H ✶ **35.** (For students who have studied calculus) Prove that $\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$ exists.

36. (For students who have studied calculus) Define $x_0, x_1, x_2, \ldots$ as follows:

$$x_k = \sqrt{2 + x_{k-1}} \quad \text{for all integers } k \geq 1$$
$$x_0 = 0$$

Find $\lim_{n\to\infty} x_n$. (Assume that the limit exists.)

**37.** *Compound Interest*: Suppose a certain amount of money is deposited in an account paying 4% annual interest compounded quarterly. For each positive integer $n$, let $R_n =$ the amount on deposit at the end of the $n$th quarter, assuming no additional deposits or withdrawals, and let $R_0$ be the initial amount deposited.
a. Find a recurrence relation for $R_0, R_1, R_2, \ldots$.
b. If $R_0 = \$5000$, find the amount of money on deposit at the end of one year.
c. Find the APR for the account.

38. *Compound Interest*: Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer $n$, let $S_n =$ the amount on deposit at the end of the $n$th month, and let $S_0$ be the initial amount deposited.
a. Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year.
b. If $S_0 = \$10,000$, find the amount of money on deposit at the end of one year.
c. Find the APR for the account.

**39.** With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one- and two-stair increments. For each integer $n \geq 1$, if the staircase consists of $n$ stairs, let $c_n$ be the number of different ways to climb the staircase. Find a recurrence relation for $c_1, c_2, c_3, \ldots$.

40. A set of blocks contains blocks of heights 1, 2, and 4 centimeters. Imagine constructing towers by piling blocks of different heights directly on top of one another. (A tower of height 6 cm could be obtained using six 1-cm blocks, three 2-cm blocks one 2-cm block with one 4-cm block on top, one 4-cm block with one 2-cm block on top, and so forth.) Let $t$ be the number of ways to construct a tower of height $n$ cm using blocks from the set. (Assume an unlimited supply of blocks of each size.) Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

**41.** Use the recursive definition of summation, together with mathematical induction, to prove the generalized distributive law that for all positive integers $n$, if $a_1, a_2, \ldots, a_n$ and $c$ are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right).$$

42. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers $n$, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\prod_{i=1}^{n}(a_i b_i) = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right).$$

43. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers $n$, if $a_1, a_2, \ldots, a_n$ and $c$ are real numbers, then

$$\prod_{i=1}^{n}(ca_i) = c^n\left(\prod_{i=1}^{n} a_i\right).$$

H **44.** The triangle inequality for absolute value states that for all real numbers $a$ and $b$, $|a + b| \leq |a| + |b|$. Use the recursive definition of summation, the triangle inequality, the definition of absolute value, and mathematical induction to prove that for all positive integers $n$, if $a_1, a_2, \ldots, a_n$ are real numbers, then

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

## Answers for Test Yourself

1. recurrence relation; initial conditions 2. earlier terms 3. values of the first few terms 4. that the smaller subproblems have already been solved; solve the initial problem 5. sequence

# 5.7 Solving Recurrence Relations by Iteration

*The keener one's sense of logical deduction, the less often one makes hard and fast inferences.* — Bertrand Russell, 1872–1970

Suppose you have a sequence that satisfies a certain recurrence relation and initial conditions. It is often helpful to know an explicit formula for the sequence, especially if you need to compute terms with very large subscripts or if you need to examine general properties of the sequence. Such an explicit formula is called a **solution** to the recurrence relation. In this section, we discuss methods for solving recurrence relations. For example, in the text and exercises of this section, we will show that the Tower of Hanoi sequence of Example 5.6.5 satisfies the formula

$$m_n = 2^n - 1,$$

and that the compound interest sequence of Example 5.6.7 satisfies

$$A_n = (1.04)^n \cdot \$100,000.$$

## The Method of Iteration

The most basic method for finding an explicit formula for a recursively defined sequence is **iteration**. Iteration works as follows: Given a sequence $a_0, a_1, a_2, \ldots$ defined by a recurrence relation and initial conditions, you start from the initial conditions and calculate successive terms of the sequence until you see a pattern developing. At that point you guess an explicit formula.

### Example 5.7.1 Finding an Explicit Formula

Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively as follows: For all integers $k \geq 1$,

$$\text{(1)} \quad a_k = a_{k-1} + 2 \qquad \text{recurrence relation}$$

$$\text{(2)} \quad a_0 = 1 \qquad \text{initial condition.}$$

Use iteration to guess an explicit formula for the sequence.

