Answers for Test Yourself

1. the set of all outcomes of the random process or experiment 2. a subset of the sample space 3. number of outcomes in the event; total number of outcomes 4. $n - m + 1$

9.2 Possibility Trees and the Multiplication Rule

Don't believe anything unless you have thought it through for yourself.
— Anna Pell Wheeler, 1883–1966

A tree structure is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order. The following example shows how to use such a structure to count the number of different outcomes of a tournament.

Example 9.2.1 Possibilities for Tournament Play

Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games. One way in which this tournament can be played is for A to win the first game, B to win the second, and A to win the third and fourth games. Denote this by writing $A–B–A–A$.

a. How many ways can the tournament be played?

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

Solution

a. The possible ways for the tournament to be played are represented by the distinct paths from "root" (the start) to "leaf" (a terminal point) in the tree shown sideways in Figure 9.2.1. The label on each branching point indicates the winner of the game. The notations in parentheses indicate the winner of the tournament.

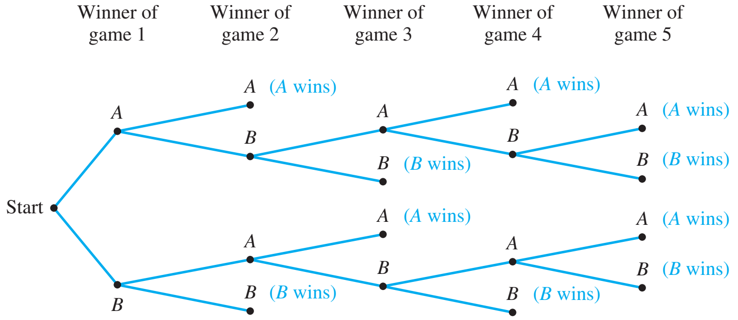


Figure 9.2.1 **The Outcomes of a Tournament**

The fact that there are ten paths from the root of the tree to its leaves shows that there are ten possible ways for the tournament to be played. They are (moving from the top down): $A–A$, $A–B–A–A$, $A–B–A–B–A$, $A–B–A–B–B$, $A–B–B$, $B–A–A$, $B–A–B–A–A$, $B–A–B–A–B$, $B–A–B–B$, and $B–B$. In five cases A wins, and in the other five B wins. The least number of games that must be played to determine a winner is two, and the most that will need to be played is five.

b. Since all the possible ways of playing the tournament listed in part (a) are assumed to be equally likely, and the listing shows that five games are needed in four different cases (A–B–A–B–A, A–B–A–B–B, B–A–B–A–B, and B–A–B–A–A), the probability that five games are needed is $4/10 = 2/5 = 40\%$. ■

The Multiplication Rule

Consider the following example. Suppose a computer installation has four input/output units (A, B, C, and D) and three central processing units (X, Y, and Z). Any input/output unit can be paired with any central processing unit. How many ways are there to pair an input/output unit with a central processing unit?

To answer this question, imagine the pairing of the two types of units as a two-step operation:

Step 1: Choose the input/output unit.

Step 2: Choose the central processing unit.

The possible outcomes of this operation are illustrated in the possibility tree of Figure 9.2.2.

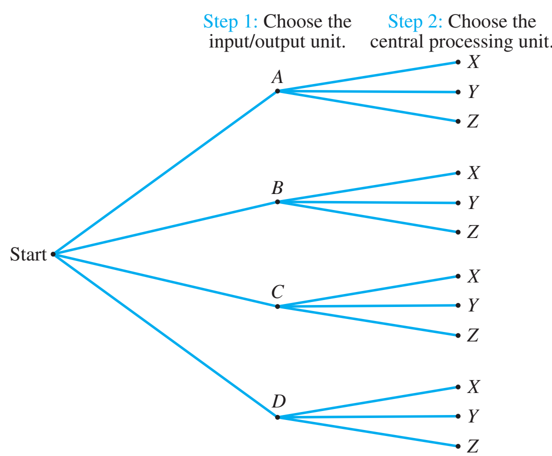


Figure 9.2.2 Pairing Objects Using a Possibility Tree

The topmost path from "root" to "leaf" indicates that input/output unit A is to be paired with central processing unit X. The next lower branch indicates that input/output unit A is to be paired with central processing unit Y. And so forth.

Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is

$$3 + 3 + 3 + 3 = 4 \cdot 3 = 12.$$

The idea behind this example can be used to prove the following rule. A formal proof uses mathematical induction and is left to the exercises.

