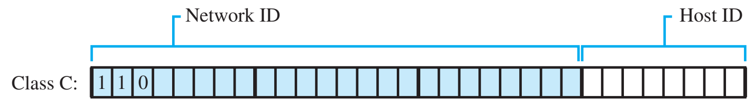


a. Check that the dotted decimal form of 10001100 11000000 00100000 10001000 is 140.192.32.136.

b. How many Class B networks can there be?

c. What is the dotted decimal form of the IP address for a computer in a Class B network?

d. How many host IDs can there be for a Class B network?

Solution

a. $10001100 = 1 \cdot 2^7 + 1 \cdot 2^3 + 1 \cdot 2^2 = 128 + 8 + 4 = 140$
$11000000 = 1 \cdot 2^7 + 1 \cdot 2^6 = 128 + 64 = 192$
$00100000 = 1 \cdot 2^5 = 32$
$10001000 = 1 \cdot 2^7 + 1 \cdot 2^3 = 128 + 8 = 136$

b. The network ID for a Class B network consists of 16 bits and begins with 10. Because there are two choices for each of the remaining 14 positions (either 0 or 1), the total number of possible network IDs is 2^{14}, or 16,384.

c. The network ID part of a Class B IP address goes from

$$10000000\ 00000000 \text{ to } 10111111\ 11111111.$$

As dotted decimals, these numbers range from 128.0 to 191.255 because $10000000_2 = 128_{10}$, $00000000_2 = 0_{10}$, $10111111_2 = 191_{10}$, and $11111111_2 = 255_{10}$. Thus the dotted decimal form of the IP address of a computer in a Class B network is $w.x.y.z$, where $128 \le w \le 191$, $0 \le x \le 255$, $0 \le y \le 255$, and $0 \le z \le 255$. However, y and z are not allowed both to be 0 or both to be 255 because host IDs may not consist of either all 0's or all 1's.

d. For a class B network, 16 bits are used for host IDs. Having two choices (either 0 or 1) for each of 16 positions gives a potential total of 2^{16}, or 65,536, host IDs. But because two of these are not allowed (all 0's and all 1's), the total number of host IDs is 65,534. ■

The Inclusion/Exclusion Rule

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a union of sets when some of the sets overlap. For simplicity, begin by looking at a union of two sets A and B, as shown in Figure 9.3.5.

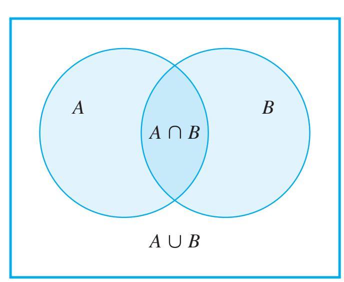


Figure 9.3.5

First observe that the number of elements in $A \cup B$ varies according to the number of elements the two sets have in common. If A and B have no elements in common, then $N(A \cup B) = N(A) + N(B)$. If A and B coincide, then $N(A \cup B) = N(A)$. Thus any general formula for $N(A \cup B)$ must contain a reference to the number of elements the two sets have in common, $N(A \cap B)$, as well as to $N(A)$ and $N(B)$.

The simplest way to derive a formula for $N(A \cup B)$ is to reason as follows: The number $N(A)$ counts the elements that are in A and not in B and also the elements that are in both A and B. Similarly, the number $N(B)$ counts the elements that are in B and not in A and also the elements that are in both A and B. Hence when the two numbers $N(A)$ and $N(B)$ are added, the elements that are in both A and B are counted twice. To get an accurate count of the elements in $A \cup B$, it is necessary to subtract the number of elements that are in both A and B. Because these are the elements in $A \cap B$,

Note An alternative proof is outlined in exercise 46 at the end of this section.

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

A similar analysis gives a formula for the number of elements in a union of three sets, as shown in Theorem 9.3.3.

Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

It can be shown using mathematical induction (see exercise 48 at the end of this section) that formulas analogous to those of Theorem 9.3.3 hold for unions of any finite number of sets.

Example 9.3.6 Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Solution

a. Let A = the set of all integers from 1 through 1,000 that are multiples of 3.
Let B = the set of all integers from 1 through 1,000 that are multiples of 5.