**Solution** Recall that to say

$$a_k = a_{k-1} + 2 \quad \text{for all integers } k \geq 1$$

means

$$a_{\square} = a_{\square - 1} + 2 \qquad \text{no matter what positive integer is placed into the box } \square.$$

In particular,

$$a_1 = a_0 + 2,$$
$$a_2 = a_1 + 2,$$
$$a_3 = a_2 + 2,$$

and so forth. Now use the initial condition to begin a process of successive substitutions into these equations, not just of numbers (as was done in Section 5.6) but of *numerical expressions*.

The reason for using numerical expressions rather than numbers is that in these problems you are seeking a numerical pattern that underlies a general formula. The secret of success is to leave most of the arithmetic undone. However, you do need to eliminate parentheses as you go from one step to the next. Otherwise, you will soon end up with a bewilderingly large nest of parentheses. Also, it is nearly always helpful to use shorthand notations for regrouping additions, subtractions, and multiplications of numbers that repeat. Thus, for instance, you would write

$$5 \cdot 2 \qquad \text{instead of } 2 + 2 + 2 + 2 + 2$$

and

$$2^5 \qquad \text{instead of } 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

Notice that you don't lose any information about the number patterns when you use these shorthand notations.

Here's how the process works for the given sequence:

$$a_0 = 1$$ the initial condition

$$a_1 = a_0 + 2 = 1 + 2$$ by substitution

$$a_2 = a_1 + 2 = (1 + 2) + 2 \qquad = 1 + 2 + 2$$ eliminate parentheses

$$a_3 = a_2 + 2 = (1 + 2 + 2) + 2 \qquad = 1 + 2 + 2 + 2$$ eliminate parentheses again; write $3 \cdot 2$ instead of $2 + 2 + 2$?

$$a_4 = a_3 + 2 = (1 + 2 + 2 + 2) + 2 = 1 + 2 + 2 + 2 + 2$$ eliminate parentheses again; definitely write $4 \cdot 2$ instead of $2 + 2 + 2 + 2$—the length of the string of 2's is getting out of hand.

**Tip** Do no arithmetic *except*

- replace $n \cdot 1$ and $1 \cdot n$ by $n$
- reformat repeated numbers
- get rid of parentheses

Since it appears helpful to use the shorthand $k \cdot 2$ in place of $2 + 2 + \cdots + 2$ ($k$ times), we do so, starting again from $a_0$.

$$a_0 = 1 \qquad = 1 + 0 \cdot 2$$ the initial condition

$$a_1 = a_0 + 2 = 1 + 2 \qquad = 1 + 1 \cdot 2$$ by substitution

$$a_2 = a_1 + 2 = (1 + 2) + 2 \quad = 1 + 2 \cdot 2$$

$$a_3 = a_2 + 2 = (1 + 2 \cdot 2) + 2 = 1 + 3 \cdot 2$$

$$a_4 = a_3 + 2 = (1 + 3 \cdot 2) + 2 = 1 + 4 \cdot 2$$ At this point it certainly seems likely that the general pattern is $1 + n \cdot 2$; check whether the next calculation supports this.

$$a_5 = a_4 + 2 = (1 + 4 \cdot 2) + 2 = 1 + 5 \cdot 2$$ It does! So go ahead and write an answer. It's only a guess, after all.

$$\vdots$$

*Guess:* $a_n = 1 + n \cdot 2 = 1 + 2n$

The answer obtained for this problem is just a guess. To be sure of the correctness of this guess, you will need to check it by mathematical induction. Later in this section, we will show how to do this. ■

A sequence like the one in Example 5.7.1, in which each term equals the previous term plus a fixed constant, is called an *arithmetic sequence*. In the exercises at the end of this section you are asked to show that the $n$th term of an arithmetic sequence always equals the initial value of the sequence plus $n$ times the fixed constant.

**• Definition**

A sequence $a_0,\ a_1,\ a_2, \ldots$ is called an **arithmetic sequence** if, and only if, there is a constant $d$ such that

$$a_k = a_{k-1} + d \quad \text{for all integers } k \geq 1.$$

It follows that,

$$a_n = a_0 + dn \quad \text{for all integers } n \geq 0.$$

### Example 5.7.2 An Arithmetic Sequence

Under the force of gravity, an object falling in a vacuum falls about 9.8 meters per second (m/sec) faster each second than it fell the second before. Thus, neglecting air resistance, a skydiver's speed upon leaving an airplane is approximately 9.8 m/sec one second after departure, $9.8 + 9.8 = 19.6$ m/sec two seconds after departure, and so forth. If air resistance is neglected, how fast would the skydiver be falling 60 seconds after leaving the airplane?