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

⋮

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

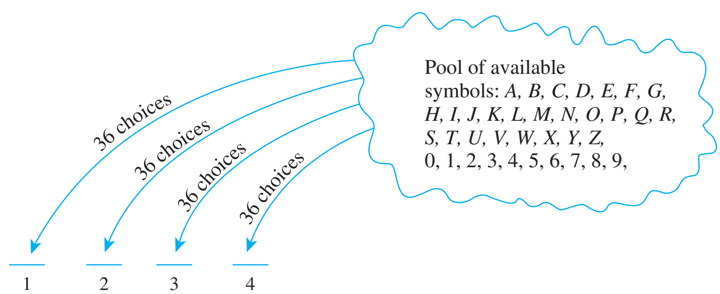
then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

To apply the multiplication rule, think of the objects you are trying to count as the output of a multistep operation. The possible ways to perform a step may depend on how preceding steps were performed, but the *number* of ways to perform each step must be constant regardless of the action taken in prior steps.

Example 9.2.2 Number of Personal Identification Numbers (PINs)

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed. How many different PINs are possible?

Solution Typical PINs are CARE, 3387, B32B, and so forth. You can think of forming a PIN as a four-step operation to fill in each of the four symbols in sequence.



Step 1: Choose the first symbol.

Step 2: Choose the second symbol.

Step 3: Choose the third symbol.

Step 4: Choose the fourth symbol.

There is a fixed number of ways to perform each step, namely 36, regardless of how preceding steps were performed. And so, by the multiplication rule, there are $36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1{,}679{,}616$ PINs in all. ■

Another way to look at the PINs of Example 9.2.2 is as ordered 4-tuples. For example, you can think of the PIN M2ZM as the ordered 4-tuple (M, 2, Z, M). Therefore, the total number of PINs is the same as the total number of ordered 4-tuples whose elements are either letters of the alphabet or digits. One of the most important uses of the multiplication rule is to derive a general formula for the number of elements in any Cartesian product of a finite number of finite sets. In Example 9.2.3, this is done for a Cartesian product of four sets.

Example 9.2.3 The Number of Elements in a Cartesian Product

Suppose A_1, A_2, A_3, and A_4 are sets with n_1, n_2, n_3, and n_4 elements, respectively. Show that the set $A_1 \times A_2 \times A_3 \times A_4$ has $n_1 n_2 n_3 n_4$ elements.

Solution Each element in $A_1 \times A_2 \times A_3 \times A_4$ is an ordered 4-tuple of the form (a_1, a_2, a_3, a_4), where $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, and $a_4 \in A_4$. Imagine the process of constructing these ordered tuples as a four-step operation:

Step 1: Choose the first element of the 4-tuple.

Step 2: Choose the second element of the 4-tuple.

Step 3: Choose the third element of the 4-tuple.

Step 4: Choose the fourth element of the 4-tuple.

There are n_1 ways to perform step 1, n_2 ways to perform step 2, n_3 ways to perform step 3, and n_4 ways to perform step 4. Hence, by the multiplication rule, there are $n_1 n_2 n_3 n_4$ ways to perform the entire operation. Therefore, there are $n_1 n_2 n_3 n_4$ distinct 4-tuples in $A_1 \times A_2 \times A_3 \times A_4$. ■

Example 9.2.4 Number of PINs without Repetition

In Example 9.2.2 we formed PINs using four symbols, either letters of the alphabet or digits, and supposing that letters could be repeated. Now suppose that repetition is not allowed.

a. How many different PINs are there?

b. If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbol?

Solution

a. Again think of forming a PIN as a four-step operation: Choose the first symbol, then the second, then the third, and then the fourth. There are 36 ways to choose the first symbol, 35 ways to choose the second (since the first symbol cannot be used again), 34 ways to choose the third (since the first two symbols cannot be reused), and 33 ways to choose the fourth (since the first three symbols cannot be reused). Thus, the multiplication rule can be applied to conclude that there are $36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$ different PINs with no repeated symbol.

b. By part (a) there are 1,413,720 PINs with no repeated symbol, and by Example 9.2.2 there are 1,679,616 PINs in all. Thus the probability that a PIN chosen at random contains no repeated symbol is $\frac{1{,}413{,}720}{1{,}679{,}616} \cong .8417$. In other words, approximately 84% of PINs have no repeated symbol. ■

Any circuit with two input signals P and Q has an input/output table consisting of four rows corresponding to the four possible assignments of values to P and Q: 11, 10, 01, and 00. The next example shows that there are only 16 distinct ways in which such a circuit can function.

Example 9.2.5 Number of Input/Output Tables for a Circuit with Two Input Signals

Consider the set of all circuits with two input signals P and Q. For each such circuit an input/output table can be constructed, but, as shown in Section 2.4, two such input/output tables may have the same values. How many distinct input/output tables can be constructed for circuits with input/output signals P and Q?

Solution Fix the order of the input values for P and Q. Then two input/output tables are distinct if their output values differ in at least one row. For example, the input/output tables shown below are distinct, because their output values differ in the first row.