Then

$A \cup B$ = the set of all integers from 1 through 1,000 that are multiples of 3 or multiples of 5

and

$A \cap B$ = the set of all integers from 1 through 1,000 that are multiples of both 3 and 5
= the set of all integers from 1 through 1,000 that are multiples of 15.

[Now calculate $N(A)$, $N(B)$, and $N(A \cap B)$ and use the inclusion/exclusion rule to solve for $N(A \cup B)$.]

Because every third integer from 3 through 999 is a multiple of 3, each can be represented in the form $3k$, for some integer k from 1 through 333. Hence there are 333 multiples of 3 from 1 through 1,000, and so $N(A) = 333$.

$$\boxed{\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 996 & 997 & 998 & 999 \\ & & \updownarrow & & & \updownarrow & & \updownarrow & & & \updownarrow \\ & & 3\cdot 1 & & & 3\cdot 2 & & 3\cdot 332 & & & 3\cdot 333 \end{array}}$$

Similarly, each multiple of 5 from 1 through 1,000 has the form $5k$, for some integer k from 1 through 200.

$$\boxed{\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots & 995 & 996 & 997 & 998 & 999 & 1{,}000 \\ & & & & \updownarrow & & & & & \updownarrow & & \updownarrow & & & & & \updownarrow \\ & & & & 5\cdot 1 & & & & & 5\cdot 2 & & 5\cdot 199 & & & & & 5\cdot 200 \end{array}}$$

Thus there are 200 multiples of 5 from 1 through 1,000 and $N(B) = 200$.
Finally, each multiple of 15 from 1 through 1,000 has the form $15k$, for some integer k from 1 through 66 (since $990 = 66 \cdot 15$).

$$\boxed{\begin{array}{cccccccccccc} 1 & 2 & \ldots & 15 & \ldots & 30 & \ldots & 975 & \ldots & 990 & \ldots & 999 & 1{,}000 \\ & & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & & \\ & & & 15\cdot 1 & & 15\cdot 2 & & 15\cdot 65 & & 15\cdot 66 & & & \end{array}}$$

Hence there are 66 multiples of 15 from 1 through 1,000, and $N(A \cap B) = 66$.
It follows by the inclusion/exclusion rule that

$$\begin{aligned} N(A \cup B) &= N(A) + N(B) - N(A \cap B) \\ &= 333 + 200 - 66 \\ &= 467. \end{aligned}$$

Thus, 467 integers from 1 through 1,000 are multiples of 3 or multiples of 5.

b. There are 1,000 integers from 1 through 1,000, and by part (a), 467 of these are multiples of 3 or multiples of 5. Thus, by the set difference rule, there are $1{,}000 - 467 = 533$ that are neither multiples of 3 nor multiples of 5. ■

Note that the solution to part (b) of Example 9.3.6 hid a use of De Morgan's law. The number of elements that are neither in A nor in B is $N(A^c \cap B^c)$, and by De Morgan's law, $A^c \cap B^c = (A \cup B)^c$. So $N((A \cup B)^c)$ was then calculated using the set difference rule: $N((A \cup B)^c) = N(U) - N(A \cup B)$, where the universe U was the set of all integers from 1 through 1,000. Exercises 37–39 at the end of this section explore this technique further.

Example 9.3.7 Counting the Number of Elements in an Intersection

A professor in a discrete mathematics class passes out a form asking students to check all the mathematics and computer science courses they have recently taken. The finding is that out of a total of 50 students in the class,

30 took precalculus;
18 took calculus;
26 took Java;
9 took both precalculus and calculus;
16 took both precalculus and Java;
8 took both calculus and Java;
47 took at least one of the three courses.

Note that when we write "30 students took precalculus," we mean that the total number of students who took precalculus is 30, and we allow for the possibility that some of these students may have taken one or both of the other courses. If we want to say that 30 students took precalculus *only* (and not either of the other courses), we will say so explicitly.

a. How many students did not take any of the three courses?

b. How many students took all three courses?

c. How many students took precalculus and calculus but not Java? How many students took precalculus but neither calculus nor Java?