**Solution** Let $s_n$ be the skydiver's speed in m/sec $n$ seconds after exiting the airplane if there were no air resistance. Thus $s_0$ is the initial speed, and since the diver would travel 9.8 m/sec faster each second than the second before,

$$s_k = s_{k-1} + 9.8 \text{ m/sec} \quad \text{for all integers } k \geq 1.$$

It follows that $s_0, s_1, s_2, \ldots$ is an arithmetic sequence with a fixed constant of 9.8, and thus

$$s_n = s_0 + (9.8)n \quad \text{for each integer } n \geq 0.$$

Hence sixty seconds after exiting and neglecting air resistance, the skydiver would travel at a speed of

$$s_{60} = 0 + (9.8)(60) = 588 \text{ m/sec}.$$

Note that 588 m/sec is over half a kilometer per second or over a third of a mile per second, which is very fast for a human being to travel. Happily for the skydiver, taking air resistance into account cuts the speed considerably. ■

In an arithmetic sequence, each term equals the previous term plus a fixed constant. In a geometric sequence, each term equals the previous term *times* a fixed constant. Geometric sequences arise in a large variety of applications, such as compound interest certain models of population growth, radioactive decay, and the number of operations needed to execute certain computer algorithms.

### Example 5.7.3 The Explicit Formula for a Geometric Sequence

Let $r$ be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = ra_{k-1} \quad \text{for all integers } k \geq 1,$$
$$a_0 = a.$$

Use iteration to guess an explicit formula for this sequence.

**Solution**

$$a_0 = a$$

$$a_1 = ra_0 = ra$$

$$a_2 = ra_1 = r\,(ra) = r^2a$$

$$a_3 = ra_2 = r\,(r^2a) = r^3a$$

$$a_4 = ra_3 = r\,(r^3a) = r^4a$$

$$\vdots$$

*Guess:* $a_n = r^n a = ar^n$ for any arbitrary integer $n \geq 0$

In the exercises at the end of this section, you are asked to prove that this formula is correct. ■

**• Definition**

A sequence $a_0, a_1, a_2, \ldots$ is called a **geometric sequence** if, and only if, there is a constant $r$ such that

$$a_k = ra_{k-1} \quad \text{for all integers } k \geq 1.$$

It follows that,

$$a_n = a_0 r^n \quad \text{for all integers } n \geq 0.$$

## Example 5.7.4 A Geometric Sequence

As shown in Example 5.6.7, if a bank pays interest at a rate of 4% per year compounded annually and $A_n$ denotes the amount in the account at the end of year $n$, then $A_k = (1.04)A_{k-1}$, for all integers $k \geq 1$, assuming no deposits or withdrawals during the year. Suppose the initial amount deposited is \$100,000, and assume that no additional deposits or withdrawals are made.

a. How much will the account be worth at the end of 21 years?

b. In how many years will the account be worth \$1,000,000?

**Solution**

a. $A_0,\ A_1,\ A_2, \ldots$ is a geometric sequence with initial value 100,000 and constant multiplier 1.04. Hence,

$$A_n = \$100{,}000 \cdot (1.04)^n \quad \text{for all integers } n \geq 0.$$

After 21 years, the amount in the account will be

$$A_{21} = \$100{,}000 \cdot (1.04)^{21} \cong \$227,876.81.$$

This is the same answer as that obtained in Example 5.6.7 but is computed much more easily (at least if a calculator with a powering key, such as $\boxed{\wedge}$ or $\boxed{x^y}$, is used).

b. Let $t$ be the number of years needed for the account to grow to \$1,000,000. Then

$$\$1{,}000{,}000 = \$100{,}000 \cdot (1.04)^t.$$

Dividing both sides by 100,000 gives

$$10 = (1.04)^t,$$

and taking natural logarithms of both sides results in

$$\ln(10) = \ln(1.04)^t.$$

**Note** Properties of logarithms are reviewed in Section 7.2.

Then

$$\ln(10) \cong t\ \ln(1.04) \qquad \text{because } \log_b(x^a) = a \log_b(x) \text{ (see exercise 35 of Section 7.2)}$$

and so

$$t = \frac{\ln(10)}{\ln(1.04)} \cong 58.7$$

Hence the account will grow to \$1,000,000 in approximately 58.7 years. ■

An important property of a geometric sequence with constant multiplier greater than 1 is that its terms increase very rapidly in size as the subscripts get larger and larger. For instance, the first ten terms of a geometric sequence with a constant multiplier of 10 are

$$1,\ 10,\ 10^2,\ 10^3,\ 10^4,\ 10^5,\ 10^6,\ 10^7,\ 10^8,\ 10^9.$$

Thus, by its tenth term, the sequence already has the value $10^9 = 1{,}000{,}000{,}000 = 1$ billion. The following box indicates some quantities that are approximately equal to certain powers of 10.