P	Q	**Output**
1	1	1
1	0	0
0	1	1
0	0	0

P	Q	**Output**
1	1	0
1	0	0
0	1	1
0	0	0

For a fixed ordering of input values, you can obtain a complete input/output table by filling in the entries in the output column. You can think of this as a four-step operation:

Step 1: Fill in the output value for the first row.

Step 2: Fill in the output value for the second row.

Step 3: Fill in the output value for the third row.

Step 4: Fill in the output value for the fourth row.

Each step can be performed in exactly two ways: either a 1 or a 0 can be filled in. Hence, by the multiplication rule, there are

$$2 \cdot 2 \cdot 2 \cdot 2 = 16$$

ways to perform the entire operation. It follows that there are $2^4 = 16$ distinct input/output tables for a circuit with two input signals P and Q. This means that such a circuit can function in only 16 distinct ways. ■

Recall from Section 5.9 that if S is a nonempty, finite set of characters, then a string over S is a finite sequence of elements of S. The number of characters in a string is called the **length** of the string. The **null string over S** is the "string" with no characters. It is usually denoted ε and is said to have length 0.

Observe that in Examples 9.2.2 and 9.2.4, the set of all PINs of length 4 is the same as the set of all strings of length 4 over the set

$$S = \{x \mid x \text{ is a letter of the alphabet or } x \text{ is a digit}\}.$$

Also observe that another way to think of Example 9.2.5 is to realize that there are as many input/output tables for a circuit with two input signals as there are bit strings of length 4 (written vertically) that can be used to fill in the output values. As another example, here is a listing of all bit strings of length 3:

000, 001, 010, 100, 011, 101, 110, 111.

Example 9.2.6 Counting the Number of Iterations of a Nested Loop

Consider the following nested loop:

for $i := 1$ **to** 4
  for $j := 1$ **to** 3
    [*Statements in body of inner loop.*
    None contain branching statements
    that lead out of the inner loop.]
  next j
next i

How many times will the inner loop be iterated when the algorithm is implemented and run?

Solution The outer loop is iterated four times, and during each iteration of the outer loop, there are three iterations of the inner loop. Hence by the multiplication rule, the total number of iterations of the inner loop is $4 \cdot 3 = 12$. This is illustrated by the trace table below.

i	1 →		→	2 →		→	3 →		→	4 →		→
j	1	2	3	1	2	3	1	2	3	1	2	3

$3 + 3 + 3 + 3 = 12$ ■

When the Multiplication Rule Is Difficult or Impossible to Apply

Consider the following problem:

> Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

It is natural to try to solve this problem using the multiplication rule. A person might answer as follows:

> There are three choices for president (all except Ann), three choices for treasurer (all except the one chosen as president), and two choices for secretary (Cyd or Dan). Therefore, by the multiplication rule, there are $3 \cdot 3 \cdot 2 = 18$ choices in all.

Unfortunately, this analysis is incorrect. The number of ways to choose the secretary varies depending on who is chosen for president and treasurer. For instance, if Bob is chosen for president and Ann for treasurer, then there are two choices for secretary: Cyd and Dan. But if Bob is chosen for president and Cyd for treasurer, then there is just one choice for secretary: Dan. The clearest way to see all the possible choices is to construct the possibility tree, as is shown in Figure 9.2.3.

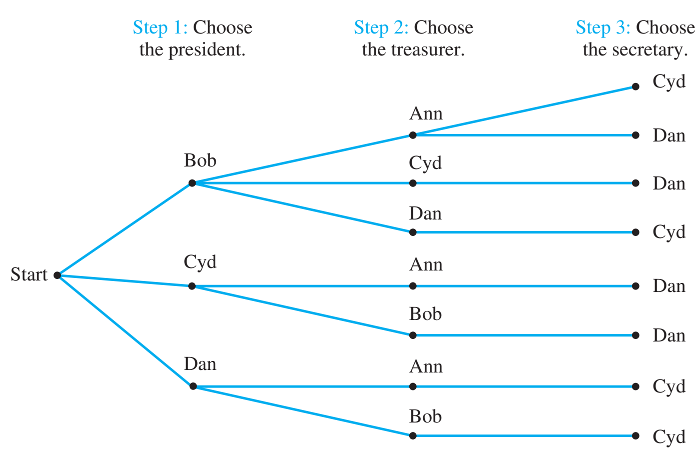


Figure 9.2.3

From the tree it is easy to see that there are only eight ways to choose a president, treasurer, and secretary so as to satisfy the given conditions.

Another way to solve this problem is somewhat surprising. It turns out that the steps can be reordered in a slightly different way so that the number of ways to perform each step is constant regardless of the way previous steps were performed.

Example 9.2.7 A More Subtle Use of the Multiplication Rule

Reorder the steps for choosing the officers in the previous example so that the total number of ways to choose officers can be computed using the multiplication rule.

