CHAPTER 9

COUNTING AND PROBABILITY

"It's as easy as 1–2–3."

That's the saying. And in certain ways, counting *is* easy. But other aspects of counting aren't so simple. Have you ever agreed to meet a friend "in three days" and then realized that you and your friend might mean different things? For example, on the European continent, to meet in eight days means to meet on the same day as today one week hence; on the other hand, in English-speaking countries, to meet in seven days means to meet one week hence. The difference is that on the continent, all days including the first and the last are counted. In the English-speaking world, it's the number of 24-hour periods that are counted.

Continental countries	1	2	3	4	5	6	7	8
	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Sun
English-speaking countries	1	2	3	4	5	6	7	

The English convention for counting days follows the almost universal convention for counting hours. If it is 9 A.M. and two people anywhere in the world agree to meet in three hours, they mean that they will get back together again at 12 noon.

Musical intervals, on the other hand, are universally reckoned the way the Continentals count the days of a week. An interval of a third consists of two tones with a single tone in between, and an interval of a second consists of two adjacent tones. (See Figure 9.1.1.)

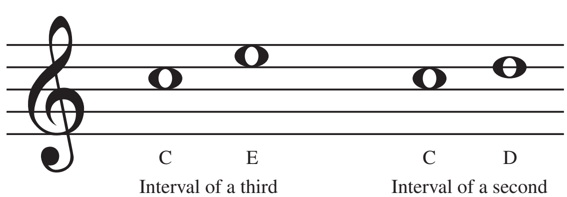
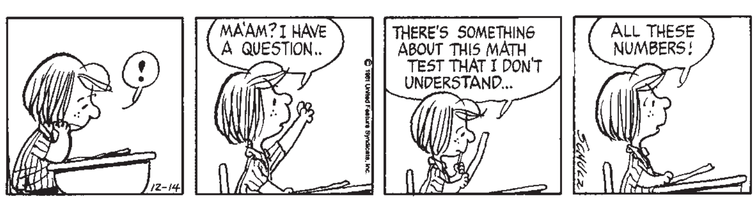


Figure 9.1.1

Of course, the complicating factor in all these examples is not how to count but rather what to count. And, indeed, in the more complex mathematical counting problems discussed in this chapter, it is what to count that is the central issue. Once one knows exactly what to count, the counting itself is as easy as 1–2–3.



Reprinted by permission of UFS, Inc.

9.1 Introduction

Imagine tossing two coins and observing whether 0, 1, or 2 heads are obtained. It would be natural to guess that each of these events occurs about one-third of the time, but in fact this is not the case. Table 9.1.1 below shows actual data obtained from tossing two quarters 50 times.

Table 9.1.1 Experimental Data Obtained from Tossing Two Quarters 50 Times

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸 𝍸 𝍸 𝍸 ||	27	54%
0 heads obtained	𝍸 𝍸 ||	12	24%

As you can see, the relative frequency of obtaining exactly 1 head was roughly twice as great as that of obtaining either 2 heads or 0 heads. It turns out that the mathematical theory of probability can be used to predict that a result like this will almost always occur. To see how, call the two coins A and B, and suppose that each is perfectly balanced. Then each has an equal chance of coming up heads or tails, and when the two are tossed together, the four outcomes pictured in Figure 9.1.2 are all equally likely.

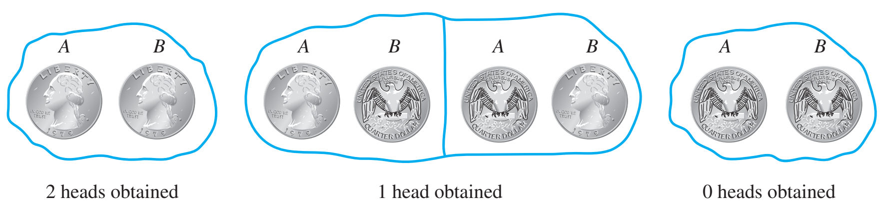


Figure 9.1.2 Equally Likely Outcomes from Tossing Two Balanced Coins

Figure 9.1.2 shows that there is a 1 in 4 chance of obtaining two heads and a 1 in 4 chance of obtaining no heads. The chance of obtaining one head, however, is 2 in 4 because either A could come up heads and B tails or B could come up heads and A tails. So if you repeatedly toss two balanced coins and record the number of heads, you should expect relative frequencies similar to those shown in Table 9.1.1.