$10^7 \cong$ number of seconds in a year
$10^9 \cong$ number of bytes of memory in a personal computer
$10^{11} \cong$ number of neurons in a human brain
$10^{17} \cong$ age of the universe in seconds (according to one theory)
$10^{31} \cong$ number of seconds to process all possible positions of a checkers game if moves are processed at a rate of 1 per billionth of a second
$10^{81} \cong$ number of atoms in the universe
$10^{111} \cong$ number of seconds to process all possible positions of a chess game if moves are processed at a rate of 1 per billionth of a second

## *Using Formulas to Simplify Solutions Obtained by Iteration*

Explicit formulas obtained by iteration can often be simplified by using formulas such as those developed in Section 5.2. For instance, according to the formula for the sum of a geometric sequence with initial term 1 (Theorem 5.2.3), for each real number $r$ except $r = 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1} \quad \text{for all integers } n \geq 0.$$

And according to the formula for the sum of the first $n$ integers (Theorem 5.2.2),

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

### Example 5.7.5 An Explicit Formula for the Tower of Hanoi Sequence

Recall that the Tower of Hanoi sequence $m_1,\ m_2,\ m_3, \ldots$ of Example 5.6.5 satisfies the recurrence relation

$$m_k = 2m_{k-1} + 1 \quad \text{for all integers } k \geq 2$$

and has the initial condition

$$m_1 = 1.$$

Use iteration to guess an explicit formula for this sequence, and make use of a formula from Section 5.2 to simplify the answer.

**Solution** By iteration

$$m_1 = 1$$

$$m_2 = 2m_1 + 1 = 2 \cdot 1 + 1 \qquad = 2^1 + 1,$$

$$m_3 = 2m_2 + 1 = 2\,(2+1) + 1 \qquad = 2^2 + 2 + 1,$$

$$m_4 = 2m_3 + 1 = 2\,(2^2 + 2 + 1) + 1 \qquad = 2^3 + 2^2 + 2 + 1,$$

$$m_5 = 2m_4 + 1 = 2\,(2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1.$$

These calculations show that each term up to $m_5$ is a sum of successive powers of 2, starting with $2^0 = 1$ and going up to $2^k$, where $k$ is 1 less than the subscript of the term. The pattern would seem to continue to higher terms because each term is obtained from the preceding one by multiplying by 2 and adding 1; multiplying by 2 raises the exponent of each component of the sum by 1, and adding 1 adds back the 1 that was lost when the previous 1 was multiplied by 2. For instance, for $n = 6$,

$$m_6 = 2m_5 + 1 = 2(2^4 + 2^3 + 2^2 + 2 + 1) + 1 = 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1.$$

Thus it seems that, in general,

$$m_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1.$$

By the formula for the sum of a geometric sequence (Theorem 5.2.3),

$$2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

Hence the explicit formula seems to be

$$m_n = 2^n - 1 \quad \text{for all integers } n \geq 1.$$

■

**Caution!** It is not true that

$\cancel{2 \cdot (2+1) + 1 = 2^2 + 1 + 1}.$

This is crossed out because it is false.

A common mistake people make when doing problems such as this is to misuse the laws of algebra. For instance, by the distributive law,

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{for all real numbers } a,\ b,\ \text{and } c.$$

Thus, in particular, for $a = 2,\ b = 2$, and $c = 1$,

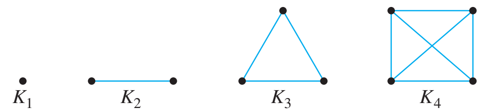
$$2 \cdot (2 + 1) = 2 \cdot 2 + 2 \cdot 1 = 2^2 + 2.$$

It follows that

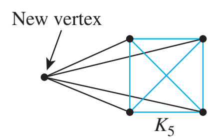
$$2 \cdot (2 + 1) + 1 = (2^2 + 2) + 1 = 2^2 + 2 + 1.$$

## Example 5.7.6 Using the Formula for the Sum of the First $n$ Positive Integers

Let $K_n$ be the picture obtained by drawing $n$ dots (which we call *vertices*) and joining each pair of vertices by a line segment (which we call an *edge*). (In Chapter 10 we discuss these objects in a more general context.) Then $K_1$, $K_2$, $K_3$, and $K_4$ are as follows:




Observe that $K_5$ may be obtained from $K_4$ by adding one vertex and drawing edges between this new vertex and all the vertices of $K_4$ (the old vertices). The reason this procedure gives the correct result is that each pair of old vertices is already joined by an edge, and adding the new edges joins each pair of vertices consisting of an old one and the new one.




Thus $\quad$ the number of edges of $K_5 = 4 +$ the number of edges of $K_4$.