Solution

a. By the difference rule, the number of students who did not take any of the three courses equals the number in the class minus the number who took at least one course. Thus the number of students who did not take any of the three courses is

$$50 - 47 = 3.$$

b. Let

$$P = \text{the set of students who took precalculus}$$
$$C = \text{the set of students who took calculus}$$
$$J = \text{the set of students who took Java.}$$

Then, by the inclusion/exclusion rule,

$$N(P \cup C \cup J) = N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$

Substituting known values, we get

$$47 = 30 + 26 + 18 - 9 - 16 - 8 + N(P \cap C \cap J).$$

Solving for $N(P \cap C \cap J)$ gives

$$N(P \cap C \cap J) = 6.$$

Hence there are six students who took all three courses. In general, if you know any seven of the eight terms in the inclusion/exclusion formula for three sets, you can solve for the eighth term.

c. To answer the questions of part (c), look at the diagram in Figure 9.3.6.

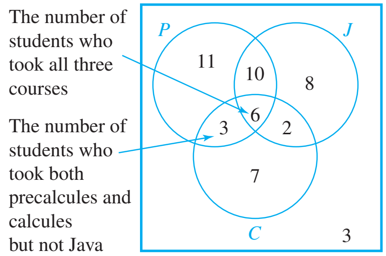


Figure 9.3.6

Since $N(P \cap C \cap J) = 6$, put the number 6 inside the innermost region. Then work outward to find the numbers of students represented by the other regions of the diagram. For example, since nine students took both precalculus and calculus and six took all three courses, $9 - 6 = 3$ students took precalculus and calculus but not Java. Similarly, since 16 students took precalculus and calculus and six took all three courses, $16 - 6 = 10$ students took precalculus and calculus but not Java. Now the total number of students who took precalculus is 30. Of these 30, three also took calculus but not Java, ten took Java but not calculus, and six took both calculus and Java. That leaves 11 students who took precalculus but neither of the other two courses.

A similar analysis can be used to fill in the numbers for the other regions of the diagram. ■

Test Yourself

1. The addition rule says that if a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$, then _____.
2. The difference rule says that if A is a finite set and B is a subset of A, then _____.
3. If S is a finite sample space and A is an event in S, then the probability of A^c equals _____.
4. The inclusion/exclusion rule for two sets says that if A and B are any finite sets, then _____.
5. The inclusion/exclusion rule for three sets says that if A, B, and C are any finite sets, then _____.

Exercise Set 9.3

1. a. How many bit strings consist of from one through four digits? (Strings of different lengths are considered distinct. Thus 10 and 0010 are distinct strings.)
   b. How many bit strings consist of from five through eight digits?

2. a. How many strings of hexadecimal digits consist of from one through three digits? (Recall that hexadecimal numbers are constructed using the 16 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.)
   b. How many strings of hexadecimal digits consist of from two through five digits?

3. a. How many integers from 1 through 999 do not have any repeated digits?
   b. How many integers from 1 through 999 have at least one repeated digit?
   c. What is the probability that an integer chosen at random from 1 through 999 has at least one repeated digit?

4. How many arrangements in a row of no more than three letters can be formed using the letters of the word *NETWORK* (with no repetitions allowed)?

5. a. How many five-digit integers (integers from 10,000 through 99,999) are divisible by 5?
   b. What is the probability that a five-digit integer chosen at random is divisible by 5?

6. In a certain state, license plates consist of from zero to three letters followed by from zero to four digits, with the provision, however, that a blank plate is not allowed.
   a. How many different license plates can the state produce?
   b. Suppose 85 letter combinations are not allowed because of their potential for giving offense. How many different license plates can the state produce?

7. In another state, all license plates consist of from four to six symbols chosen from the 26 letters of the alphabet together with the ten digits 0–9.
   a. How many license plates are possible if repetition of symbols is allowed?
   b. How many license plates do not contain any repeated symbol?
   H **c.** How many license plates have at least one repeated symbol?
   d. What is the probability that a license plate chosen at random has a repeated symbol?

8. At a certain company, passwords must be from 3–5 symbols long and composed of the 26 letters of the alphabet, the ten digits 0–9, and the 14 symbols !,@,#,$,%,^,&, *,(,),−,+,{, and }.
   a. How many passwords are possible if repetition of symbols is allowed?
   b. How many passwords contain no repeated symbols?

c. How many passwords have at least one repeated symbol?

d. What is the probability that a password chosen at random has n repeated symbol?