Solution

Step 1: Choose the secretary.

Step 2: Choose the president.

Step 3: Choose the treasurer.

There are exactly two ways to perform step 1 (either Cyd or Dan may be chosen), two ways to perform step 2 (neither Ann nor the person chosen in step 1 may be chosen but either of the other two may), and two ways to perform step 3 (either of the two people not chosen as secretary or president may be chosen as treasurer). Thus, by the multiplication rule, the total number of ways to choose officers is $2 \cdot 2 \cdot 2 = 8$. A possibility tree illustrating this sequence of choices is shown in Figure 9.2.4. Note how balanced this tree is compared with the one in Figure 9.2.3.

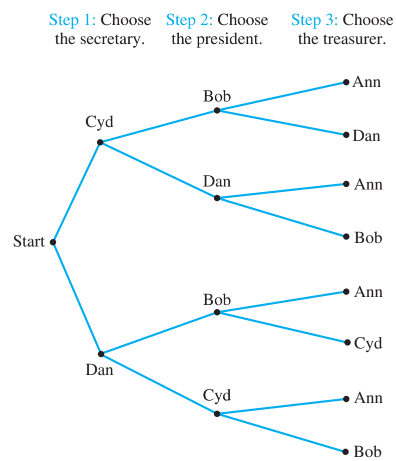


Figure 9.2.4

Permutations

A **permutation** of a set of objects is an ordering of the objects in a row. For example, the set of elements a, b, and c has six permutations.

$$abc \quad acb \quad cba \quad bac \quad bca \quad cab$$

In general, given a set of n objects, how many permutations does the set have? Imagine forming a permutation as an n-step operation:

Step 1: Choose an element to write first.

Step 2: Choose an element to write second.

⋮ ⋮

Step n: Choose an element to write nth.

Any element of the set can be chosen in step 1, so there are n ways to perform step 1. Any element except that chosen in step 1 can be chosen in step 2, so there are $n - 1$ ways to perform step 2. In general, the number of ways to perform each successive step is one less than the number of ways to perform the preceding step. At the point when the nth element is chosen, there is only one element left, so there is only one way to perform step n. Hence, by the multiplication rule, there are

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

ways to perform the entire operation. In other words, there are $n!$ permutations of a set of n elements. This reasoning is summarized in the following theorem. A formal proof uses mathematical induction and is left as an exercise.

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Example 9.2.8 Permutations of the Letters in a Word

a. How many ways can the letters in the word *COMPUTER* be arranged in a row?

b. How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

c. If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

Solution

a. All the eight letters in the word *COMPUTER* are distinct, so the number of ways in which we can arrange the letters equals the number of permutations of a set of eight elements. This equals $8! = 40{,}320$.

b. If the letter group *CO* is treated as a unit, then there are effectively only seven objects that are to be arranged in a row.

CO	M	P	U	T	E	R

Hence there are as many ways to write the letters as there are permutations of a set of seven elements, namely $7! = 5{,}040$.

c. When the letters are arranged randomly in a row, the total number of arrangements is 40,320 by part (a), and the number of arrangements with the letters *CO* next to each other (in order) as a unit is 5,040. Thus the probability is

$$\frac{5{,}040}{40{,}320} = \frac{1}{8} = 12.5\%. \qquad ■$$

Example 9.2.9 Permutations of Objects Around a Circle

At a meeting of diplomats, the six participants are to be seated around a circular table. Since the table has no ends to confer particular status, it doesn't matter who sits in which chair. But it does matter how the diplomats are seated relative to each other. In other words, two seatings are considered the same if one is a rotation of the other. How many different ways can the diplomats be seated?

Solution Call the diplomats by the letters A, B, C, D, E, and F. Since only relative position matters, you can start with any diplomat (say A), place that diplomat anywhere (say in the top seat of the diagram shown in Figure 9.2.5), and then consider all arrangements of the other diplomats around that one. B through F can be arranged in the seats around diplomat A in all possible orders. So there are $5! = 120$ ways to seat the group.

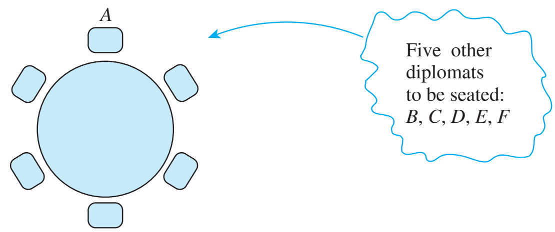


Figure 9.2.5

Permutations of Selected Elements

Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of two elements of $\{a, b, c\}$ is called a 2-*permutation* of $\{a, b, c\}$.

• Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

Theorem 9.2.3

If n and r are integers and $1 \le r \le n$, then the number of r-permutations of a set of n elements is given by the formula

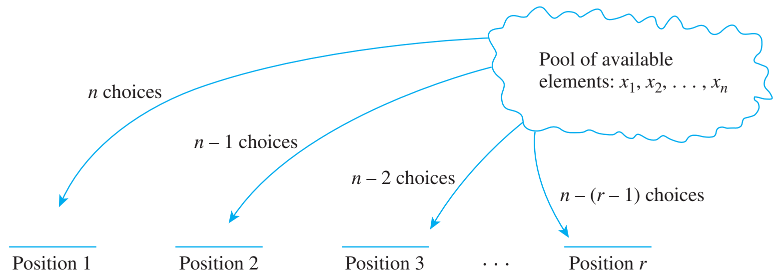
$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$

A formal proof of this theorem uses mathematical induction and is based on the multiplication rule. The idea of the proof is the following.

Suppose a set of n elements is given. Formation of an r-permutation can be thought of as an r-step process. Step 1 is to choose the element to be first. Since the set has n elements, there are n ways to perform step 1. Step 2 is to choose the element to be second. Since the element chosen in step 1 is no longer available, there are $n - 1$ ways to perform step 2. Step 3 is to choose the element to be third. Since neither of the two elements chosen in the first two steps is available, there are $n - 2$ choices for step 3. This process is repeated r times, as shown on the next page.



The number of ways to perform each successive step is one less than the number of ways to perform the preceding step. Step r is to choose the element to be rth. At the point just before step r is performed, $r-1$ elements have already been chosen, and so there are

$$n-(r-1) = n-r+1$$

left to choose from. Hence there are $n-r+1$ ways to perform step r. It follows by the multiplication rule that the number of ways to form an r-permutation is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1).$$

Note that

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$
$$= n(n-1)(n-2)\cdots(n-r+1).$$

Thus the formula can be written as

$$P(n,r) = \frac{n!}{(n-r)!}.$$

The second version of the formula is easier to remember. When you actually use it, however, first substitute the values of n and r and then immediately cancel the numerical value of $(n-r)!$ from the numerator and denominator. Because factorials become so large so fast, direct use of the second version of the formula without cancellation can overload your calculator's capacity for exact arithmetic even when n and r are quite small. For instance, if $n = 15$ and $r = 2$, then

$$\frac{n!}{(n-r)!} = \frac{15!}{13!} = \frac{1{,}307{,}674{,}368{,}000}{6{,}227{,}020{,}800}.$$

But if you cancel $(n-r)! = 13!$ from numerator and denominator before multiplying out, you obtain

$$\frac{n!}{(n-r)!} = \frac{15!}{13!} = \frac{15\cdot 14\cdot \cancel{13!}}{\cancel{13!}} = 15\cdot 14 = 210.$$

In fact, many scientific calculators allow you to compute $P(n,r)$ simply by entering the values of n and r and pressing a key or making a menu choice. Alternative notations for $P(n,r)$ that you may see in your calculator manual are ${}_nP_r$, $P_{n,r}$ and ${}^nP_r$.

Example 9.2.10 Evaluating r-Permutations

a. Evaluate $P(5,2)$.

b. How many 4-permutations are there of a set of seven objects?

c. How many 5-permutations are there of a set of five objects?

Solution

a. $P(5, 2) = \dfrac{5!}{(5-2)!} = \dfrac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 20$

b. The number of 4-permutations of a set of seven objects is

$$P(7, 4) = \frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

c. The number of 5-permutations of a set of five objects is

$$P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120.$$

Note that the definition of 0! as 1 makes this calculation come out as it should, for the number of 5-permutations of a set of five objects is certainly equal to the number of permutations of the set. ■

Example 9.2.11 Permutations of Selected Letters of a Word

a. How many different ways can three of the letters of the word *BYTES* be chosen and written in a row?

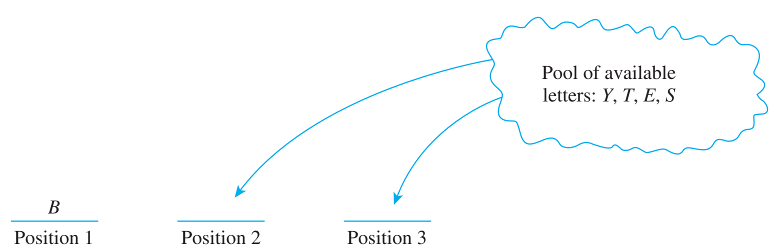
b. How many different ways can this be done if the first letter must be B?

Solution

a. The answer equals the number of 3-permutations of a set of five elements. This equals

$$P(5, 3) = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 5 \cdot 4 \cdot 3 = 60.$$

b. Since the first letter must be B, there are effectively only two letters to be chosen and placed in the other two positions. And since the B is used in the first position, there are four letters available to fill the remaining two positions.



