Exercises 32–38 refer to the sequence of Stirling numbers of the second kind.

32. Find $S_{3,4}$ by exhibiting all the partitions of $\{x_1, x_2, x_3, x_4, x_5\}$ into four subsets.

33. Use the values computed in Example 9.5.12 and the recurrence relation and initial conditions found in Example 9.5.13 to compute $S_{5,2}$.

34. Use the values computed in Example 9.5.12 and the recurrence relation and initial conditions found in Example 9.5.13 to compute $S_{5,3}$.

35. Use the results of exercises 32–34 to find the total number of different partitions of a set with five elements.

36. Use mathematical induction and the recurrence relation found in Example 9.5.13 to prove that for all integers $n \geq 2$, $S_{n,2} = 2^{n-1} - 1$.

37. Use mathematical induction and the recurrence relation found in Example 9.5.13 to prove that for all integers $n \geq 2$, $\sum_{k=2}^{k}(3^{4-k} S_{k,2}) - S_{n+1,3}$.

H 38. If X is a set with n elements and Y is a set with m elements, express the number of onto functions from X and Y using Stirling numbers of the second kind. Justify your answer.

Answers for Test Yourself

1. $\binom{n}{r}$; n choose r 2. $\binom{n}{r}$ (*Or*: n choose r) 3. the order in which they are chosen 4. $\binom{n}{r} = \frac{P(n,r)}{r!}$ 5. n or more; n or fewer 6. $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ $\left(Or: \frac{n!}{n_1!n_2!n_3!\cdots n_k!}\right)$ 7. the number of ways a set of size n can be partitioned into r subsets 8. the number of partitions of X into r subsets of which $\{x_n\}$ is one; the number of partitions of X into r subsets, none of which is $\{x_n\}$

9.6 r-Combinations with Repetition Allowed

The value of mathematics in any science lies more in disciplined analysis and abstract thinking than in particular theories and techniques. — Alan Tucker, 1982

In Section 9.5 we showed that there are $\binom{n}{r}$ r-combinations, or subsets of size r, of a set of n elements. In other words, there are $\binom{n}{r}$ ways to choose r distinct elements without regard to order from a set of n elements. For instance, there are $\binom{4}{3} = 4$ ways to choose three elements out of a set of four: {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}.

In this section we ask: How many ways are there to choose r elements without regard to order from a set of n elements *if repetition is allowed?* A good way to imagine this is to visualize the n elements as categories of objects from which multiple selections may be made. For instance, if the categories are labeled 1, 2, 3, and 4 and three elements are chosen, it is possible to choose two elements of type 3 and one of type 1, or all three of type 2, or one each of types 1, 2 and 4. We denote such choices by [3, 3, 1], [2, 2, 2], and [1, 2, 4], respectively. Note that because order does not matter, [3, 3, 1] = [3, 1, 3] = [1, 3, 3], for example.

• Definition

An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

Example 9.6.1 *r*-Combinations with Repetition Allowed

Write a complete list to find the number of 3-combinations with repetition allowed, or multisets of size 3, that can be selected from {1, 2, 3, 4}. Observe that because the order in which the elements are chosen does not matter, the elements of each selection may be written in increasing order, and writing the elements in increasing order will ensure that no combinations are overlooked.

Solution

[1, 1, 1] ; [1, 1, 2]; [1, 1, 3]; [1, 1, 4]	all combinations with 1, 1
[1, 2, 2] ; [1, 2, 3]; [1, 2, 4];	all additional combinations with 1, 2
[1, 3, 3] ; [1, 3, 4]; [1, 4, 4];	all additional combinations with 1, 3 or 1, 4
[2, 2, 2] ; [2, 2, 3]; [2, 2, 4];	all additional combinations with 2, 2
[2, 3, 3] ; [2, 3, 4]; [2, 4, 4];	all additional combinations with 2, 3 or 2, 4
[3, 3, 3] ; [3, 3, 4]; [3, 4, 4];	all additional combinations with 3, 3 or 3, 4
[4, 4, 4]	the only additional combination with 4, 4

Thus there are twenty 3-combinations with repetition allowed. ■

How could the number twenty have been predicted other than by making a complete list? Consider the numbers 1, 2, 3, and 4 as categories and imagine choosing a total of three numbers from the categories with multiple selections from any category allowed. The results of several such selections are represented by the table below.