To formalize this analysis and extend it to more complex situations, we introduce the notions of random process, sample space, event and probability. To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be. For instance, if an ordinary person performs the experiment of tossing an ordinary coin into the air and allowing it to fall flat on the ground, it can be predicted with certainty that the coin will land either heads up or tails up (so the set of outcomes can be denoted {heads, tails}), but it is not known for sure whether heads or tails will occur. We restricted this experiment to ordinary people because a skilled magician can toss a coin in a way that appears random but is not, and a physicist equipped with first-rate measuring devices may be able to analyze all the forces on the coin and correctly predict its landing position. Just a few of many examples of random processes or experiments are choosing winners in state lotteries, selecting respondents in public opinion polls, and choosing subjects to receive treatments or serve as controls in medical experiments. The set of outcomes that can result from a random process or experiment is called a *sample space.*

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

In case an experiment has finitely many outcomes and all outcomes are equally likely to occur, the *probability* of an event (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes. Strictly speaking, this result can be deduced from a set of axioms for probability formulated in 1933 by the Russian mathematician A. N. Kolmogorov. In Section 9.8 we discuss the axioms and show how to derive their consequences formally. At present, we take a naïve approach to probability and simply state the result as a principle.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of *E***, denoted ***P*(*E*)**, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

• Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

With this notation, the equally likely probability formula becomes

$$P(E) = \frac{N(E)}{N(S)}.$$

Example 9.1.1 Probabilities for a Deck of Cards

An ordinary deck of cards contains 52 cards divided into four *suits.* The *red suits* are diamonds (♦) and hearts (♥) and the *black suits* are clubs (♣) and spades (♠). Each suit contains 13 cards of the following *denominations:* 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and A (ace). The cards J, Q, and K are called *face cards.*

Mathematician Persi Diaconis, working with David Aldous in 1986 and Dave Bayer in 1992, showed that seven shuffles are needed to “thoroughly mix up” the cards in an ordinary deck. In 2000 mathematician Nick Trefethen, working with his father, Lloyd Trefethen, a mechanical engineer, used a somewhat different definition of “thoroughly mix up” to show that six shuffles will nearly always suffice. Imagine that the cards in a deck have become—by some method—so thoroughly mixed up that if you spread them out face down and pick one at random, you are as likely to get any one card as any other.

a. What is the sample space of outcomes?

b. What is the event that the chosen card is a black face card?

c. What is the probability that the chosen card is a black face card?

Solution

a. The outcomes in the sample space S are the 52 cards in the deck.

b. Let E be the event that a black face card is chosen. The outcomes in E are the jack, queen, and king of clubs and the jack, queen, and king of spades. Symbolically,

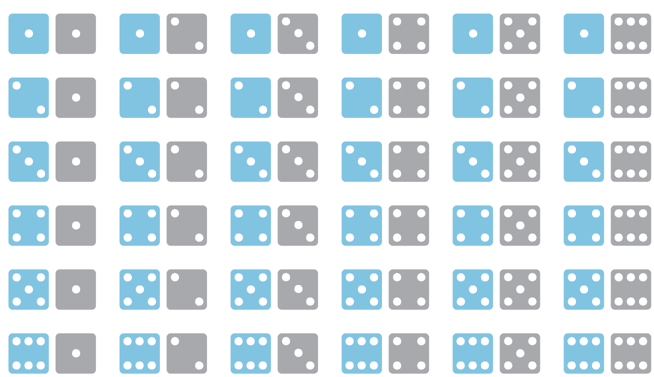
$$E = \{J♣, Q♣, K♣, J♠, Q♠, K♠\}.$$

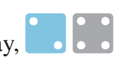
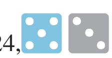
c. By part (b), $N(E) = 6$, and according to the description of the situation, all 52 outcomes in the sample space are equally likely. Therefore, by the equally likely probability formula, the probability that the chosen card is a black face card is

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \cong 11.5\%.$$ ■

Example 9.1.2 Rolling a Pair of Dice

A die is one of a pair of dice. It is a cube with six sides, each containing from one to six dots, called *pips*. Suppose a blue die and a gray die are rolled together, and the numbers of dots that occur face up on each are recorded. The possible outcomes can be listed as follows, where in each case the die on the left is blue and the one on the right is gray.