By the same reasoning, for all integers $k \geq 2$, the number of edges of $K_k$ is $k-1$ more than the number of edges of $K_{k-1}$. That is, if for each integer $n \geq 1$

$$s_n = \text{the number of edges of } K_n,$$

then
$$s_k = s_{k-1} + (k-1) \quad \text{for all integers } k \geq 2.$$

Note that $s_1$, is the number of edges in $K_1$, which is 0, and use iteration to find an explicit formula for $s_1, s_2, s_3, \ldots$.

**Solution** Because

$$s_k = s_{k-1} + (k-1) \quad \text{for } all \text{ integers } k \geq 2$$

and

$$s_1 = \overset{1-1}{0}$$

then, in particular,

$$
\begin{aligned}
s_2 &= s_1 + 1 = 0 + \overset{2-1}{1}, \\
s_3 &= s_2 + 2 = (0+1) + 2 = 0 + 1 + \overset{3-1}{2}, \\
s_4 &= s_3 + 3 = (0+1+2) + 3 = 0 + 1 + 2 + \overset{4-1}{3}, \\
s_5 &= s_4 + 4 = (0+1+2+3) + 4 = 0 + 1 + 2 + 3 + \overset{5-1}{4}, \\
&\ \vdots
\end{aligned}
$$

*Guess:* $\quad s_n = 0 + 1 + 2 + \cdots + (n-1)$.

But by Theorem 5.2.2,

$$0 + 1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)n}{2} = \frac{n(n-1)}{2}.$$

Hence it appears that

$$s_n = \frac{n(n-1)}{2}.$$

■

## Checking the Correctness of a Formula by Mathematical Induction

As you can see from some of the previous examples, the process of solving a recurrence relation by iteration can involve complicated calculations. It is all too easy to make a mistake and come up with the wrong formula. That is why it is important to confirm your calculations by checking the correctness of your formula. The most common way to do this is to use mathematical induction.

### Example 5.7.7 Using Mathematical Induction to Verify the Correctness of a Solution to a Recurrence Relation

In Example 5.6.5 we obtained a formula for the Tower of Hanoi sequence. Use mathematical induction to show that this formula is correct.

**Solution** What does it mean to show the correctness of a formula for a recursively defined sequence? Given a sequence of numbers that satisfies a certain recurrence relation and initial condition, your job is to show that each term of the sequence satisfies the proposed explicit formula. In this case, you need to prove the following statement:

> If $m_1, m_2, m_3, \ldots$ is the sequence defined by
>
> $m_k = 2m_{k-1} + 1$ for all integers $k \geq 2$, and
>
> $m_1 = 1$,
>
> then $m_n = 2^n - 1$ for all integers $n \geq 1$.

**Proof of Correctness:**

Let $m_1,\ m_2,\ m_3, \ldots$ be the sequence defined by specifying that $m_1 = 1$ and $m_k = 2m_{k+1} + 1$ for all integers $k \geq 2$, and let the property $P(n)$ be the equation

$$m_n = 2^n - 1 \qquad \leftarrow P(n)$$

We will use mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.

***Show that P(1) is true:***
To establish $P(1)$, we must show that

$$m_1 = 2^1 - 1. \qquad \leftarrow P(1)$$

But the left-hand side of $P(1)$ is

$$m_1 = 1 \qquad \text{by definition of } m_1, m_2, m_3, \ldots,$$

and the right-hand side of $P(1)$ is

$$2^1 - 1 = 2 - 1 = 1.$$

Thus the two sides of $P(1)$ equal the same quantity, and hence $P(1)$ is true.

***Show that for all integers $k \geq 1$, if P(k) is true then $P(k+1)$ is also true:*** *[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:]* Suppose that $k$ is any integer with $k \geq 1$ such that

$$m_k = 2^k - 1. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

*[We must show that $P(k+1)$ is true. That is:]* We must show that

$$m_{k+1} = 2^{k+1} - 1. \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} m_{k+1} &= 2m_{(k+1)-1} + 1 && \text{by definition of } m_1,\ m_2,\ m_3, \ldots \\ &= 2m_k + 1 \\ &= 2(2^k - 1) + 1 && \text{by substitution from the inductive hypothesis} \\ &= 2^{k+1} - 2 + 1 && \text{by the distributive law and the fact that } 2 \cdot 2^k = 2^{k-1} \\ &= 2^{k+1} - 1 && \text{by basic algebra} \end{aligned}$$

which equals the right-hand side of $P(k+1)$. *[Since the basis and inductive steps have been proved, it follows by mathematical induction that the given formula holds for all integers $n \geq 1$.]* ■

## Discovering That an Explicit Formula Is Incorrect

The following example shows how the process of trying to verify a formula by mathematical induction may reveal a mistake.

