begin{array}{l}\text{the amount in the account}\\ \text{at the end of year } n\end{array}\right]$$

and let

$$A_0 = \left[\begin{array}{l}\text{the initial amount}\\ \text{in the account}\end{array}\right] = \$100,000.$$

Then for any particular year k, substitution into equation (5.6.3) gives

$$\begin{aligned} A_k &= A_{k-1} + (0.04)\cdot A_{k-1} \\ &= (1+0.04)\cdot A_{k-1} = (1.04)\cdot A_{k-1} \qquad \text{by factoring out } A_{k-1}. \end{aligned}$$

Consequently, the values of the sequence $A_0, A_1, A_2, \ldots$ are completely specified as follows: for all integers $k \geq 1$,

$$(1) \quad A_k = (1.04)\cdot A_{k-1} \qquad \text{recurrence relation}$$

$$(2) \quad A_0 = \$100,000 \qquad \text{initial condition.}$$

The number 1.04 is called the *growth factor* of the sequence.

In the next section we derive an explicit formula for the value of the account in any year n. The value on your twenty-first birthday can also be computed by repeated substitution as follows:

$$\begin{array}{lllll} (3) \; A_1 & = 1.04\cdot A_0 & = (1.04)\cdot\$100,000 & = \$104,000 & \text{by (1) and (2)} \\ (4) \; A_2 & = 1.04\cdot A_1 & = (1.04)\cdot\$104,000 & = \$108,160 & \text{by (1) and (3)} \\ (5) \; A_3 & = 1.04\cdot A_2 & = (1.04)\cdot\$108,160 & = \$112,486.40 & \text{by (1) and (4)} \\ & \vdots & & \vdots & \\ (22) \; A_{20} & = 1.04\cdot A_{19} & \cong (1.04)\cdot\$210,684.92 & \cong \$219,112.31 & \text{by (1) and (21)} \\ (23) \; A_{21} & = 1.04\cdot A_{20} & \cong (1.04)\cdot\$219,112.31 & \cong \$227,876.81 & \text{by (1) and (22)} \end{array}$$

The amount in the account is \$227,876.81 (to the nearest cent). Fill in the dots (to check the arithmetic) and collect your money! ■

Example 5.6.8 Compound Interest with Compounding Several Times a Year

When an annual interest rate of i is compounded m times per year, the interest rate paid per period is i/m. For instance, if $3\% = 0.03$ annual interest is compounded quarterly, then the interest rate paid per quarter is $0.03/4 = 0.0075$.

For each integer $k \geq 1$, let $P_k =$ the amount on deposit at the end of the kth period, assuming no additional deposits or withdrawals. Then the interest earned during the kth period equals the amount on deposit at the end of the $(k-1)$st period times the interest rate for the period:

$$\text{interest earned during } k\text{th period} = P_{k-1}\left(\frac{i}{m}\right).$$

The amount on deposit at the end of the kth period, P_k, equals the amount at the end of the $(k-1)$st period, P_{k-1}, plus the interest earned during the kth period:

$$P_k = P_{k-1} + P_{k-1}\left(\frac{i}{m}\right) = P_{k-1}\left(1 + \frac{i}{m}\right). \tag{5.6.4}$$

Suppose \$10,000 is left on deposit at 3% compounded quarterly.

a. How much will the account be worth at the end of one year, assuming no additional deposits or withdrawals?

b. The **annual percentage rate (APR)** is the percentage increase in the value of the account over a one-year period. What is the APR for this account?