A more compact notation identifies, say,  with the notation 24,  with 53, and so forth.

a. Use the compact notation to write the sample space S of possible outcomes.

b. Use set notation to write the event E that the numbers showing face up have a sum of 6 and find the probability of this event.

Solution

a. $S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$.

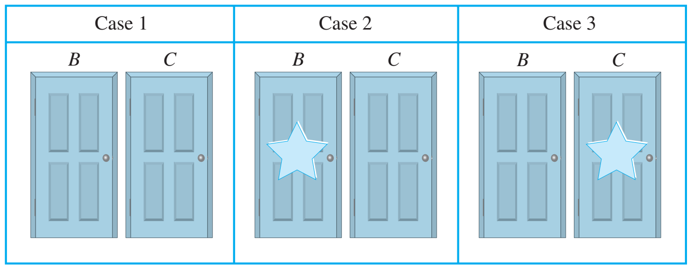
b. $E = \{15, 24, 33, 42, 51\}$.

The probability that the sum of the numbers is $6 = P(E) = \dfrac{N(E)}{N(S)} = \dfrac{5}{36}$. ■

The next example is called the Monty Hall problem after the host of an old game show, "Let's Make A Deal." When it was originally publicized in a newspaper column and on a radio show, it created tremendous controversy. Many highly educated people, even some with Ph.D.'s, submitted incorrect solutions or argued vociferously against the correct solution. Before you read the answer, think about what your own response to the situation would be.

Example 9.1.3 The Monty Hall Problem

There are three doors on the set for a game show. Let's call them A, B, and C. If you pick the right door you win the prize. You pick door A. The host of the show, Monty Hall, then opens one of the other doors and reveals that there is no prize behind it. Keeping the remaining two doors closed, he asks you whether you want to switch your choice to the other closed door or stay with your original choice of door A. What should you do if you want to maximize your chance of winning the prize: stay with door A or switch—or would the likelihood of winning be the same either way?



Solution At the point just before the host opens one of the closed doors, there is no information about the location of the prize. Thus there are three equally likely possibilities for what lies behind the doors: (Case 1) the prize is behind A (i.e., it is not behind either B or C), (Case 2) the prize is behind B; (Case 3) the prize is behind C.

Since there is no prize behind the door the host opens, in Case 1 the host could open either door and you would win by staying with your original choice: door A. In Case 2 the host must open door C, and so you would win by switching to door B. In Case 3 the host must open door B, and so you would win by switching to door C. Thus, in two of the three equally likely cases, you would win by switching from A to the other closed door. In only one of the three equally likely cases would you win by staying with your original choice. Therefore, you should switch.

A reality note: The analysis used for this solution applies only if the host *always* opens one of the closed doors and offers the contestant the choice of staying with the original choice or switching. In the original show, Monty Hall made this offer only occasionally—most often when he knew the contestant had already chosen the correct door. ■


Bettmann/CORBIS

Pierre-Simon Laplace (1749–1827)

Many of the fundamental principles of probability were formulated in the mid-1600s in an exchange of letters between Pierre de Fermat and Blaise Pascal in response to questions posed by a French nobleman interested in games of chance. In 1812, Pierre-Simon Laplace published the first general mathematical treatise on the subject and extended the range of applications to a variety of scientific and practical problems.

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list. For instance, how many integers are there from 5 through 12? To answer this question, imagine going along the list of integers from 5 to 12, counting each in turn.

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

So the answer is 8.

More generally, if m and n are integers and $m \le n$, how many integers are there from m through n? To answer this question, note that $n = m + (n - m)$, where $n - m \ge 0$ [*since* $n \ge m$]. Note also that the element $m + 0$ is the first element of the list, the element $m + 1$ is the second element, the element $m + 2$ is the third, and so forth. In general, the element $m + i$ is the $(i + 1)$st element of the list.

list:	$m (= m + 0)$	$m + 1$	$m + 2$	$\ldots$	$n (= m + (n - m))$
	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$
count:	1	2	3	$\ldots$	$(n - m) + 1$

And so the number of elements in the list is $n - m + 1$.