Category 1		Category 2		Category 3		Category 4	Result of the Selection
	|	×	|		|	× ×	1 from category 2 2 from category 4
×	|		|	×	|	×	1 each from categories 1, 3, and 4
× × ×	|		|		|		3 from category 1

As you can see, each selection of three numbers from the four categories can be represented by a string of vertical bars and crosses. Three vertical bars are used to separate the four categories, and three crosses are used to indicate how many items from each category are chosen. Each distinct string of three vertical bars and three crosses represents a distinct selection. For instance, the string

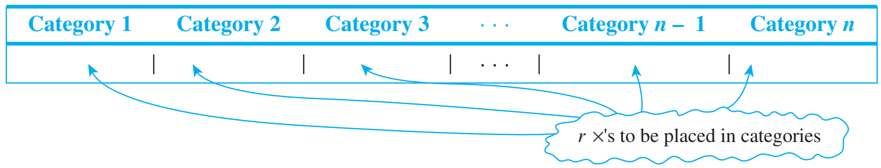
$$\times \times \mid \; \mid \times \mid$$

represents the selection: two from category 1, none from category 2, one from category 3, and none from category 4. Thus the number of distinct selections of three elements that can be formed from the set {1, 2, 3, 4} with repetition allowed equals the number of distinct strings of six symbols consisting of three |'s and three ×'s. But this equals the number of ways to select three positions out of six because once three positions have been chosen for the ×'s, the |'s are placed in the remaining three positions. Thus the answer is

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{\cancel{6} \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{3!}} = 20,$$

as was obtained earlier by a careful listing.

The analysis of this example extends to the general case. To count the number of r-combinations with repetition allowed, or multisets of size r, that can be selected from a set of n elements, think of the elements of the set as categories. Then each r-combination with repetition allowed can be represented as a string of $n-1$ vertical bars (to separate the n categories) and r crosses (to represent the r elements to be chosen). The number of $\times$'s in each category represents the number of times the element represented by that category is repeated.



The number of strings of $n-1$ vertical bars and r crosses is the number of ways to choose r positions, into which to place the r crosses, out of a total of $r+(n-1)$ positions, leaving the remaining positions for the vertical bars. But by Theorem 9.5.1, this number is $\binom{r+n-1}{r}$.

This discussion proves the following theorem.

Theorem 9.6.1

The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Example 9.6.2 Selecting 15 Cans of Soft Drinks of Five Different Types

A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.

a. How many different selections of cans of 15 soft drinks can he make?

b. If root beer is one of the types of soft drink, how many different selections include at least six cans of root beer?

c. If the store has only five cans of root beer but at least 15 cans of each other type of soft drink, how many different selections are there?

Solution

a. Think of the five different types of soft drinks as the n categories and the 15 cans of soft drinks to be chosen as the r objects (so $n=5$ and $r=15$). Each selection of cans of soft drinks is represented by a string of $5-1=4$ vertical bars (to separate the categories of soft drinks) and 15 crosses (to represent the cans selected). For instance, the string

$$\times\times\times \mid \times\times\times\times\times\times\times \mid \quad \mid \times\times\times \mid \times\times$$

represents a selection of three cans of soft drinks of type 1, seven of type 2, none of type 3, three of type 4, and two of type 5. The total number of selections of 15 cans of soft drinks of the five types is the number of strings of 19 symbols, $5 - 1 = 4$ of them | and 15 of them $\times$:

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19\cdot \cancelto{6}{18}\cdot 17\cdot \cancelto{2}{16}\cdot \cancel{15!}}{\cancel{15!}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 3{,}876.$$

b. If at least six cans of root beer are included, we can imagine choosing six such cans first and then choosing 9 additional cans. The choice of the nine additional cans can be represented as a string of 9 $\times$'s and 4 |'s. For example, if root beer is type 1, then the string $\times\times\times\,|\quad|\times\times|\times\times\times\times|$ represents a selection of three cans of root beer (in addition to the six chosen initially), none of type 2, two of type 3, four of type 4, and none of type 5. Thus the total number of selections of 15 cans of soft drinks of the five types, including at least six cans of root beer, is the number of strings of 13 symbols, $4\,(= 5 - 1)$ of them | and 9 of them $\times$:

$$\binom{9+4}{9} = \binom{13}{9} = \frac{13\cdot \cancel{12}\cdot 11\cdot \cancelto{5}{10}\cdot \cancel{9!}}{\cancel{9!}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 715.$$

c. If the store has only five cans of root beer, then the number of different selections of 15 cans of soft drinks of the five types is the same as the number of different selections that contain five or fewer cans of root beer. Let T be the set of selections for which the type of cans of root beer is unrestricted, $R_{\le 5}$ the set of selections containing five or fewer cans of root beer, and $R_{\ge 6}$ the set of selections containing six or more cans of root beer. Then

$$T = R_{\le 5} \cup R_{\ge 6} \quad \text{and} \quad R_{\le 5} \cap R_{\ge 6} = \emptyset.$$

By part (a) $N(T) = 3{,}876$ and by part (b) $N(R_{\ge 6}) = 715$. Thus, by the difference rule,

$$N(R_{\le 5}) = N(T) - N(R_{\ge 6}) = 3{,}876 - 715 = 3{,}161.$$

So the number of different selections of soft drinks is 3,161. ■

Example 9.6.3 Counting Triples (i, j, k) with $1 \le i \le j \le k \le n$

If n is a positive integer, how many triples of integers from 1 through n can be formed in which the elements of the triple are written in increasing order but are not necessarily distinct? In other words, how many triples of integers (i, j, k) are there with $1 \le i \le j \le k \le n$?

Solution Any triple of integers (i, j, k) with $1 \le i \le j \le k \le n$ can be represented as a string of $n - 1$ vertical bars and three crosses, with the positions of the crosses indicating which three integers from 1 to n are included in the triple. The table below illustrates this for $n = 5$.

Category 1	2	3	4	5	Result of the Selection
		$\times\times$		$\times$	(3, 3, 5)
$\times$	$\times$		$\times$		(1, 2, 4)

Thus the number of such triples is the same as the number of strings of $(n-1)$ |'s and 3 ×'s, which is

$$\binom{3+(n-1)}{3} = \binom{n+2}{3} = \frac{(n+2)!}{3!(n+2-3)!}$$
$$= \frac{(n+2)(n+1)n\cancel{(n-1)!}}{3!\cancel{(n-1)!}} = \frac{n(n+1)(n+2)}{6}.$$

■

Note that in Examples 9.6.2 and 9.6.3 the reasoning behind Theorem 9.6.1 was used rather than the statement of the theorem itself. Alternatively, in either example we could invoke Theorem 9.6.1 directly by recognizing that the items to be counted either are r-combinations with repetition allowed or are the same in number as such combinations. For instance, in Example 9.6.3 we might observe that there are exactly as many triples of integers (i, j, k) with $1 \le i \le j \le k \le n$ as there are 3-combinations of integers from 1 through n with repetition allowed because the elements of any such 3-combination can be written in increasing order in only one way.