Solution

a. For each integer $n \geq 1$, let $P_n =$ the amount on deposit after n consecutive quarters, assuming no additional deposits or withdrawals, and let P_0 be the initial \$10,000. Then by equation (5.6.4) with $i = 0.03$ and $m = 4$, a recurrence relation for the sequence $P_0, P_1, P_2, \ldots$ is

$$(1) \quad P_k = P_{k-1}(1 + 0.0075) = (1.0075) \cdot P_{k-1} \qquad \text{for all integers k} \geq 1.$$

The amount on deposit at the end of one year (four quarters), P_4, can be found by successive substitution:

$$(2) \quad P_0 = \$10,000$$

$$(3) \quad P_1 = 1.0075 \cdot P_0 = (1.0075) \cdot \$10,000.00 = \$10,075.00 \qquad \text{by (1) and (2)}$$

$$(4) \quad P_2 = 1.0075 \cdot P_1 = (1.0075) \cdot \$10,075.00 = \$10,150.56 \qquad \text{by (1) and (3)}$$

$$(5) \quad P_3 = 1.0075 \cdot P_2 \cong (1.0075) \cdot \$10,150.56 = \$10,226.69 \qquad \text{by (1) and (4)}$$

$$(6) \quad P_4 = 1.0075 \cdot P_3 \cong (1.0075) \cdot \$10,226.69 = \$10,303.39 \qquad \text{by (1) and (5)}$$

Hence after one year there is \$10,303.39 (to the nearest cent) in the account.

b. The percentage increase in the value of the account, or APR, is

$$\frac{10303.39 - 10000}{10000} = 0.03034 = 3.034\%.$$

■

Recursive Definitions of Sum and Product

Addition and multiplication are called *binary* operations because only two numbers can be added or multiplied at a time. Careful definitions of sums and products of more than two numbers use recursion.

• Definition

Given numbers $a_1, a_2, \ldots, a_n$, where n is a positive integer, the **summation from $i = 1$ to n of the a_i**, denoted $\sum_{i=1}^{n} a_i$, is defined as follows:

$$\sum_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n, \quad \text{if } n > 1.$$

The **product from $i = 1$ to n of the a_i**, denoted $\prod_{i=1}^{n} a_i$, is defined by

$$\prod_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n, \quad \text{if } n > 1.$$

The effect of these definitions is to specify an *order* in which sums and products of more than two numbers are computed. For example,

$$\sum_{i=1}^{4} a_i = \left(\sum_{i=1}^{3} a_i\right) + a_4 = \left(\left(\sum_{i=1}^{2} a_i\right) + a_3\right) + a_4 = ((a_1 + a_2) + a_3) + a_4.$$

The recursive definitions are used with mathematical induction to establish various properties of general finite sums and products.

Example 5.6.9 A Sum of Sums

Prove that for any positive integer n, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

Solution The proof is by mathematical induction. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \qquad \leftarrow P(n)$$

We must show that $P(n)$ is true for all integers $n \geq 0$. We do this by mathematical induction on n.

Show that P(1) is true: To establish $P(1)$, we must show that

$$\sum_{i=1}^{1}(a_i + b_i) = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i. \qquad \leftarrow P(1)$$

But

$$\begin{aligned}\sum_{i=1}^{1}(a_i + b_i) &= a_1 + b_1 && \text{by definition of } \Sigma\\ &= \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i && \text{also by definition of } \Sigma.\end{aligned}$$

Hence $P(1)$ is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is also true: Suppose $a_1, a_2, \ldots, a_k, a_{k+1}$ and $b_1, b_2, \ldots, b_k, b_{k+1}$ are real numbers and that for some $k \geq 1$

$$\sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i. \qquad \begin{array}{l}\leftarrow P(k)\\ \text{inductive hypothesis}\end{array}$$

We must show that

$$\sum_{i=1}^{k+1}(a_i + b_i) = \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of this equation equals the right-hand side.]

