9.7 Pascal's Formula and the Binomial Theorem

I'm very well acquainted, too, with matters mathematical, I understand equations both the simple and quadratical. About binomial theorem I am teaming with a lot of news, With many cheerful facts about the square of the hypotenuse.

—William S. Gilbert, *The Pirates of Penzance,* 1880

In this section we derive several formulas for values of $\binom{n}{r}$ The most important is Pascal's formula, which is the basis for Pascal's triangle and is a crucial component of one of the proofs of the binomial theorem. We offer two distinct proofs for both Pascal's formula and the binomial theorem. One of them is called "algebraic" because it relies to a great extent on algebraic manipulation, and the other is called "combinatorial," because it is based on the kind of counting arguments we have been discussing in this chapter.

Example 9.7.1 Values of $\binom{n}{n}$, $\binom{n}{n-1}$, $\binom{n}{n-2}$

Think of Theorem 9.5.1 as a general template: Regardless of what nonnegative numbers are placed in the boxes, if the number in the lower box is no greater than the number in the top box, then

$$\binom{\square}{\Diamond} = \frac{\square!}{\Diamond!(\square - \Diamond)!}.$$

Use Theorem 9.5.1 to show that for all integers $n \geq 0$,

$$\binom{n}{n} = 1 \qquad 9.7.1$$

$$\binom{n}{n-1} = n, \quad \text{if } n \geq 1 \qquad 9.7.2$$

$$\binom{n}{n-2} = \frac{n(n-1)}{2}, \quad \text{if } n \geq 2. \qquad 9.7.3$$

Solution

$$\binom{n}{n} = \frac{\cancel{n!}}{\cancel{n!}(n-n)!} = \frac{1}{0!} = 1 \qquad \text{since } 0! = 1 \text{ by definition}$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)!(n-(n-1))!}$$

$$= \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}(n-n+1)!} = \frac{n}{1} = n$$

$$\binom{n}{n-2} = \frac{n!}{(n-2)!(n-(n-2))!}$$

$$= \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)!}2!} = \frac{n(n-1)}{2}$$

■

Note that the result derived algebraically above, that $\binom{n}{n}$ equals 1, agrees with the fact that a set with n elements has just one subset of size n, namely itself. Similarly, exercise 1 at the end of the section asks you to show algebraically that $\binom{n}{0} = 1$, which agrees with the fact that a set with n elements has one subset, the empty set, of size 0. In exercise 2 you are also asked to show algebraically that $\binom{n}{1} = n$. This result agrees with the fact that there are n subsets of size 1 that can be chosen from a set with n elements, namely the subsets consisting of each element taken alone.

Example 9.7.2 $\binom{n}{r} = \binom{n}{n-r}$

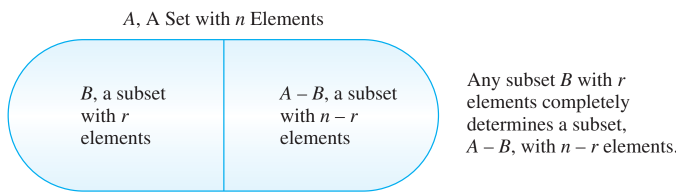
In exercise 5 at the end of the section you are asked to verify algebraically that

$$\binom{n}{r} = \binom{n}{n-r}$$

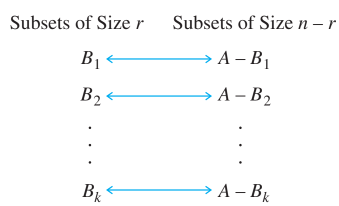
for all nonnegative integers n and r with $r \le n$.

An alternative way to deduce this formula is to interpret it as saying that a set A with n elements has exactly as many subsets of size r as it has subsets of size $n - r$. Derive the formula using this reasoning.

Solution Observe that any subset of size r can be specified either by saying which r elements lie in the subset or by saying which $n - r$ elements lie outside the subset.



Suppose A has k subsets of size r: $B_1, B_2, \ldots, B_k$. Then each B_i can be paired up with exactly one set of size $n - r$, namely its complement $A - B_i$ as shown below.