9. **a.** Consider the following algorithm segment:

    for $i := 1$ **to** 4
        for $j := 1$ **to** i
        [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
        next j
    next i

    How many times will the inner loop be iterated when the algorithm is implemented and run?

    b. Let n be a positive integer, and consider the following algorithm segment:

    for $i := 1$ **to** n
        for $j := 1$ **to** i
        [*Statements in body of inner loop. None contain branching statements that lead outside the loop.*]
        next j
    next i

    How many times will the inner loop be iterated when the algorithm is implemented and run?

✶ 10. A calculator has an eight-digit display and a decimal point that is located at the extreme right of the number displayed, at the extreme left, or between any pair of digits. The calculator can also display a minus sign at the extreme left of the number. How many distinct numbers can the calculator display? (Note that certain numbers are equal, such as 1.9, 1.90, and 01.900, and should, therefore, not be counted twice.)

11. a. How many ways can the letters of the word *QUICK* be arranged in a row?

b. How many ways can the letters of the word *QUICK* be arranged in a row if the Q and the U must remain next to each other in the order QU?

c. How many ways can the letters of the word *QUICK* be arranged in a row if the letters QU must remain together but may be in either the order QU or the order UQ?

12. a. How many ways can the letters of the word *THEORY* be arranged in a row?

b. How many ways can the letters of the word *THEORY* be arranged in a row if T and H must remain next to each other as either TH or HT?

13. A group of eight people are attending the movies together.

a. Two of the eight insist on sitting side-by-side. In how many ways can the eight be seated together in a row?

b. Two of the people do not like each other and do not want to sit side-by-side. Now how many ways can the eight be seated together in a row?

14. An early compiler recognized variable names according to the following rules: Numeric variable names had to begin with a letter, and then the letter could be followed by another letter or a digit or by nothing at all. String variable names had to begin with the symbol $ followed by a letter, which could then be followed by another letter or a digit or by nothing at all. How many distinct variable names were recognized by this compiler?

H **15.** Identifiers in a certain database language must begin with a letter, and then the letter may be followed by other characters, which can be letters, digits, or underscores (_). However, 82 keywords (all consisting of 15 or fewer characters) are reserved and cannot be used as identifiers. How many identifiers with 30 or fewer characters are possible? (Write the answer using summation notation and evaluate it using a formula from Section 5.2.)

16. a. If any seven digits could be used to form a telephone number, how many seven-digit telephone numbers would not have any repeated digits?

b. How many seven-digit telephone numbers would have at least one repeated digit?

c. What is the probability that a randomly chosen seven-digit telephone number would have at least one repeated digit?

17. a. How many strings of four hexadecimal digits do not have any repeated digits?

b. How many strings of four hexadecimal digits have at least one repeated digit?

c. What is the probability that a randomly chosen string of four hexadecimal digits has at least one repeated digit?

18. Just as the difference rule gives rise to a formula for the probability of the complement of an event, so the addition and inclusion/exclusion rules give rise to formulas for the probability of the union of mutually disjoint events and for a general union of (not necessarily mutually exclusive) events.

a. Prove that for mutually disjoint events A and B,

$$P(A \cup B) = P(A) + P(B).$$

b. Prove that for any events A and B.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

H **19.** A combination lock requires three selections of numbers, each from 1 through 39. Suppose the lock is constructed in such a way that no number can be used twice in a row but the same number may occur both first and third. For example, 20 13 20 would be acceptable, but 20 20 13 would not. How many different combinations are possible?

✶20. **a.** How many integers from 1 through 100,000 contain the digit 6 exactly once?
b. How many integers from 1 through 100,000 contain the digit 6 at least once?
c. If an integer is chosen at random from 1 through 100,000, what is the probability that it contains two or more occurrences of the digit 6?

H ✶**21.** Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have nonadjacent desks? (*Hint:* First find the probability that the couple will have adjacent desks, and then subtract this number from 1.)

✶22. Consider strings of length n over the set $\{a, b, c, d\}$.
a. How many such strings contain at least one pair of adjacent characters that are the same?
b. If a string of length ten over $\{a, b, c, d\}$ is chosen at random, what is the probability that it contains at least one pair of adjacent characters that are the same?

23. **a.** How many integers from 1 through 1,000 are multiples of 4 or multiples of 7?
b. Suppose an integer from 1 through 1,000 is chosen at random. Use the result of part (a) to find the probability that the integer is a multiple of 4 or a multiple of 7.
c. How many integers from 1 through 1,000 are neither multiples of 4 nor multiples of 7?