Hence the answer is the number of 2-permutations of a set of four elements, which is

$$P(4, 2) = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 4 \cdot 3 = 12.$$ ■

In many applications of the mathematics of counting, it is necessary to be skillful in working algebraically with quantities of the form $P(n, r)$. The next example shows a kind of problem that gives practice in developing such skill.

Example 9.2.12 Proving a Property of $P(n, r)$

Prove that for all integers $n \geq 2$,

$$P(n, 2) + P(n, 1) = n^2.$$

Solution Suppose n is an integer that is greater than or equal to 2. By Theorem 9.2.3,

$$P(n,2) = \frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1)$$

and

$$P(n,1) = \frac{n!}{(n-1)!} = \frac{n\cdot\cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

Hence

$$P(n,2) + P(n,1) = n\cdot(n-1) + n = n^2 - n + n = n^2,$$

which is what we needed to show. ■

Test Yourself

1. The multiplication rule says that if an operation can be performed in k steps and, for each i with $1 \le i \le k$, the ith step can be performed in n_i ways (regardless of how previous steps were performed), then the operation as a whole can be performed in _____.
2. A permutation of a set of elements is _____.
3. The number of permutations of a set of n elements equals _____.
4. An r-permutation of a set of n elements is _____.
5. The number of r-permutations of a set of n elements is denoted _____.
6. One formula for the number of r-permutations of a set of n elements is _____ and another formula is _____.

Exercise Set 9.2

In 1–4, use the fact that in baseball's World Series, the first team to win four games wins the series.

1. Suppose team A wins the first three games. How many ways can the series be completed? (Draw a tree.)

2. Suppose team A wins the first two games. How many ways can the series be completed? (Draw a tree.)

3. How many ways can a World Series be played if team A wins four games in a row?

4. How many ways can a World Series be played if no team wins two games in a row?

5. In a competition between players X and Y, the first player to win three games in a row or a total of four games wins. How many ways can the competition be played if X wins the first game and Y wins the second and third games? (Draw a tree.)

6. One urn contains two black balls (labeled B_1 and B_2) and one white ball. A second urn contains one black ball and two white balls (labeled W_1 and W_2). Suppose the following experiment is performed: One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.
   a. Construct the possibility tree showing all possible outcomes of this experiment.
   b. What is the total number of outcomes of this experiment?
   c. What is the probability that two black balls are chosen?
   d. What is the probability that two balls of opposite color are chosen?

7. One urn contains one blue ball (labeled B_1) and three red balls (labeled R_1, R_2, and R_3). A second urn contains two red balls (R_4 and R_5) and two blue balls (B_2 and B_3). An experiment is performed in which one of the two urns is chosen at random and then two balls are randomly chosen from it, one after the other without replacement.
   a. Construct the possibility tree showing all possible outcomes of this experiment.
   b. What is the total number of outcomes of this experiment?
   c. What is the probability that two red balls are chosen?

8. A person buying a personal computer system is offered a choice of three models of the basic unit, two models of keyboard, and two models of printer. How many distinct systems can be purchased?

9. Suppose there are three roads from city A to city B and five roads from city B to city C.
   a. How many ways is it possible to travel from city A to city C via city B?
   b. How many different round-trip routes are there from city A to B to C to B and back to A?
   c. How many different routes are there from city A to B to C to B and back to A in which no road is traversed twice?

10. Suppose there are three routes from North Point to Boulder Creek, two routes from Boulder Creek to Beaver Dam, two routes from Beaver Dam to Star Lake, and four routes directly from Boulder Creek to Star Lake. (Draw a sketch.)
    a. How many routes from North Point to Star Lake pass through Beaver Dam?
    b. How many routes from North Point to Star Lake bypass Beaver Dam?

11. **a.** A bit string is a finite sequence of 0's and 1's. How many bit strings have length 8?
    b. How many bit strings of length 8 begin with three 0's?
    c. How many bit strings of length 8 begin and end with a 1?

12. Hexadecimal numbers are made using the sixteen digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. They are denoted by the subscript 16. For example, $9A2D_{16}$ and $BC54_{16}$ are hexadecimal numbers.
    a. How many hexadecimal numbers begin with one of the digits 3 through B, end with one of the digits 5 through F, and are 5 digits long?
    b. How many hexadecimal numbers begin with one of the digits 4 through D, end with one of the digits 2 through E, and are 6 digits long?

13. A coin is tossed four times. Each time the result *H* for heads or *T* for tails is recorded. An outcome of *HHTT* means that heads were obtained on the first two tosses and tails on the second two. Assume that heads and tails are equally likely on each toss.
    a. How many distinct outcomes are possible?
    b. What is the probability that exactly two heads occur?
    c. What is the probability that exactly one head occurs?