All subsets of size r are listed in the left-hand column, and all subsets of size $n - r$ are listed in the right-hand column. The number of subsets of size r equals the number of subsets of size $n - r$, and so $\binom{n}{r} = \binom{n}{n-r}$. ■

The type of reasoning used in this example is called *combinatorial*, because it is obtained by counting things that are combined in different ways. A number of theorems have both combinatorial proofs and proofs that are purely algebraic.

Pascal's Formula

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Blaise Pascal (1623–1662)

Pascal's formula, named after the seventeenth-century French mathematician and philosopher Blaise Pascal, is one of the most famous and useful in combinatorics (which is the formal term for the study of counting and listing problems). It relates the value of $\binom{n+1}{r}$ to the values of $\binom{n}{r-1}$ and $\binom{n}{r}$. Specifically, it says that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

whenever n and r are positive integers with $r \le n$. This formula makes it easy to compute higher combinations in terms of lower ones: If all the values of $\binom{n}{r}$ are known, then the values of $\binom{n+1}{r}$ can be computed for all r such that $0 < r \le n$.

Pascal's triangle, shown in Table 9.7.1, is a geometric version of Pascal's formula. Sometimes it is simply called the arithmetic triangle because it was used centuries before Pascal by Chinese and Persian mathematicians. But Pascal discovered it independently, and ever since 1654, when he published a treatise that explored many of its features, it has generally been known as Pascal's triangle.

Table 9.7.1 **Pascal's Triangle for $\left(\text{Values of } \binom{n}{r}\right)$**

$n \backslash r$	**0**	**1**	**2**	**3**	**4**	**5**	**...**	**$r-1$**	**r**	**...**
0	1							·	·	...
1	1	1						·	·	...
2	1	2	1					·	·	...
3	1	3	3	1				·	·	...
4	1	4	6 +	4	1			·	·	...
5	1	5	10	= 10	5	1		·	·	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮⋮⋮
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	...	$\binom{n}{r-1}$ +	$\binom{n}{r}$	...
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$	...		= $\binom{n+1}{r}$	...
·		·	·	·	·	·		·	·	...
·		·	·	·	·	·		·	·	...
·		·	·	·	·	·		·	·	...

Each entry in the triangle is a value of $\binom{n}{r}$. Pascal's formula translates into the fact that the entry in row $n+1$, column r equals the sum of the entry in row n, column $r-1$ plus the entry in row n, column r. That is, the entry in a given interior position equals the sum of the two entries directly above and to the above left. The left-most and right-most entries in each row are 1 because $\binom{n}{n} = 1$ by Example 9.7.1 and $\binom{n}{0} = 1$ by exercise 1 at the end of this section.

Example 9.7.3 Calculating $\binom{n}{r}$ Using Pascal's Triangle

Use Pascal's triangle to compute the values of

$$\binom{6}{2} \quad \text{and} \quad \binom{6}{3}.$$

Solution By construction, the value in row n, column r of Pascal's triangle is the value of $\binom{n}{r}$, for every pair of positive integers n and r with $r \le n$. By Pascal's formula, $\binom{n+1}{r}$ can be computed by adding together $\binom{n}{r-1}$ and $\binom{n}{r}$, which are located directly above and above left of $\binom{n+1}{r}$. Thus,

$$\binom{6}{2} = \binom{5}{1} + \binom{5}{2} = 5 + 10 = 15 \quad \text{and}$$

$$\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20.$$

■

Pascal's formula can be derived by two entirely different arguments. One is algebraic; it uses the formula for the number of r-combinations obtained in Theorem 9.5.1. The other is combinatorial; it uses the definition of the number of r-combinations as the number of subsets of size r taken from a set with a certain number of elements. We give both proofs since both approaches have applications in many other situations.