But the left-hand side of the equation is

$$\begin{aligned}\sum_{i=1}^{k+1}(a_i + b_i) &= \sum_{i=1}^{k}(a_i + b_i) + (a_{k+1} + b_{k+1}) && \text{by definition of } \Sigma\\ &= \left(\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i\right) + (a_{k+1} + b_{k+1}) && \text{by inductive hypothesis}\\ &= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) + \left(\sum_{i=1}^{k} b_i + b_{k+1}\right) && \text{by the associative and cummutative laws of algebra}\\ &= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i && \text{by definition of } \Sigma\end{aligned}$$

which equals the right-hand side of the equation. *[This is what was to be shown.]* ■

Test Yourself

1. A recursive definition for a sequence consists of a _____ and _____.

2. A recurrence relation is an equation that defines each later term of a sequence by reference to _____ in the sequence.

3. Initial conditions for a recursive definition of a sequence consist of one or more of the _____ of the sequence.

4. To solve a problem recursively means to divide the problem into smaller subproblems of the same type as the initial problem, to suppose _____, and to figure out how to use the supposition to _____.

5. A crucial step for solving a problem recursively is to define a _____ in terms of which the recurrence relation and initial conditions can be specified.

Exercise Set 5.6

Find the first four terms of each of the recursively defined sequences in 1–8.

1. $a_k = 2a_{k-1} + k$, for all integers $k \geq 2$
$a_1 = 1$

2. $b_k = b_{k-1} + 3k$, for all integers $k \geq 2$
$b_1 = 1$

3. $c_k = k(c_{k-1})^2$, for all integers $k \geq 1$
$c_0 = 1$

4. $d_k = k(d_{k-1})^2$, for all integers $k \geq 1$
$d_0 = 3$

5. $s_k = s_{k-1} + 2s_{k-2}$, for all integers $k \geq 2$
$s_0 = 1,\ s_1 = 1$

6. $t_k = t_{k-1} + 2t_{k-2}$, for all integers $k \geq 2$
$t_0 = -1,\ t_1 = 2$

7. $u_k = ku_{k-1} - u_{k-2}$, for all integers $k \geq 3$
$u_1 = 1,\ u_2 = 1$

8. $v_k = v_{k-1} + v_{k-2} + 1$, for all integers $k \geq 3$
$v_1 = 1,\ v_2 = 3$

9. Let $a_0, a_1, a_2, \ldots$ be defined by the formula $a_n = 3n + 1$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = a_{k-1} + 3$, for all integers $k \geq 1$.

10. Let $b_0, b_1, b_2, \ldots$ be defined by the formula $b_n = 4^n$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $b_k = 4b_{k-1}$, for all integers $k \geq 1$.

11. Let $c_0, c_1, c_2, \ldots$ be defined by the formula $c_n = 2^n - 1$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$c_k = 2c_{k-1} + 1.$$

12. Let $s_0, s_1, s_2, \ldots$ be defined by the formula $s_n = \dfrac{(-1)^n}{n!}$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$s_k = \frac{-s_{k-1}}{k}.$$

13. Let $t_0, t_1, t_2, \ldots$ be defined by the formula $t_n = 2 + n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$t_k = 2t_{k-1} - t_{k-2}.$$

14. Let $d_0, d_1, d_2, \ldots$ be defined by the formula $d_n = 3^n - 2^n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$d_k = 5d_{k-1} - 6d_{k-2}.$$

***H* 15.** For the sequence of Catalan numbers defined in Example 5.6.4, prove that for all integers $n \geq 1$,

$$C_n = \frac{1}{4n+2}\binom{2n+2}{n+1}.$$

16. Use the recurrence relation and values for the Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ discussed in Example 5.6.5 to compute m_7 and m_8.

17. *Tower of Hanoi with Adjacency Requirement*: Suppose that in addition to the requirement that they never move a larger disk on top of a smaller one, the priests who move the disks of the Tower of Hanoi are also allowed only to move disks one by one from one pole to an *adjacent* pole. Assume poles A and C are at the two ends of the row and pole B is in the middle. Let

$$a_n = \left[\begin{array}{l}\text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } C\end{array}\right].$$

a. Find a_1, a_2, and a_3. b. Find a_4.
c. Find a recurrence relation for $a_1, a_2, a_3, \ldots$.