### Example 5.7.8 Using Verification by Mathematical Induction to Find a Mistake

Let $c_0,\ c_1,\ c_2, \ldots$ be the sequence defined as follows:

$$c_k = 2c_{k-1} + k \qquad \text{for all integers } k \geq 1,$$
$$c_0 = 1.$$

Suppose your calculations suggest that $c_0,\ c_1,\ c_2, \ldots$ satisfies the following explicit formula:

$$c_n = 2^n + n \qquad \text{for all integers } n \geq 0.$$

Is this formula correct?

**Solution** Start to prove the statement by mathematical induction and see what develops. The proposed formula passes the basis step of the inductive proof with no trouble, for on the one hand, $c_0 = 1$ by definition and on the other hand, $2^0 + 0 = 1 + 0 = 1$ also.

In the inductive step, you suppose

$$c_k = 2^k + k \qquad \text{for some integer } k \geq 0 \qquad \text{This is the inductive hypothesis.}$$

and then you must show that

$$c_{k+1} = 2^{k+1} + (k+1).$$

To do this, you start with $c_{k+1}$, substitute from the recurrence relation, and then use the inductive hypothesis as follows:

$$\begin{aligned} c_{k+1} &= 2c_k + (k+1) && \text{by the recurrence relation} \\ &= 2(2^k + k) + (k+1) && \text{by substitution from the inductive hypothesis} \\ &= 2^{(k+1)} + 3k + 1 && \text{by basic algebra} \end{aligned}$$

To finish the verification, therefore, you need to show that

$$2^{k+1} + 3k + 1 = 2^{k+1} + (k+1).$$

Now this equation is equivalent to

$$2k = 0 \quad \text{by subtracting } 2^{k+1} + k + 1 \text{ from both sides}.$$

which is equivalent to

$$k = 0 \quad \text{by dividing both sides by 2.}$$

But this is false since $k$ may be *any* nonnegative integer.

Observe that when $k = 0$, then $k + 1 = 1$, and

$$c_1 = 2 \cdot 1 + 1 = 3 \quad \text{and} \quad 2^1 + 1 = 3.$$

Thus the formula gives the correct value for $c_1$. However, when $k = 1$, then $k + 1 = 2$, and

$$c_2 = 2 \cdot 3 + 2 = 8 \quad \text{whereas} \quad 2^2 + 2 = 4 + 2 = 6.$$

So the formula does not give the correct value for $c_2$. Hence the sequence $c_0, c_1, c_2, \ldots$ does not satisfy the proposed formula. ■

Once you have foud a proposed formula to be false, you should look back at your calculations to see where you made a mistake, correct it, and try again.

## Test Yourself

1. To use iteration to find an explicit formula for a recursively defined sequence, start with the _____ and use successive substitution into the _____ to look for a numerical pattern.
2. At every step of the iteration process, it is important to eliminate _____.
3. If a single number, say $a$, is added to itself $k$ times in one of the steps of the iteration, replace the sum by the expression _____.
4. If a single number, say $a$, is multiplied by itself $k$ times in one of the steps of the iteration, replace the product by the expression _____.
5. A general arithmetic sequence $a_0, a_1, a_2, \ldots$ with initial value $a_0$ and fixed constant $d$ satisfies the recurrence relation _____ and has the explicit formula _____.
6. A general geometric sequence $a_0, a_1, a_2, \ldots$ with initial value $a_0$ and fixed constant $r$ satisfies the recurrence relation _____ and has the explicit formula _____.
7. When an explicit formula for a recursively defined sequence has been obtained by iteration, its correctness can be checked by _____.

## Exercise Set 5.7

1. The formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

is true for all integers $n \geq 1$. Use this fact to solve each of the following problems:

**a.** If $k$ is an integer and $k \geq 2$, find a formula for the expression $1 + 2 + 3 + \cdots + (k-1)$.

**b.** If $n$ is an integer and $n \geq 1$, find a formula for the expression $3 + 2 + 4 + 6 + 8 + \cdots + 2n$.

c. If $n$ is an integer and $n \geq 1$, find a formula for the expression $3 + 3 \cdot 2 + 3 \cdot 3 + \cdots + 3 \cdot n + n$.

2. The formula

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

is true for all real numbers $r$ except $r = 1$ and for all integers $n \geq 0$. Use this fact to solve each of the following problems:

a. If $i$ is an integer and $i \geq 1$, find a formula for the expression $1 + 2 + 2^2 + \cdots + 2^{i-1}$.

b. If $n$ is an integer and $n \geq 1$, find a formula for the expression $3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1$.

c. If $n$ is an integer and $n \geq 2$, find a formula for the expression $2^n + 2^{n-2} \cdot 3 + 2^{n-3} \cdot 3 + \cdots + 2^2 \cdot 3 + 2 \cdot 3 + 3$

d. If $n$ is an integer and $n \geq 1$, find a formula for the expression

$$2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n.$$

In each of 3–15 a sequence is defined recursively. Use iteration to guess an explicit formula for the sequence. Use the formulas from Section 5.2 to simplify your answers whenever possible.