This general result is important enough to be restated as a theorem, the formal proof of which uses mathematical induction. (See exercise 28 at the end of this section.) The heart of the proof is the observation that if the list $m, m + 1, \ldots, k$ has $k - m + 1$ numbers, then the list $m, m + 1, \ldots, k, k + 1$ has $(k - m + 1) + 1 = (k + 1) - m + 1$ numbers.

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \le n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 9.1.4 Counting the Elements of a Sublist

a. How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

b. What is the probability that a randomly chosen three-digit integer is divisible by 5?

Solution

a. Imagine writing the three-digit integers in a row, noting those that are multiples of 5 and drawing arrows between each such integer and its corresponding multiple of 5.

100	101	102	103	104	105	106	107	108	109	110	$\cdots$	994	995	996	997	998	999
$\updownarrow$					$\updownarrow$					$\updownarrow$			$\updownarrow$				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

From the sketch it is clear that there are as many three-digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive. By Theorem 9.1.1, there are $199 - 20 + 1$, or 180, such integers. Hence there are 180 three-digit integers that are divisible by 5.

b. By Theorem 9.1.1 the total number of integers from 100 through 999 is $999 - 100 + 1 = 900$. By part (a), 180 of these are divisible by 5. Hence the probability that a randomly chosen three-digit integer is divisible by 5 is $180/900 = 1/5$. ■

Example 9.1.5 Application: Counting Elements of a One-Dimensional Array

Analysis of many computer algorithms requires skill at counting the elements of a one-dimensional array. Let $A[1], A[2], \ldots, A[n]$ be a one-dimensional array, where n is a positive integer.

a. Suppose the array is cut at a middle value $A[m]$ so that two subarrays are formed:

$$(1)\ A[1], A[2], \ldots, A[m] \quad \text{and} \quad (2)\ A[m+1], A[m+2], \ldots, A[n].$$

How many elements does each subarray have?

b. What is the probability that a randomly chosen element of the array has an even subscript

(i) if n is even? (ii) if n is odd?

Solution

a. Array (1) has the same number of elements as the list of integers from 1 through m. So by Theorem 9.1.1, it has m, or $m - 1 + 1$, elements. Array (2) has the same number of elements as the list of integers from $m + 1$ through n. So by Theorem 9.1.1, it has $n - m$, or $n - (m + 1) + 1$, elements.

b. (i) If n is even, each even subscript starting with 2 and ending with n can be matched up with an integer from 1 to $n/2$.

1	2	3	4	5	6	7	8	9	10	$\cdots$	n
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$
	$2\cdot 1$		$2\cdot 2$		$2\cdot 3$		$2\cdot 4$		$2\cdot 5$		$2\cdot n/2$

So there are $n/2$ array elements with even subscripts. Since the entire array has n elements, the probability that a randomly chosen element has an even subscript is $\dfrac{n/2}{n} = \dfrac{1}{2}$.

(ii) If n is odd, then the greatest even subscript of the array is $n - 1$. So there are as many even subscripts between 1 and n as there are from 2 through $n - 1$. Then the reasoning of (i) can be used to conclude that there are $(n - 1)/2$ array elements with even subscripts.

1	2	3	4	5	6	$\cdots$	$n-1$	n
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$	
	$2\cdot 1$		$2\cdot 2$		$2\cdot 3$	$\cdots$	$2\cdot(n-1)/2$	

Since the entire array has n elements, the probability that a randomly chosen element has an even subscript is $\dfrac{(n-1)/2}{n} = \dfrac{n-1}{2n}$. Observe that as n gets larger and larger, this probability gets closer and closer to 1/2.

Note that the answers to (i) and (ii) can be combined using the floor notation. By Theorem 4.5.2, the number of array elements with even subscripts is $\lfloor n/2 \rfloor$, so the probability that a randomly chosen element has an even subscript is $\dfrac{\lfloor n/2 \rfloor}{n}$. ■

Test Yourself

Answers to Test Yourself questions are located at the end of each section.

1. A sample space of a random process or experiment is _____.
2. An event in a sample space is _____.
3. To compute the probability of an event using the equally likely probability formula, you take the ratio of the _____ to the _____.
4. If $m \le n$, the number of integers from m to n inclusive is _____.

Exercise Set 9.1*

1. Toss two coins 30 times and make a table showing the relative frequencies of 0, 1, and 2 heads. How do your values compare with those shown in Table 9.1.1?