18. *Tower of Hanoi with Adjacency Requirement*: Suppose the same situation as in exercise 17. Let

$$b_n = \left[\begin{array}{l}\text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } B\end{array}\right].$$

a. Find b_1, b_2, and b_3. **b.** Find b_4.

c. Show that $b_k = a_{k-1} + 1 + b_{k-1}$ for all integers $k \geq 2$, where $a_1, a_2, a_3, \ldots$ is the sequence defined in exercise 17.

d. Show that $b_k \leq 3b_{k-1} + 1$ for all integers $k \geq 2$.

H ✶ **e.** Show that $b_k = 3b_{k-1} + 1$ for all integers $k \geq 2$.

19. *Four-Pole Tower of Hanoi*: Suppose that the Tower of Hanoi problem has four poles in a row instead of three. Disks can be transferred one by one from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Let s_n be the minimum number of moves needed to transfer the entire tower of n disks from the left-most to the right-most pole.

a. Find s_1, s_2, and s_3. **b.** Find s_4.

c. Show that $s_k \leq 2s_{k-2} + 3$ for all integers $k \geq 3$.

20. *Tower of Hanoi Poles in a Circle:* Suppose that instead of being lined up in a row, the three poles for the original Tower of Hanoi are placed in a circle. The monks move the disks one by one from one pole to another, but they may only move disks one over in a clockwise direction and they may never move a larger disk on top of a smaller one. Let c_n be the minimum number of moves needed to transfer a pile of n disks from one pole to the next adjacent pole in the clockwise direction.

a. Justify the inequality $c_k \leq 4c_{k-1} + 1$ for all integers $k \geq 2$.

b. The expression $4c_{k-1} + 1$ is not the minimum number of moves needed to transfer a pile of k disks from one pole to another. Explain, for example, why $c_3 \neq 4c_2 + 1$.

21. *Double Tower of Hanoi*: In this variation of the Tower of Hanoi there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially one of the poles contains all the disks placed on top of each other in pairs of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let t_n be the minimum number of moves needed to transfer a tower of $2n$ disks from one pole to another.

a. Find t_1 and t_2. **b.** Find t_3.

c. Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

22. *Fibonacci Variation*: A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions (which are more realistic than Fibonacci's):

(1) Rabbit pairs are not fertile during their first month of life but thereafter give birth to four new male/female pairs at the end of every month.

(2) No rabbits die.

a. Let r_n = the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $r_0 = 1$. Find a recurrence relation for $r_0, r_1, r_2, \ldots$.

b. Compute $r_0, r_1, r_2, r_3, r_4, r_5$, and r_6.

c. How many rabbits will there be at the end of the year?

23. *Fibonacci Variation*: A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

(1) Rabbit pairs are not fertile during their first *two* months of life, but thereafter give birth to three new male/female pairs at the end of every month.

(2) No rabbits die.

a. Let s_n = the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $s_0 = 1$. Find a recurrence relation for $s_0, s_1, s_2, \ldots$.

b. Compute s_0, s_1, s_2, s_3, s_4, and s_5.

c. How many rabbits will there be at the end of the year?

In 24–34, $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

24. Use the recurrence relation and values for $F_0, F_1, F_2, \ldots$ given in Example 5.6.6 to compute F_{13} and F_{14}.

25. The Fibonacci sequence satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$, for all integers $k \geq 2$.

a. Explain why the following is true:

$$F_{k+1} = F_k + F_{k-1} \quad \text{for all integers } k \geq 1.$$

b. Write an equation expressing F_{k+2} in terms of F_{k+1} and F_k.

c. Write an equation expressing F_{k+3} in terms of F_{k+2} and F_{k+1}

26. Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for all integers $k \geq 4$.