24. a. How many integers from 1 through 1,000 are multiples of 2 or multiples of 9?
b. Suppose an integer from 1 through 1,000 is chosen at random. Use the result of part (a) to find the probability that the integer is a multiple of 2 or a multiple of 9.
c. How many integers from 1 through 1,000 are neither multiples of 2 nor multiples of 9?

25. *Counting Strings*:
a. Make a list of all bit strings of lengths zero, one, two, three, and four that do not contain the bit pattern 111.
b. For each integer $n \geq 0$, let $d_n =$ the number of bit strings of length n that do not contain the bit pattern 111. Find d_0, d_1, d_2, d_3, and d_4.
c. Find a recurrence relation for $d_0, d_1, d_2, \ldots$.
d. Use the results of parts (b) and (c) to find the number of bit strings of length five that do not contain the pattern 111.

26. *Counting Strings*: Consider the set of all strings of a's, b's, and c's.
a. Make a list of all of these strings of lengths zero, one, two, and three that do not contain the pattern aa.
b. For each integer $n \geq 0$, let $s_n =$ the number of strings of a's, b's, and c's of length n that do not contain the pattern aa. Find s_0, s_1, s_2, and s_3.
H **c.** Find a recurrence relation for $s_0, s_1, s_2, \ldots$.
d. Use the results of parts (b) and (c) to find the number of strings of a's, b's, and c's of length four that do not contain the pattern aa.
H **e.** Use the technique described in Section 5.8 to find an explicit formula for $s_0, s_1, s_2, \ldots$.

27. For each integer $n \geq 0$, let a_k be the number of bit strings of length n that do not contain the pattern 101.
a. Show that $a_k = a_{k-1} + a_{k-3} + a_{k-4} + \cdots + a_0 + 2$, for all integers $k \geq 3$.
b. Use the result of part (a) to show that if $k \geq 3$, then $a_k = 2a_{k-1} - a_{k-2} + a_{k-3}$.

✶28. For each integer $n \geq 2$ let a_n be the number of permutations of $\{1, 2, 3, \ldots, n\}$ in which no number is more than one place removed from its "natural" position. Thus $a_1 = 1$ since the one permutation of $\{1\}$, namely 1, does not move 1 from its natural position. Also $a_2 = 2$ since neither of the two permutations of $\{1,2\}$, namely 12 and 21, moves either number more than one place from its natural position.
a. Find a_3.
b. Find a recurrence relation for $a_1, a_2, a_3, \ldots$.

29. Refer to Example 9.3.5.
a. Write the following IP address in dotted decimal form:

11001010 00111000 01101011 11101110

b. How many Class A networks can there be?
c. What is the dotted decimal form of the IP address for a computer in a Class A network?
d. How many host IDs can there be for a Class A network?
e. How many Class C networks can there be?
f. What is the dotted decimal form of the IP address for a computer in a Class C network?
g. How many host IDs can there be for a Class C network?
h. How can you tell, by looking at the first of the four numbers in the dotted decimal form of an IP address, what kind of network the address is from? Explain.
i. An IP address is 140.192.32.136. What class of network does it come from?
j. An IP address is 202.56.107.238. What class of network does it come from?

✶30. A row in a classroom has n seats. Let s_n be the number of ways nonempty sets of students can sit in the row so that no student is seated directly adjacent to any other student. (For instance, a row of three seats could contain a single student in any of the seats or a pair of students in the two outer seats. Thus $s_3 = 4$.) Find a recurrence relation for $s_1, s_2, s_3, \ldots$.

31. Assume that birthdays are equally likely to occur in any one of the 12 months of the year.
a. Given a group of four people, A, B, C, and D, what is the total number of ways in which birth months could be associated with A, B, C, and D? (For instance, A and B might have been born in May, C in September, and D in February. As another example, A might have been born in January, B in June, C in March, and D in October.)

b. How many ways could birth months be associated with A, B, C, and D so that no two people would share the same birth month?

c. How many ways could birth months be associated with A, B, C, and D so that at least two people would share the same birth month?

d. What is the probability that at least two people out of A, B, C, and D share the same birth month?

e. How large must n be so that in any group of n people, the probability that two or more share the same birth month is at least 50%?