Example 9.6.4 Counting Iterations of a Loop

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run? (Assume n is a positive integer.)

for $k := 1$ **to** n
  for $j := 1$ **to** k
    for $i := 1$ **to** j
      [*Statements in the body of the inner loop,*
      none containing branching statements that lead
      outside the loop]
    next i
  next j
next k

Solution Construct a trace table for the values of k, j, and i for which the statements in the body of the innermost loop are executed. (See the table that follows.) Because i goes from 1 to j, it is always the case that $i \le j$. Similarly, because j goes from 1 to k, it is always the case that $j \le k$. To focus on the details of the table construction, consider what happens when $k = 3$. In this case, j takes each value 1, 2, and 3. When $j = 1$, i can only take the value 1 (because $i \le j$). When $j = 2$, i takes each value 1 and 2 (again because $i \le j$). When $j = 3$, i takes each value 1, 2, and 3 (yet again because $i \le j$).

k	1	2		→	3					→	⋯	n						→
j	1	1	2	→	1	2	→	3		→	⋯	1	2	→	⋯	n		→
i	1	1	1	2	1	1	2	1	2	3	⋯	1	1	2	⋯	1	⋯	n

Observe that there is one iteration of the innermost loop for each column of this table, and there is one column of the table for each triple of integers (i, j, k) with $1 \le i \le j \le k \le n$. But Example 9.6.3 showed that the number of such triples is $[n(n+1)(n+2)]/6$. Thus there are $[n(n+1)(n+2)]/6$ iterations of the innermost loop. ■

The solution in Example 9.6.4 is the most elegant and generalizable one. (See exercises 8 and 9.) An alternative solution using summations is outlined in exercise 21.

Example 9.6.5 The Number of Integral Solutions of an Equation

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if x_1, x_2, x_3, and x_4 are nonnegative integers?

Solution Think of the number 10 as divided into ten individual units and the variables x_1, x_2, x_3, and x_4 as four categories into which these units are placed. The number of units in each category x_i indicates the value of x_i in a solution of the equation. Each solution can, then, be represented by a string of three vertical bars (to separate the four categories) and ten crosses (to represent the ten individual units). For example, in the following table, the two crosses under x_1, five crosses under x_2, and three crosses under x_4 represent the solution $x_1 = 2$, $x_2 = 5$, $x_3 = 0$, and $x_4 = 3$.

Categories x_1 | x_2 | x_3 | x_4	Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
× × | × × × × × | | × × ×	$x_1 = 2$, $x_2 = 5$, $x_3 = 0$, and $x_4 = 3$
× × × × | × × × × × × | |	$x_1 = 4$, $x_2 = 6$, $x_3 = 0$, and $x_4 = 0$

Therefore, there are as many solutions to the equation as there are strings of ten crosses and three vertical bars, namely

$$\binom{10+3}{10} = \binom{13}{10} = \frac{13!}{10!(13-10)!} = \frac{13\cdot 12\cdot 11\cdot \cancel{10!}}{\cancel{10!}\cdot 3\cdot 2\cdot 1} = 286. \quad \blacksquare$$

Example 9.6.6 illustrates a variation on Example 9.6.5.

Example 9.6.6 Additional Constraints on the Number of Solutions

How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each $x_i \geq 1$?

Solution In this case imagine starting by putting one cross in each of the four categories. Then distribute the remaining six crosses among the categories. Such a distribution can be represented by a string of three vertical bars and six crosses. For example, the string

× × × | | × × | ×

indicates that there are three more crosses in category x_1 in addition to the one cross already there (so $x_1 = 4$), no more crosses in category x_2 in addition to the one already there (so $x_2 = 1$), two more crosses in category x_3 in addition to the one already there (so $x_3 = 3$), and one more cross in category x_4 in addition to the one already there (so $x_4 = 2$). It follows that the number of solutions to the equation that satisfy the given condition is the same as the number of strings of three vertical bars and six crosses, namely

$$\binom{6+3}{6} = \binom{9}{6} = \frac{9!}{6!(9-6)!} = \frac{9\cdot 8\cdot 7\cdot 6!}{6!\cdot 3\cdot 2\cdot 1} = 84.$$

An alternative solution to this example is based on the observation that since each $x_i \geq 1$, we may introduce new variables $y_i = x_i - 1$ for each $i = 1, 2, 3, 4$. Then each $y_i \geq 0$, and $y_1 + y_2 + y_3 + y_4 = 6$. Thus the number of solutions of $y_1+y_2+y_3+y_4=6$ in nonnegative integers is the same as the number of solutions of $x_1 + x_2 + x_3 + x_4 = 10$ in positive integers. $\blacksquare$

Which Formula to Use?