27. Prove that $F_k^2 - F_{k-1}^2 = F_k F_{k-1} - F_{k+1} F_{k-1}$, for all integers $k \geq 1$.

28. Prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_k F_{k-1}$, for all integers $k \geq 1$.

29. Prove that $F_{k+1}^2 - F_k^2 = F_{k-1} F_{k+2}$, for all integers $k \geq 1$.

30. Use mathematical induction to prove that for all integers $n \geq 0$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$.

✶ 31. Use strong mathematical induction to prove that $F_n < 2^n$ for all integers $n \geq 1$.

H ✶ **32.** Let $F_0, F_1, F_2, \ldots$ be the Fibonacci sequence defined in Section 5.6. Prove that for all integers $n \geq 0$, $\gcd(F_{n+1}, F_n) = 1$.

33. It turns out that the Fibonacci sequence satisfies the following explicit formula: For all integers $F_n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Verify that the sequence defined by this formula satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$ for all integers $k \geq 2$.

H **34.** (For students who have studied calculus) Find $\lim_{n \to \infty}\left(\frac{F_{n+1}}{F_n}\right)$, assuming that the limit exists.

H ✶35. (For students who have studied calculus) Prove that $\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$ exists.

36. (For students who have studied calculus) Define $x_0, x_1, x_2, \ldots$ as follows:

$$x_k = \sqrt{2 + x_{k-1}} \qquad \text{for all integers } k \geq 1$$
$$x_0 = 0$$

Find $\lim_{n\to\infty} x_n$. (Assume that the limit exists.)

37. *Compound Interest*: Suppose a certain amount of money is deposited in an account paying 4% annual interest compounded quarterly. For each positive integer n, let $R_n =$ the amount on deposit at the end of the nth quarter, assuming no additional deposits or withdrawals, and let R_0 be the initial amount deposited.

a. Find a recurrence relation for $R_0, R_1, R_2, \ldots$.

b. If $R_0 = \$5000$, find the amount of money on deposit at the end of one year.

c. Find the APR for the account.

38. *Compound Interest*: Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer n, let $S_n =$ the amount on deposit at the end of the nth month, and let S_0 be the initial amount deposited.

a. Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year.

b. If $S_0 = \$10,000$, find the amount of money on deposit at the end of one year.

c. Find the APR for the account.

39. With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one- and two-stair increments. For each integer $n \geq 1$, if the staircase consists of n stairs, let c_n be the number of different ways to climb the staircase. Find a recurrence relation for $c_1, c_2, c_3, \ldots$.

40. A set of blocks contains blocks of heights 1, 2, and 4 centimeters. Imagine constructing towers by piling blocks of different heights directly on top of one another. (A tower of height 6 cm could be obtained using six 1-cm blocks, three 2-cm blocks one 2-cm block with one 4-cm block on top, one 4-cm block with one 2-cm block on top, and so forth.) Let t be the number of ways to construct a tower of height n cm using blocks from the set. (Assume an unlimited supply of blocks of each size.) Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

41. Use the recursive definition of summation, together with mathematical induction, to prove the generalized distributive law that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and c are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right).$$

42. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\prod_{i=1}^{n}(a_i b_i) = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right).$$

43. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and c are real numbers, then

$$\prod_{i=1}^{n}(ca_i) = c^n\left(\prod_{i=1}^{n} a_i\right).$$

H 44. The triangle inequality for absolute value states that for all real numbers a and b, $|a+b| \leq |a| + |b|$. Use the recursive definition of summation, the triangle inequality, the definition of absolute value, and mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ are real numbers, then

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

Answers for Test Yourself

1. recurrence relation; initial conditions 2. earlier terms 3. values of the first few terms 4. that the smaller subproblems have already been solved; solve the initial problem 5. sequence

5.7 Solving Recurrence Relations by Iteration

The keener one's sense of logical deduction, the less often one makes hard and fast inferences. — Bertrand Russell, 1872–1970

Suppose you have a sequence that satisfies a certain recurrence relation and initial conditions. It is often helpful to know an explicit formula for the sequence, especially if