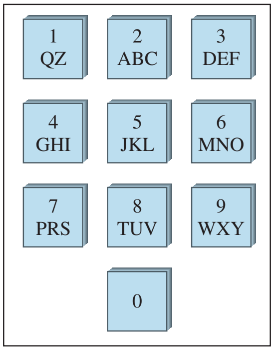
14. Suppose that in a certain state, all automobile license plates have four letters followed by three digits.
    a. How many different license plates are possible?
    b. How many license plates could begin with *A* and end in 0?
    c. How many license plates could begin with *TGIF*?
    d. How many license plates are possible in which all the letters and digits are distinct?
    e. How many license plates could begin with *AB* and have all letters and digits distinct?

15. A combination lock requires three selections of numbers, each from 1 through 30.
    a. How many different combinations are possible?
    b. Suppose the locks are constructed in such a way that no number may be used twice. How many different combinations are possible?

16. a. How many integers are there from 10 through 99?
    b. How many odd integers are there from 10 through 99?
    c. How many integers from 10 through 99 have distinct digits?
    d. How many odd integers from 10 through 99 have distinct digits?
    e. What is the probability that a randomly chosen two-digit integer has distinct digits? has distinct digits and is odd?

17. a. How many integers are there from 1000 through 9999?
    b. How many odd integers are there from 1000 through 9999?
    c. How many integers from 1000 through 9999 have distinct digits?
    d. How many odd integers from 1000 through 9999 have distinct digits?
    e. What is the probability that a randomly chosen four-digit integer has distinct digits? has distinct digits and is odd?

18. The diagram below shows the keypad for an automatic teller machine. As you can see, the same sequence of keys represents a variety of different PINs. For instance, 2133, AZDE, and BQ3F are all keyed in exactly the same way.



    a. How many different PINs are represented by the same sequence of keys as 2133?
    b. How many different PINs are represented by the same sequence of keys as 5031?
    c. At an automatic teller machine, each PIN corresponds to a four-digit numeric sequence. For instance, TWJM corresponds to 8956. How many such numeric sequences contain no repeated digit?

19. Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that Bob is not qualified to be treasurer and Cyd's other commitments make it impossible for her to be secretary. How many ways can the officers be chosen? Can the multiplication rule be used to solve this problem?

20. Modify Example 9.2.4 by supposing that a PIN must not begin with any of the letters A–M and must end with a digit. Continue to assume that no symbol may be used more than once and that the total number of PINs is to be determined.
   a. Find the error in the following "solution."

   "Constructing a PIN is a four-step process.

   Step 1: Choose the left-most symbol.

   Step 2: Choose the second symbol from the left.

   Step 3: Choose the third symbol from the left.

   Step 4: Choose the right-most symbol.

   Because none of the thirteen letters from A through M may be chosen in step 1, there are $36 - 13 = 23$ ways to perform step 1. There are 35 ways to perform step 2 and 34 ways to perform step 3 because previously used symbols may not be used. Since the symbol chosen in step 4 must be a previously unused digit, there are $10 - 3 = 7$ ways to perform step 4. Thus there are $23 \cdot 35 \cdot 34 \cdot 7 = 191{,}590$ different PINs that satisfy the given conditions."

   b. Reorder steps 1–4 in part (a) as follows:

   Step 1: Choose the right-most symbol.

   Step 2: Choose the left-most symbol.

   Step 3: Choose the second symbol from the left.

   Step 4: Choose the third symbol from the left.

   Use the multiplication rule to find the number of PINs that satisfy the given conditions.

H **21.** Suppose A is a set with m elements and B is a set with n elements.
   a. How many relations are there from A to B? Explain.
   b. How many functions are there from A to B? Explain.
   c. What fraction of the relations from A to B are functions?

22. **a.** How many functions are there from a set with three elements to a set with four elements?
   b. How many functions are there from a set with five elements to a set with two elements?
   c. How many functions are there from a set with m elements to a set with n elements, where m and n are positive integers?

23. In Section 2.5 we showed how integers can be represented by strings of 0's and 1's inside a digital computer. In fact, through various coding schemes, strings of 0's and 1's can be used to represent all kinds of symbols. One commonly used code is the Extended Binary-Coded Decimal Interchange Code (EBCDIC) in which each symbol has an 8-bit representation. How many distinct symbols can be represented by this code?

In each of 24–28, determine how many times the innermost loop will be iterated when the algorithm segment is implemented and run. (Assume that $m, n, p, a, b, c,$ and d are all positive integers.)