Theorem 9.7.1 Pascal's Formula

Let n and r be positive integers and suppose $r \leq n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Proof (algebraic version):

Let n and r be positive integers with $r \leq n$. By Theorem 9.5.1,

$$\begin{aligned}\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}.\end{aligned}$$

To add these fractions, a common denominator is needed, so multiply the numerator and denominator of the left-hand fraction by r and multiply the numerator and denominator of the right-hand fraction by $(n-r+1)$. Then

$$\begin{aligned}\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!}\cdot\frac{r}{r} + \frac{n!}{r!(n-r)!}\cdot\frac{(n-r+1)}{(n-r+1)} \\ &= \frac{n!\cdot r}{(n-r+1)!r(r-1)!} + \frac{n\cdot n! - n!\cdot r + n!}{(n-r+1)(n-r)!r!} \\ &= \frac{\cancel{n!\cdot r} + n!\cdot n - \cancel{n!\cdot r} + n!}{(n-r+1)!r!} = \frac{n!(n+1)}{(n+1-r)!r!} \\ &= \frac{(n+1)!}{((n+1)-r)!r!} = \binom{n+1}{r}.\end{aligned}$$

Proof (combinatorial version):

Let n and r be positive integers with $r \leq n$. Suppose S is a set with $n+1$ elements. The number of subsets of S of size r can be calculated by thinking of S as consisting of two pieces: one with n elements $\{x_1, x_2, \ldots, x_n\}$ and the other with one element $\{x_{n+1}\}$.

Any subset of S with r elements either contains x_{n+1} or it does not. If it contains x_{n+1}, then it contains $r-1$ elements from the set $\{x_1, x_2, \ldots, x_n\}$. If it does not contain x_{n+1}, then it contains r elements from the set $\{x_1, x_2, \ldots, x_n\}$.

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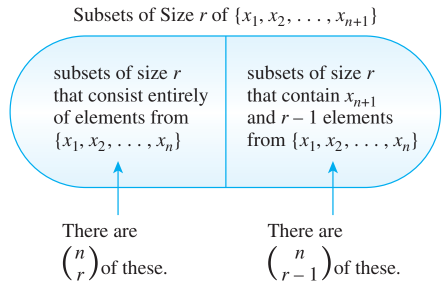

By the addition rule,

$$\begin{bmatrix}\text{number of subsets of} \\ \{x_1, x_2, \ldots, x_n, x_{n+1}\} \\ \text{of size } r\end{bmatrix} = \begin{bmatrix}\text{number of subsets of} \\ \{x_1, x_2, \ldots, x_n\} \\ \text{of size } r-1\end{bmatrix} + \begin{bmatrix}\text{number of subsets of} \\ \{x_1, x_2, \ldots, x_n\} \\ \text{of size } r\end{bmatrix}.$$

By Theorem 9.5.1, the set $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ has $\binom{n+1}{r}$ subsets of size r, the set $\{x_1, x_2, \ldots, x_n\}$ has $\binom{n}{r-1}$ subsets of size $r-1$, and the set $\{x_1, x_2, \ldots, x_n\}$ has $\binom{n}{r}$ subsets of size r. Thus

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r},$$

as was to be shown.

Example 9.7.4 Deriving New Formulas from Pascal's Formula

Use Pascal's formula to derive a formula for $\binom{n+2}{r}$ in terms of values of $\binom{n}{r}$, $\binom{n}{r-1}$, and $\binom{n}{r-2}$. Assume n and r are nonnegative integers and $2 \le r \le n$.

Solution By Pascal's formula,

$$\binom{n+2}{r} = \binom{n+1}{r-1} + \binom{n+1}{r}.$$

Now apply Pascal's formula to $\binom{n+1}{r-1}$ and $\binom{n+1}{r}$ and substitute into the above to obtain

$$\binom{n+2}{r} = \left[\binom{n}{r-2} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r}\right].$$

Combining the two middle terms gives

$$\binom{n+2}{r} = \binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}$$

for all nonnegative integers n and r such that $2 \le r \le n$. ■

The Binomial Theorem

In algebra a sum of two terms, such as $a + b$, is called a **binomial.** The *binomial theorem* gives an expression for the powers of a binomial $(a + b)^n$, for each positive integer n and all real numbers a and b.