Sections 9.2, 9.3, 9.5, and 9.6 have discussed four different ways of choosing k elements from n. The order in which the choices are made may or may not matter, and repetition may or may not be allowed. The following table summarizes which formula to use in which situation.

	Order Matters	**Order Does Not Matter**
Repetition Is Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Is Not Allowed	$P(n,k)$	$\binom{n}{k}$

Test Yourself

1. Given a set $X = \{x_1, x_2, \ldots, x_n\}$, an r-combination with repetition allowed, or a multiset of size r, chosen from X is _____, which is denoted _____.

2. If $X = \{x_1, x_2, \ldots, x_n\}$, the number of r-combinations with repetition allowed (or multisets of size r) chosen from X is _____.

3. When choosing k elements from a set of n elements, order may or may not matter and repetition may or may not be allowed.
   - The number of ways to choose the k elements when repetition is allowed and order matters is _____.
   - The number of ways to choose the k elements when repetition is not allowed and order matters is _____.
   - The number of ways to choose the k elements when repetition is not allowed and order does not matter is _____.
   - The number of ways to choose the k elements when repetition is allowed and order does not matter is _____.

Exercise Set 9.6

1. a. According to Theorem 9.6.1, how many 5-combinations with repetition allowed can be chosen from a set of three elements?
   b. List all of the 5-combinations that can be chosen with repetition allowed from $\{1, 2, 3\}$.

2. a. According to Theorem 9.6.1, how many multisets of size four can be chosen from a set of three elements?
   b. List all of the multisets of size four that can be chosen from the set $\{x, y, z\}$.

3. A bakery produces six different kinds of pastry, one of which is eclairs. Assume there are at least 20 pastries of each kind.
   a. How many different selections of twenty pastries are there?
   b. How many different selections of twenty pastries are there if at least three must be eclairs?
   c. How many different selections of twenty pastries contain at most two eclairs?

4. A camera shop stocks eight different types of batteries, one of which is type A7b. Assume there are at least 30 batteries of each type.
   a. How many ways can a total inventory of 30 batteries be distributed among the eight different types?
   b. How many ways can a total inventory of 30 batteries be distributed among the eight different types if the inventory must include at least four A76 batteries?
   c. How many ways can a total inventory of 30 batteries be distributed among the eight different types if the inventory includes at most three A7b batteries?

5. If n is a positive integer, how many 4-tuples of integers from 1 through n can be formed in which the elements of the 4-tuple are written in increasing order but are not necessarily distinct? In other words, how many 4-tuples of integers (i, j, k, m) are there with $1 \le i \le j \le k \le m \le n$?

6. If n is a positive integer, how many 5-tuples of integers from 1 through n can be formed in which the elements of the 5-tuple are written in decreasing order but are not necessarily distinct? In other words, how many 5-tuples of integers (h, i, j, k, m) are there with $n \ge h \ge i \ge j \ge k \ge m \ge 1$?

7. Another way to count the number of nonnegative integral solutions to an equation of the form $x_1 + x_2 + \cdots + x_n = m$ is to reduce the problem to one of finding the number of n-tuples $(y_1, y_2, \ldots, y_n)$ with $0 \le y_1 \le y_2 \le \cdots \le y_n \le m$. The reduction results from letting $y_i = x_1 + x_2 + \cdots + x_i$ for each $i = 1, 2, \ldots, n$. Use this approach to derive a general formula for the number of nonnegative integral solutions to $x_1 + x_2 + \cdots + x_n = m$.