2. In the example of tossing two quarters, what is the probability that at least one head is obtained? that coin A is a head? that coins A and B are either both heads or both tails?

In 3–6 use the sample space given in Example 9.1.1. Write each event as a set, and compute its probability.

3. The event that the chosen card is red and is not a face card.

4. The event that the chosen card is black and has an even number on it.

5. The event that the denomination of the chosen card is at least 10 (counting aces high).

6. The event that the denomination of the chosen card is at most 4 (counting aces high).

In 7–10, use the sample space given in Example 9.1.2. Write each of the following events as a set and compute its probability.

7. The event that the sum of the numbers showing face up is 8.

8. The event that the numbers showing face up are the same.

9. The event that the sum of the numbers showing face up is at most 6.

10. The event that the sum of the numbers showing face up is at least 9.

11. Suppose that a coin is tossed three times and the side showing face up on each toss is noted. Suppose also that on each toss heads and tails are equally likely. Let HHT indicate the outcome heads on the first two tosses and tails on the third, THT the outcome tails on the first and third tosses and heads on the second, and so forth.
    a. List the eight elements in the sample space whose outcomes are all the possible head–tail sequences obtained in the three tosses.
    b. Write each of the following events as a set and find its probability:
        (i) The event that exactly one toss results in a head.
        (ii) The event that at least two tosses result in a head.
        (iii) The event that no head is obtained.

12. Suppose that each child born is equally likely to be a boy or a girl. Consider a family with exactly three children. Let BBG indicate that the first two children born are boys and the third child is a girl, let GBG indicate that the first and third children born are girls and the second is a boy, and so forth.
    a. List the eight elements in the sample space whose outcomes are all possible genders of the three children.
    b. Write each of the events in the next column as a set and find its probability.
        (i) The event that exactly one child is a girl.
        (ii) The event that at least two children are girls.
        (iii) The event that no child is a girl.

13. Suppose that on a true/false exam you have no idea at all about the answers to three questions. You choose answers randomly and therefore have a 50–50 chance of being correct on any one question. Let CCW indicate that you were correct on the first two questions and wrong on the third, let WCW indicate that you were wrong on the first and third questions and correct on the second, and so forth.
    a. List the elements in the sample space whose outcomes are all possible sequences of correct and incorrect responses on your part.
    b. Write each of the following events as a set and find its probability:
        (i) The event that exactly one answer is correct.
        (ii) The event that at least two answers are correct.
        (iii) The event that no answer is correct.

14. Three people have been exposed to a certain illness. Once exposed, a person has a 50–50 chance of actually becoming ill.
    a. What is the probability that exactly one of the people becomes ill?
    b. What is the probability that at least two of the people become ill?
    c. What is the probability that none of the three people becomes ill?

15. When discussing counting and probability, we often consider situations that may appear frivolous or of little practical value, such as tossing coins, choosing cards, or rolling dice. The reason is that these relatively simple examples serve as models for a wide variety of more complex situations in the real world. In light of this remark, comment on the relationship between your answer to exercise 11 and your answers to exercises 12–14.

16. Two faces of a six-sided die are painted red, two are painted blue, and two are painted yellow. The die is rolled three times, and the colors that appear face up on the first, second, and third rolls are recorded.
    a. Let BBR denote the outcome where the color appearing face up on the first and second rolls is blue and the color appearing face up on the third roll is red. Because there are as many faces of one color as of any other, the outcomes of this experiment are equally likely. List all 27 possible outcomes.
    b. Consider the event that all three rolls produce different colors. One outcome in this event is RBY and another RYB. List all outcomes in the event. What is the probability of the event?

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

c. Consider the event that two of the colors that appear face up are the same. One outcome in this event is *RRB* and another is *RBR*. List all outcomes in the event. What is the probability of the event?

17. Consider the situation described in exercise 16.
 a. Find the probability of the event that exactly one of the colors that appears face up is red.
 b. Find the probability of the event that at least one of the colors that appears face up is red.