Consider what happens when you calculate the first few powers of $a + b$. According to the distributive law of algebra, you take the sum of the products of all combinations of individual terms:

$$(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb,$$
$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= aaa + aab + aba + abb + baa + bab + bba + bbb,$$
$$(a+b)^4 = \underbrace{(a+b)}_{\substack{\text{1st}\\\text{factor}}}\underbrace{(a+b)}_{\substack{\text{2nd}\\\text{factor}}}\underbrace{(a+b)}_{\substack{\text{3rd}\\\text{factor}}}\underbrace{(a+b)}_{\substack{\text{4th}\\\text{factor}}}$$
$$= aaaa + aaab + aaba + aabb + abaa + abab + abba + abbb$$
$$+ baaa + baab + baba + babb + bbaa + bbab + bbba + bbbb.$$

Now focus on the expansion of $(a+b)^4$. (It is concrete, and yet it has all the features of the general case.) A typical term of this expansion is obtained by multiplying one of the two terms from the first factor times one of the two terms from the second factor times one of the two terms from the third factor times one of the two terms from the fourth factor. For example, the term *abab* is obtained by multiplying the a's and b's marked with arrows below.

$$(\overset{\downarrow}{a} + b)(a + \overset{\downarrow}{b})(\overset{\downarrow}{a} + b)(a + \overset{\downarrow}{b})$$

Since there are two possible values—a or b—for each term selected from one of the four factors, there are $2^4 = 16$ terms in the expansion of $(a+b)^4$.

Now some terms in the expansion are "like terms" and can be combined. Consider all possible orderings of three a's and one b, for example. By the techniques of Section 9.5, there are $\binom{4}{1} = 4$ of them. And each of the four occurs as a term in the expansion of $(a+b)^4$:

$$aaab \quad aaba \quad abaa \quad baaa.$$

By the commutative and associative laws of algebra, each such term equals a^3b, so all four are "like terms." When the like terms are combined, therefore, the coefficient of a^3b equals $\binom{4}{1}$.

Similarly, the expansion of $(a+b)^4$ contains the $\binom{4}{2} = 6$ different orderings of two a's and two b's,

$$aabb \quad abab \quad abba \quad baab \quad baba \quad bbaa,$$

all of which equal a^2b^2, so the coefficient of a^2b^2 equals $\binom{4}{2}$. By a similar analysis, the coefficient of ab^3 equals $\binom{4}{3}$. Also, since there is only one way to order four a's, the coefficient of a^4 is 1 (which equals $\binom{4}{0}$), and since there is only one way to order four b's, the coefficient of b^4 is 1 (which equals $\binom{4}{4}$). Thus, when all of the like terms are combined,

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

The binomial theorem generalizes this formula to an arbitrary nonnegative integer n.

Theorem 9.7.2 Binomial Theorem

Given any real numbers a and b and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n.$$

Note that the second expression equals the first because $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, for all nonnegative integers n, provided that $b^0 = 1$ and $a^{n-n} = 1$.

It is instructive to see two proofs of the binomial theorem: an algebraic proof and a combinatorial proof. Both require a precise definition of integer power.

• Definition

For any real number a and any nonnegative integer n, the **nonnegative integer powers of *a*** are defined as follows:

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a \cdot a^{n-1} & \text{if } n > 0 \end{cases}$$

Note This is the definition of 0^0 given by Donald E. Knuth in *The Art of Computer Programming, Volume 1: Fundamental Algorithms,* Third Edition (Reading, Mass.: Addison-Wesley, 1997), p. 57.

In some mathematical contexts, 0^0 is left undefined. Defining it to be 1, as is done here, makes it possible to write general formulas such as $\sum_{i=0}^{n} x^i = \frac{1}{1-x}$ without having to exclude values of the variables that result in the expression 0^0.

The algebraic version of the binomial theorem uses mathematical induction and calls upon Pascal's formula at a crucial point.