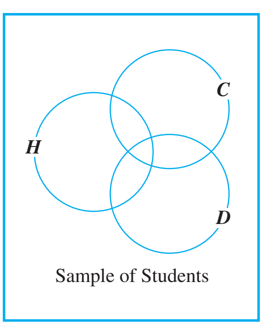
H **32.** Assuming that all years have 365 days and all birthdays occur with equal probability, how large must n be so that in any randomly chosen group of n people, the probability that two or more have the same birthday is at least 1/2? (This is called the **birthday problem.** Many people find the answer surprising.)

33. A college conducted a survey to explore the academic interests and achievements of its students. It asked students to place checks beside the numbers of all the statements that were true of them. Statement #1 was "I was on the honor roll last term," statement #2 was "I belong to an academic club, such as the math club or the Spanish club," and statement #3 was "I am majoring in at least two subjects." Out of a sample of 100 students, 28 checked #1, 26 checked #2, and 14 checked #3, 8 checked both #1 and #2, 4 checked both #1 and #3, 3 checked both #2 and #3, and 2 checked all three statements.

a. How many students checked at least one of the statements?

b. How many students checked none of the statements?

c. Let H be the set of students who checked #1, C the set of students who checked #2, and D the set of students who checked #3. Fill in the numbers for all eight regions of the diagram below.



d. How many students checked #1 and #2 but not #3?

e. How many students checked #2 and #3 but not #1?

f. How many students checked #2 but neither of the other two?

34. A study was done to determine the efficacy of three different drugs—A, B, and C—in relieving headache pain. Over the period covered by the study, 50 subjects were given the chance to use all three drugs. The following results were obtained:

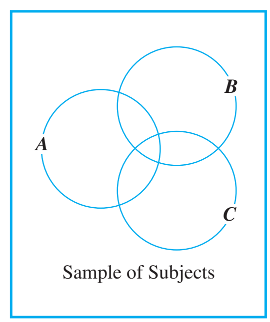
21 reported relief from drug A.
21 reported relief from drug B.
31 reported relief from drug C.
9 reported relief from both drugs A and B.
14 reported relief from both drugs A and C.
15 reported relief from both drugs B and C.
41 reported relief from at least one of the drugs.

Note that some of the 21 subjects who reported relief from drug A may also have reported relief from drugs B or C. A similar occurrence may be true for the other data.

a. How many people got relief from none of the drugs?

b. How many people got relief from all three drugs?

c. Let A be the set of all subjects who got relief from drug A, B the set of all subjects who got relief from drug B, and C the set of all subjects who got relief from drug C. Fill in the numbers for all eight regions of the diagram below.



d. How many subjects got relief from A only?

35. An interesting use of the inclusion/exclusion rule is to check survey numbers for consistency. For example, suppose a public opinion polltaker reports that out of a national sample of 1,200 adults, 675 are married, 682 are from 20 to 30 years old, 684 are female, 195 are married and are from 20 to 30 years old, 467 are married females, 318 are females from 20 to 30 years old, and 165 are married females from 20 to 30 years old. Are the polltaker's figures consistent? Could they have occurred as a result of an actual sample survey?

36. Fill in the reasons for each step below. If A and B are sets in a finite universe U, then

$$
\begin{aligned}
N(A \cap B) &= N(U) - N((A \cap B)^c) && \underline{\quad (a) \quad} \\
&= N(U) - N(A^c \cup B^c) && \underline{\quad (b) \quad} \\
&= N(U) - (N(A^c) + N(B^c) - N(A^c \cap B^c)) && \underline{\quad (c) \quad}.
\end{aligned}
$$

For each of exercises 37–39 below, the number of elements in a certain set can be found by computing the number in some larger universe that are not in the set and subtracting this from the total. In each case, as indicated by exercise 34, De Morgan's laws and the inclusion/exclusion rule can be used to compute the number that are not in the set.

37. How many positive integers less than 1,000 have no common factors with 1,000?

✶38. How many permutations of *abcde* are there in which the first character is a, b, or c and the last character is c, d, or e?

✶39. How many integers from 1 through 999,999 contain each of the digits 1, 2, and 3 at least once? (*Hint:* For each $i = 1, 2,$ and 3, let A_i be the set of all integers from 1 through 999,999 that do not contain the digit i.)

For 40 and 41, use the definition of the Euler phi function ϕ on page 396.