18. An urn contains two blue balls (denoted B_1 and B_2) and one white ball (denoted W). One ball is drawn, its color is recorded, and it is replaced in the urn. Then another ball is drawn, and its color is recorded.
 a. Let B_1W denote the outcome that the first ball drawn is B_1 and the second ball drawn is W. Because the first ball is replaced before the second ball is drawn, the outcomes of the experiment are equally likely. List all nine possible outcomes of the experiment.
 b. Consider the event that the two balls that are drawn are both blue. List all outcomes in the event. What is the probability of the event?
 c. Consider the event that the two balls that are drawn are of different colors. List all outcomes in the event. What is the probability of the event?

19. An urn contains two blue balls (denoted B_1 and B_2) and three white balls (denoted W_1, W_2, and W_3). One ball is drawn, its color is recorded, and it is replaced in the urn. Then another ball is drawn and its color is recorded.
 a. Let B_1W_2 denote the outcome that the first ball drawn is B_1 and the second ball drawn is W_2. Because the first ball is replaced before the second ball is drawn, the outcomes of the experiment are equally likely. List all 25 possible outcomes of the experiment.
 b. Consider the event that the first ball that is drawn is blue. List all outcomes in the event. What is the probability of the event?
 c. Consider the event that only white balls are drawn. List all outcomes in the event. What is the probability of the event?

20. Refer to Example 9.1.3. Suppose you are appearing on a game show with a prize behind one of five closed doors: A, B, C, D, and E. If you pick the right door, you win the prize. You pick door A. The game show host then opens one of the other doors and reveals that there is no prize behind it. Then the host gives you the option of staying with your original choice of door A or switching to one of the other doors that is still closed.
 a. If you stick with your original choice, what is the probability that you will win the prize?
 b. If you switch to another door, what is the probability that you will win the prize?

21. a. How many positive two-digit integers are multiples of 3?
 b. What is the probability that a randomly chosen positive two-digit integer is a multiple of 3?
 c. What is the probability that a randomly chosen positive two-digit integer is a multiple of 4?

22. a. How many positive three-digit integers are multiples of 6?
 b. What is the probability that a randomly chosen positive three-digit integer is a multiple of 6?
 c. What is the probability that a randomly chosen positive three-digit integer is a multiple of 7?

23. Suppose $A[1], A[2], A[3], \ldots, A[n]$ is a one-dimensional array and $n \geq 50$.
 a. How many elements are in the array?
 b. How many elements are in the subarray

$$A[4], A[5], \ldots, A[39]?$$

 c. If $3 \leq m \leq n$, what is the probability that a randomly chosen array element is in the subarray

$$A[3], A[4], \ldots, A[m]?$$

 d. What is the probability that a randomly chosen array element is in the subarray shown below if $n = 39$?

$$A[\lfloor n/2 \rfloor], A[\lfloor n/2 \rfloor + 1], \ldots, A[n]$$

24. Suppose $A[1], A[2], \ldots, A[n]$ is a one-dimensional array and $n \geq 2$. Consider the subarray

$$A[1], A[2], \ldots, A[\lfloor n/2 \rfloor].$$

 a. How many elements are in the subarray (i) if n is even? and (ii) if n is odd?
 b. What is the probability that a randomly chosen array element is in the subarray (i) if n is even? and (ii) if n is odd?

25. Suppose $A[1], A[2], \ldots, A[n]$ is a one-dimensional array and $n \geq 2$. Consider the subarray

$$A[\lfloor n/2 \rfloor], A[\lfloor n/2 \rfloor + 1], \ldots, A[n].$$

 a. How many elements are in the subarray (i) if n is even? and (ii) if n is odd?
 b. What is the probability that a randomly chosen array element is in the subarray (i) if n is even? and (ii) if n is odd?

26. What is the 27th element in the one-dimensional array $A[42], A[43], \ldots, A[100]$?

27. What is the 62nd element in the one-dimensional array $B[29], B[30], \ldots, B[100]$?

28. If the largest of 56 consecutive integers is 279, what is the smallest?

29. If the largest of 87 consecutive integers is 326, what is the smallest?

30. How many even integers are between 1 and 1,001?

31. How many integers that are multiples of 3 are between 1 and 1,001?

32. A certain non-leap year has 365 days, and January 1 occurs on a Monday.
 a. How many Sundays are in the year?
 b. How many Mondays are in the year?

✶ 33. Prove Theorem 9.1.1. (Let m be any integer and prove the theorem by mathematical induction on n.)