Proof of the Binomial Theorem (algebraic version):

Suppose a and b are real numbers. We use mathematical induction and let the property $P(n)$ be the equation

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \qquad \leftarrow P(n)$$

Show that P(0) is true: When $n = 0$, the binomial theorem states that:

$$(a+b)^0 = \sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^k. \qquad \leftarrow P(0)$$

But the left-hand side is $(a+b)^0 = 1$ *[by definition of power]*, and the right-hand side is

$$\sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^k = \binom{0}{0} a^{0-0} b^0$$
$$= \frac{0!}{0! \cdot (0-0)!} \cdot 1 \cdot 1 = \frac{1}{1 \cdot 1} = 1$$

also *[since* $0! = 1$, $a^0 = 1$, *and* $b^0 = 1$*]*. Hence $P(0)$ is true.

Show that for all integers m≥0, if P(m) is true then P(m+1) is true: Let an integer $m \geq 0$ be given, and suppose $P(m)$ is true. That is, suppose

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k.$$

$P(m)$
inductive hypothesis.

We need to show that $P(m+1)$ is true:

$$(a+b)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^{(m+1)-k} b^k.$$

$P(m+1)$

Now, by definition of the $(m+1)$st power,

$$(a+b)^{m+1} = (a+b)\cdot(a+b)^m,$$

so by substitution from the inductive hypothesis,

$$\begin{aligned}(a+b)^{m+1} &= (a+b)\cdot\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k \\ &= a\cdot\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k + b\cdot\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k \\ &= \sum_{k=0}^{m}\binom{m}{k}a^{m+1-k}b^k + \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^{k+1}\end{aligned}$$

by the generalized distributive law and the facts that $a\cdot a^{m-k} = a^{1+m-k} = a^{m+1-k}$ and $b\cdot b^k = b^{1+k} = b^{k+1}$.

We transform the second summation on the right-hand side by making the change of variable $j = k+1$. When $k = 0$, then $j = 1$. When $k = m$, then $j = m+1$. And since $k = j-1$, the general term is

$$\binom{m}{k}a^{m-k}b^{k+1} = \binom{m}{j-1}a^{m-(j-1)}b^j = \binom{m}{j-1}a^{m+1-j}b^j.$$

Hence the second summation on the right-hand side above is

$$\sum_{j=1}^{m+1}\binom{m}{j-1}a^{m+1-j}b^j.$$

But the j in this summation is a dummy variable; it can be replaced by the letter k, as long as the replacement is made everywhere the j occurs:

$$\sum_{j=1}^{m+1}\binom{m}{j-1}a^{m+1-j}b^j = \sum_{k=1}^{m+1}\binom{m}{k-1}a^{m+1-k}b^k.$$

Substituting back, we get

$$(a+b)^{m+1} = \sum_{k=0}^{m}\binom{m}{k}a^{m+1-k}b^k + \sum_{k=1}^{m+1}\binom{m}{k-1}a^{m+1-k}b^k.$$

[The reason for the above maneuvers was to make the powers of a and b agree so that we can add the summations together term by term, except for the first and the last terms, which we must write separately.]

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Thus

$$(a+b)^{m+1} = \binom{m}{0} a^{m+1-0}b^0 + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] a^{m+1-k}b^k + \binom{m}{(m+1)-1} a^{m+1-(m+1)}b^{m+1}$$

$$= a^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] a^{m+1-k}b^k + b^{m+1}$$

since $a^0 = b^0 = 1$ and $\binom{m}{0} = \binom{m}{m} = 1.$

But

$$\left[\binom{m}{k} + \binom{m}{k-1}\right] = \binom{m+1}{k}$$

by Pascal's formula.

Hence

$$(a+b)^{m+1} = a^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k} a^{(m+1)-k}b^k + b^{m+1}$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k} a^{(m+1)-k}b^k$$

because $\binom{m+1}{0} = \binom{m+1}{m+1} = 1$

which is what we needed to show.

It is instructive to write out the product $(a+b)\cdot(a+b)^m$ without using the summation notation but using the inductive hypothesis about $(a+b)^m$:

$$(a+b)^{m+1} = (a+b)\cdot\left[a^m + \binom{m}{1}a^{m-1}b + \cdots + \binom{m}{k-1}a^{m-(k-1)}b^{k-1} + \binom{m}{k}a^{m-k}b^k + \cdots + \binom{m}{m-1}ab^{m-1} + b^m\right].$$

You will see that the first and last coefficients are clearly 1 and that the term containing $a^{m+1-k}b^k$ is obtained from multiplying $a^{m-k}b^k$ by a and $a^{m-(k-1)}b^{k-1}$ by b [*because* $m+1-k = m-(k-1)$]. Hence the coefficient of $a^{m+1-k}b^k$ equals the sum of $\binom{m}{k}$ and $\binom{m}{k-1}$. This is the crux of the algebraic proof.