***H* 40.** Use the inclusion/exclusion principle to prove the following: If $n = pq$, where p and q are distinct prime numbers, then $\varphi(n) = (p-1)(q-1)$.

41. Use the inclusion/exclusion principle to prove the following: If $n = pqr$, where p, q, and r are distinct prime numbers, then $\varphi(n) = (p-1)(q-1)(r-1)$.

42. A gambler decides to play successive games of blackjack until he loses three times in a row. (Thus the gambler could play five games by losing the first, winning the second, and losing the final three or by winning the first two and losing the final three. These possibilities can be symbolized as *LWLLL* and *WWLLL*.) Let g_n be the number of ways the gambler can play n games.
   a. Find g_3, g_4, and g_5.
   b. Find g_6.
   ***H* c.** Find a recurrence relation for $g_3, g_4, g_5, \ldots$.

✶43. A *derangement* of the set $\{1, 2, \ldots, n\}$ is a permutation that moves every element of the set away from its "natural" position. Thus 21 is a derangement of $\{1, 2\}$, and 231 and 312 are derangements of $\{1, 2, 3\}$. For each positive integer n, let d_n be the number of derangements of the set $\{1, 2, \ldots, n\}$.
   a. Find d_1, d_2, and d_3.
   b. Find d_4.
   ***H* c.** Find a recurrence relation for $d_1, d_2, d_3, \ldots$.

44. Note that a product $x_1x_2x_3$ may be parenthesized in two different ways: $(x_1x_2)x_3$ and $x_1(x_2x_3)$. Similarly, there are several different ways to parenthesize $x_1x_2x_3x_4$. Two such ways are $(x_1x_2)(x_3x_4)$ and $x_1((x_2x_3)x_4)$. Let P_n be the number of different ways to parenthesize the product $x_1x_2 \ldots x_4$. Show that if $P_1 = 1$, then

$$P_n = \sum_{k=1}^{n-1} P_k P_{n-k} \quad \text{for all integers } n \geq 2.$$

(It turns out that the sequence $P_1, P_2, P_3, \ldots$ is the same as the sequence of Catalan numbers: $P_n = C_{n-1}$ for all integers $n \geq 1$. See Example 5.6.4.)

45. Use mathematical induction to prove Theorem 9.3.1.

46. Prove the inclusion/exclusion rule for two sets A and B by showing that $A \cup B$ can be partitioned into $A \cap B$, $A - (A \cap B)$, and $B - (A \cap B)$, and then using the addition and difference rules.

47. Prove the inclusion/exclusion rule for three sets.

***H* ✶48.** Use mathematical induction to prove the general inclusion/exclusion rule:

If $A_1, A_2, \ldots, A_n$ are finite sets, then

$$\begin{aligned} N(A_1 \cup A_2 \cup \cdots \cup A_n) &= \sum_{1 \leq i \leq n} N(A_i) - \sum_{1 \leq i < j \leq n} N(A_i \cap A_j) \\ &\quad + \sum_{1 \leq i < j < k \leq n} N(A_i \cap A_j \cap A_k) \\ &\quad - \cdots + (-1)^{n+1} N(A_1 \cap A_2 \cap \cdots \cap A_n). \end{aligned}$$

(The notation $\sum_{1 \leq i < j \leq n} N(A_i \cap A_j)$ means that quantities of the form $N(A_i \cap A_j)$ are to be added together for all integers i and j with $1 \leq i < j \leq n$.)

✶49. A circular disk is cut into n distinct sectors, each shaped like a piece of pie and all meeting at the center point of the disk. Each sector is to be painted red, green, yellow, or blue in such a way that no two adjacent sectors are painted the same color. Let S_n be the number of ways to paint the disk.
   ***H* a.** Find a recurrence relation for S_k in terms of S_{k-1} and S_{k-2} for each integer $k \geq 4$.
   b. Find an explicit formula for S_n for $n \geq 2$.

Answers for Test Yourself

1. the number of elements in A equals $N(A_1) + N(A_2) + \ldots + N(A_n)$ 2. the number of elements in $A - B$ is the difference between the number of elements in A and the number of elements in B, that is, $N(A - B) = N(A) - N(B)$. 3. $1 - P(A)$ 4. $N(A \cup B) = N(A) + N(B) - N(A \cap B)$ 5. $N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$