24. **for** $i := 1$ **to** 30
      for $j := 1$ **to** 15
         [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
      next j
   next i

25. **for** $j := 1$ **to** m
      for $k := 1$ **to** n
         [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
      next k
   next j

26. **for** $i := 1$ **to** m
      for $j := 1$ **to** n
         for $k := 1$ **to** p
            [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
         next k
      next j
   next i

27. **for** $i := 5$ **to** 50
      for $j := 10$ **to** 20
         [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
      next j
   next i

28. Assume $a \le b$ and $c \le d$.
   for $i := a$ **to** b
      for $j := c$ **to** d
         [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
      next j
   next i

H ✶ **29.** Consider the numbers 1 through 99,999 in their ordinary decimal representations. How many contain exactly one of each of the digits 2, 3, 4, and 5?

✶30. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ where $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $k_1, k_2, \ldots, k_m$ are positive integers. How many ways can n be written as a product of two positive integers that have no common factors
   a. assuming that order matters (i.e., $8 \cdot 15$ and $15 \cdot 8$ are regarded as different)?
   b. assuming that order does not matter (i.e., $8 \cdot 15$ and $15 \cdot 8$ are regarded as the same)?

✶31. **a.** If p is a prime number and a is a positive integer, how many distinct positive divisors does p^a have?
   b. If p and q are distinct prime numbers and a and b are positive integers, how many distinct positive divisors does $p^a q^b$ have?
   c. If p, q, and r are distinct prime numbers and a, b, and c are positive integers, how many distinct positive divisors does $p^a q^b r^c$ have?
   d. If $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $a_1, a_2, \ldots, a_m$ are positive integers, how many distinct positive divisors does $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ have?
   e. What is the smallest positive integer with exactly 12 divisors?

32. **a.** How many ways can the letters of the word *ALGORITHM* be arranged in a row?
   b. How many ways can the letters of the word *ALGORITHM* be arranged in a row if A and L must remain together (in order) as a unit?
   c. How many ways can the letters of the word *ALGORITHM* be arranged in a row if the letters *GOR* must remain together (in order) as a unit?

33. Six people attend the theater together and sit in a row with exactly six seats.
   a. How many ways can they be seated together in the row?
   b. Suppose one of the six is a doctor who must sit on the aisle in case she is paged. How many ways can the people be seated together in the row with the doctor in an aisle seat?
   c. Suppose the six people consist of three married couples and each couple wants to sit together with the husband on the left. How many ways can the six be seated together in the row?

34. Five people are to be seated around a circular table. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

35. Write all the 2-permutations of $\{W, X, Y, Z\}$.

36. Write all the 3-permutations of $\{s, t, u, v\}$.

37. Evaluate the following quantities.
   a. $P(6, 4)$ b. $P(6, 6)$ c. $P(6, 3)$ d. $P(6, 1)$

38. **a.** How many 3-permutations are there of a set of five objects?
   b. How many 2-permutations are there of a set of eight objects?

39. **a.** How many ways can three of the letters of the word *ALGORITHM* be selected and written in a row?
   b. How many ways can six of the letters of the word *ALGORITHM* be selected and written in a row?
   c. How many ways can six of the letters of the word *ALGORITHM* be selected and written in a row if the first letter must be A?
   d. How many ways can six of the letters of the word *ALGORITHM* be selected and written in a row if the first two letters must be *OR*?

40. Prove that for all integers $n \geq 2$, $P(n+1, 3) = n^3 - n$.

41. Prove that for all integers $n \geq 2$,

$$P(n+1, 2) - P(n, 2) = 2P(n, 1).$$

42. Prove that for all integers $n \geq 3$,

$$P(n+1, 3) - P(n, 3) = 3P(n, 2).$$

43. Prove that for all integers $n \geq 2$, $P(n, n) = P(n, n-1)$.

44. Prove Theorem 9.2.1 by mathematical induction.

H **45.** Prove Theorem 9.2.2 by mathematical induction.

✶46. Prove Theorem 9.2.3 by mathematical induction.

47. A permutation on a set can be regarded as a function from the set to itself. For instance, one permutation of $\{1, 2, 3, 4\}$ is 2341. It can be identified with the function that sends each position number to the number occupying that position. Since position 1 is occupied by 2, 1 is sent to 2 or $1 \to 2$; since position 2 is occupied by 3, 2 is sent to 3 or $2 \to 3$; and so forth. The entire permutation can be written using arrows as follows:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{array}$$

   a. Use arrows to write each of the six permutations of $\{1, 2, 3\}$.
   b. Use arrows to write each of the permutations of $\{1, 2, 3, 4\}$ that keep 2 and 4 fixed.
   c. Which permutations of $\{1, 2, 3\}$ keep no elements fixed?
   d. Use arrows to write all permutations of $\{1, 2, 3, 4\}$ that keep no elements fixed.

Answers for Test Yourself

1. $n_1 n_2 \cdots n_k$ ways 2. an ordering of the elements of the set in a row 3. $n!$ 4. an ordered selection of r of the elements of the set 5. $P(n, r)$ 6. $n(n-1)(n-2)\cdots(n-r+1)$; $\frac{n!}{(n-r)!}$