3. $a_k = ka_{k-1}$, for all integers $k \geq 1$
$a_0 = 1$

4. $b_k = \dfrac{b_{k-1}}{1 + b_{k-1}}$, for all integers $k \geq 1$
$b_0 = 1$

5. $c_k = 3c_{k-1} + 1$, for all integers $k \geq 2$
$c_1 = 1$

H 6. $d_k = 2d_{k-1} + 3$, for all integers $k \geq 2$
$d_t = 2$

7. $e_k = 4e_{k-1} + 5$, for all integers $k \geq 1$
$e_0 = 2$

8. $f_k = f_{k-1} + 2^k$, for all integers $k \geq 2$
$f_1 = 1$

H 9. $g_k = \dfrac{g_{k-1}}{g_{k-1} + 2}$, for all integers $k \geq 2$
$g_1 = 1$

10. $h_k = 2^k - h_{k-1}$, for all integers $k \geq 1$
$h_0 = 1$

11. $p_k = p_{k-1} + 2 \cdot 3^k$
$p_1 = 2$

12. $s_k = s_{k-1} + 2k$, for all integers $k \geq 1$
$s_0 = 3$

13. $t_k = t_{k-1} + 3k + 1$, for all integers $k \geq 1$
$t_0 = 0$

✶ 14. $x_k = 3x_{k-1} + k$, for all integers $k \geq 2$
$x_1 = 1$

15. $y_k = y_{k-1} + k^2$, for all integers $k \geq 2$
$y_1 = 1$

16. Solve the recurrence relation obtained as the answer to exercise 18(c) of Section 5.6.

17. Solve the recurrence relation obtained as the answer to exercise 21(c) of Section 5.6.

18. Suppose $d$ is a fixed constant and $a_0, a_1, a_2, \ldots$ is a sequence that satisfies the recurrence relation $a_k = a_{k-1} + d$, for all integers $k \geq 1$. Use mathematical induction to prove that $a_n = a_0 + nd$, for all integers $n \geq 0$.

19. A worker is promised a bonus if he can increase his productivity by 2 units a day every day for a period of 30 days. If on day 0 he produces 170 units, how many units must he produce on day 30 to qualify for the bonus?

20. A runner targets herself to improve her time on a certain course by 3 seconds a day. If on day 0 she runs the course in 3 minutes, how fast must she run it on day 14 to stay on target?

21. Suppose $r$ is a fixed constant and $a_0, a_1, a_2 \ldots$ is a sequence that satisfies the recurrence relation $a_k = ra_{k-1}$, for all integers $k \geq 1$ and $a_0 = a$. Use mathematical induction to prove that $a_n = ar^n$, for all integers $n \geq 0$.

22. As shown in Example 5.6.8, if a bank pays interest at a rate of $i$ compounded $m$ times a year, then the amount of money $P_k$ at the end of $k$ time periods (where one time period $= 1/m$th of a year) satisfies the recurrence relation $P_k = [1 + (i/m)]P_{k-1}$ with initial condition $P_0 =$ the initial amount deposited. Find an explicit formula for $P_n$.

23. Suppose the population of a country increases at a steady rate of 3% per year. If the population is 50 million at a certain time, what will it be 25 years later?

24. A chain letter works as follows: One person sends a copy of the letter to five friends, each of whom sends a copy to five friends, each of whom sends a copy to five friends, and so forth. How many people will have received copies of the letter after the twentieth repetition of this process, assuming no person receives more than one copy?

25. A certain computer algorithm executes twice as many operations when it is run with an input of size $k$ as when it is run with an input of size $k - 1$ (where $k$ is an integer that is greater than 1). When the algorithm is run with an input of size 1, it executes seven operations. How many operations does it execute when it is run with an input of size 25?

26. A person saving for retirement makes an initial deposit of \$1,000 to a bank account earning interest at a rate of 3% per year compounded monthly, and each month she adds an additional \$200 to the account.

a. For each nonnegative integer $n$, let $A_n$ be the amount in the account at the end of $n$ months. Find a recurrence relation relating $A_k$ to $A_{k-1}$.

H b. Use iteration to find an explicit formula for $A_n$.

c. Use mathematical induction to prove the correctness of the formula you obtained in part (b).

d. How much will the account be worth at the end of 20 years? At the end of 40 years?

H e. In how many years will the account be worth \$10,000?