In 8 and 9, how many times will the innermost loop be iterated when the algorithm segment is implemented and run? Assume n, m, k, and j are positive integers.

8. **for** $m := 1$ **to** n
  for $k := 1$ **to** m
    for $j := 1$ **to** k
      for $i := 1$ **to** j
        [*Statements in the body of the inner loop, none containing branching statements that lead outside the loop*]
      next i
    next j
  next k
next m

9. **for** $k := 1$ **to** n
  for $j := k$ **to** n
    for $i := j$ **to** n
      [*Statements in the body of the inner loop, none containing branching statements that lead outside the loop*]
    next i
  next j
next k

In 10–14, find how many solutions there are to the given equation that satisfy the given condition.

10. $x_1 + x_2 + x_3 = 20$, each x_i is a nonnegative integer.

11. $x_1 + x_2 + x_3 = 20$, each x_i is a positive integer.

12. $y_1 + y_2 + y_3 + y_4 = 30$, each y_i is a nonnegative integer.

13. $y_1 + y_2 + y_3 + y_4 = 30$, each y_i is an integer that is at least 2.

14. $a + b + c + d + e = 500$, each of a, b, c, d, and e is an integer that is at least 10.

✶ 15. For how many integers from 1 through 99,999 is the sum of their digits equal to 10?

16. Consider the situation in Example 9.6.2.

a. Suppose the store has only six cans of lemonade but at least 15 cans of each of the other four types, of soft drink. In how many different ways can five cans of soft drink be selected?

b. Suppose that the store has only five cans of root beer and only six cans of lemonade but at least 15 cans of each of the other three types of soft drink. In how many different ways can five cans of soft drink be selected?

***H* 17.** a. A store sells 8 kinds of balloons with at least 30 of each kind. How many different combinations of 30 balloons can be chosen?

b. If the store has only 12 red balloons but at least 30 of each other kind of balloon, how many combinations of balloons can be chosen?

c. If the store has only 8 blue balloons but at least 30 of each other kind of balloon, how many combinations of balloons can be chosen?

d. If the store has only 12 red balloons and only 8 blue balloons but at least 30 of each other kind of balloon, how many combinations of balloons can be chosen?

18. A large pile of coins consists of pennies, nickels, dimes, and quarters.

a. How many different collections of 30 coins can be chosen if there are at least 30 of each kind of coin?

b. If the pile contains only 15 quarters but at least 30 of each other kind of coin, how many collections of 30 coins can be chosen?

c. If the pile contains only 20 dimes but at least 30 of each other kind of coin, how many collections of 30 coins can be chosen?

d. If the pile contains only 15 quarters and only 20 dimes but at least 30 of each other kind of coin, how many collections of 30 coins can be chosen?

***H* 19.** Suppose the bakery in exercise 3 has only ten eclairs but has at least twenty of each of the other kinds of pastry.

a. How many different selections of twenty pastries are there?

b. Suppose in addition to having only ten eclairs, the bakery has only eight napoleon slices. How many different selections of twenty pastries are there?

20. Suppose the camera shop in exercise 4 can obtain at most ten A76 batteries but can get at least 30 of each of the other types.

a. How many ways can a total inventory of 30 batteries be distributed among the eight different types?

b. Suppose that in addition to being able to obtain only ten A76 batteries, the store can get only six of type D303. How many ways can a total inventory of 30 batteries be distributed among the eight different types?

21. Observe that the number of columns in the trace table for Example 9.6.4 can be expressed as the sum

$$1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + n).$$

Explain why this is so, and show how this sum simplifies to the same expression given in the solution of Example 9.6.4. *Hint*: Use a formula from the exercise set for Section 5.2.

Answers for Test Yourself

1. an unordered selection of elements taken from X with repetition allowed; $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other 2. $\binom{r+n-1}{r}$ 3. n^k; $n(n-1)(n-2)\cdots(n-k+1)$ $(Or: P(n,k))$; $\binom{n}{k}$; $\binom{k+n-1}{k}$