27. A person borrows \$3,000 on a bank credit card at a nominal rate of 18% per year, which is actually charged at a rate of 1.5% per month.
   H **a.** What is the annual percentage rate (APR) for the card? (See Example 5.6.8 for a definition of APR.)
   b. Assume that the person does not place any additional charges on the card and pays the bank \$150 each month to pay off the loan. Let $B_n$ be the balance owed on the card after $n$ months. Find an explicit formula for $B_n$.
   H **c.** How long will be required to pay off the debt?
   d. What is the total amount of money the person will have paid for the loan?

In 28–42 use mathematical induction to verify the correctness of the formula you obtained in the referenced exercise.

**28.** Exercise 3 29. Exercise 4 **30.** Exercise 5
31. Exercise 6 32. Exercise 7 33. Exercise 8
34. Exercise 9 H **35.** Exercise 10 36. Exercise 11
H **37.** Exercise 12 38. Exercise 13 **39.** Exercise 14
40. Exercise 15 41. Exercise 16 42. Exercise 17

In each of 43–49 a sequence is defined recursively. (a) Use iteration to guess an explicit formula for the sequence. (b) Use strong mathematical induction to verify that the formula of part (a) is correct.

**43.** $a_k = \dfrac{a_{k-1}}{2a_{k-1} - 1}$, for all integers $k \geq 1$
$a_0 = 2$

44. $b_k = \dfrac{2}{b_{k-1}}$, for all integers $k \geq 2$
$b_1 = 1$

**45.** $v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2$, for all integers $k \geq 2$,
$v_1 = 1$.

H **46.** $s_k = 2s_{k-2}$, for all integers $k \geq 2$,
$s_0 = 1$, $s_1 = 2$.

47. $t_k = k - t_{k-1}$, for all integers $k \geq 1$,
$t_0 = 0$.

H **48.** $w_k = w_{k-2} + k$, for all integers $k \geq 3$,
$w_1 = 1$, $w_2 = 2$.

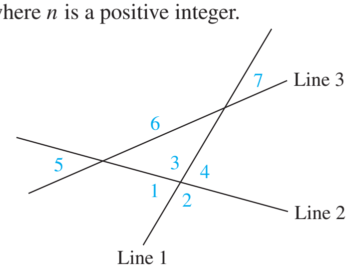
H **49.** $u_k = u_{k-2} \cdot u_{k-1}$, for all integers $k \geq 2$,
$u_0 = u_1 = 2$.

In 50 and 51 determine whether the given recursively defined sequence satisfies the explicit formula $a_n = (n-1)^2$, for all integers $n \geq 1$.

**50.** $a_k = 2a_{k-1} + k - 1$, for all integers $k \geq 2$
$a_1 = 0$

51. $a_k = (a_{k-1} + 1)^2$, for all integers $k \geq 2$
$a_1 = 0$

52. A single line divides a plane into two regions. Two lines (by crossing) can divide a plane into four regions; three lines can divide it into seven regions (see the figure). Let $P_n$ be the maximum number of regions into which $n$ lines divide a plane, where $n$ is a positive integer.




   **a.** Derive a recurrence relation for $P_k$ in terms of $P_{k-1}$, for all integers $k \geq 2$.
   b. Use iteration to guess an explicit formula for $P_n$.

**53.** Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ for small values of $n$ (up to about 5 or 6). Conjecture explicit formulas for the entries in this matrix, and prove your conjecture using mathematical induction.

54. In economics the behavior of an economy from one period to another is often modeled by recurrence relations. Let $Y_k$ be the income in period $k$ and $C_k$ be the consumption in period $k$. In one economic model, income in any period is assumed to be the sum of consumption in that period plus investment and government expenditures (which are assumed to be constant from period to period), and consumption in each period is assumed to be a linear function of the income of the preceding period. That is,

$Y_k = C_k + E$ where $E$ is the sum of investment plus government expenditures
$C_k = c + mY_{k-1}$ where $c$ and $m$ are constants.

Substituting the second equation into the first gives $Y_k = E + c + mY_{k-1}$.
   a. Use iteration on the above recurrence relation to obtain
$$Y_n = (E + c)\left(\frac{m^n - 1}{m - 1}\right) + m^n Y_0$$
   for all integers n≥1.
   b. (For students who have studied calculus) Show that if $0 < m < 1$, then $\lim_{n \to \infty} Y_n = \dfrac{E + c}{1 - m}$.

## Answers for Test Yourself

1. initial conditions; recurrence relation 2. parentheses 3. $k \cdot a$ 4. $a^k$ 5. $a_k = a_{k-1} + d$; $a_n = a_0 + dn$ 6. $a_k = ra_{k-1}$; $a_n = a_0 r^n$ 7. mathematical induction