If n and r are nonnegative integers and $r \le n$, then $\binom{n}{r}$ is called a **binomial coefficient** because it is one of the coefficients in the expansion of the binomial expression $(a+b)^n$.

The combinatorial proof of the binomial theorem follows.

Proof of Binomial Theorem (combinatorial version):

[The combinatorial argument used here to prove the binomial theorem works only for $n \ge 1$. *If we were giving only this combinatorial proof, we would have to prove the case* $n = 0$ *separately. Since we have already given a complete algebraic proof that includes the case* $n = 0$, *we do not prove it again here.]*

Let a and b be real numbers and n an integer that is at least 1. The expression $(a+b)^n$ can be expanded into products of n letters, where each letter is either a or b.

For each $k = 0, 1, 2, \ldots, n$, the product

$$a^{n-k}b^k = \underbrace{a \cdot a \cdot a \cdots a}_{n-k \text{ factors}} \cdot \underbrace{b \cdot b \cdot b \cdots b}_{k \text{ factors}}$$

occurs as a term in the sum the same number of times as there are orderings of $(n-k)$ a's and k b's. But this number is $\binom{n}{k}$, the number of ways to choose k positions into which to place the b's. *[The other $n-k$ positions will be filled by a's.]* Hence, when like terms are combined, the coefficient of $a^{n-k}b^k$ in the sum is $\binom{n}{k}$. Thus

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.$$

This is what was to be proved.

Example 9.7.5 Substituting into the Binomial Theorem

Expand the following expressions using the binomial theorem:

a. $(a+b)^5$ b. $(x-4y)^4$

Solution

a.
$$\begin{aligned}(a+b)^5 &= \sum_{k=0}^{5} \binom{5}{k} a^{5-k}b^k \\ &= a^5 + \binom{5}{1} a^{5-1}b^1 + \binom{5}{2} a^{5-2}b^2 + \binom{5}{3} a^{5-3}b^3 + \binom{5}{4} a^{5-4}b^4 + b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

b. Observe that $(x-4y)^4 = (x+(-4y))^4$. So let $a = x$ and $b = (-4y)$, and substitute into the binomial theorem.

$$\begin{aligned}(x-4y)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^{4-k}(-4y)^k \\ &= x^4 + \binom{4}{1} x^{4-1}(-4y)^1 + \binom{4}{2} x^{4-2}(-4y)^2 + \binom{4}{3} x^{4-3}(-4y)^3 + (-4y)^4 \\ &= x^4 + 4x^3(-4y) + 6x^2(16y^2) + 4x^1(-64y^3) + (256y^4) \\ &= x^4 - 16x^3y + 96x^2y^2 - 256xy^3 + 256y^4\end{aligned}$$

■

Example 9.7.6 Deriving Another Combinatorial Identity from the Binomial Theorem

Use the binomial theorem to show that

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

for all integers $n \geq 0$.

Solution Since $2 = 1 + 1$, $2^n = (1+1)^n$. Apply the binomial theorem to this expression by letting $a = 1$ and $b = 1$. Then

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 \cdot 1$$

since $1^{n-k} = 1$ and $1^k = 1$. Consequently,

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$ ■

Example 9.7.7 Using a Combinatorial Argument to Derive the Identity

According to Theorem 6.3.1, a set with n elements has 2^n subsets. Apply this fact to give a combinatorial argument to justify the identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n.$$

Solution Suppose S is a set with n elements. Then every subset of S has some number of elements k, where k is between 0 and n. It follows that the total number of subsets of S, $N(\mathscr{P}(S))$, can be expressed as the following sum:

$$\begin{bmatrix}\text{number of}\\ \text{subsets}\\ \text{of } S\end{bmatrix} = \begin{bmatrix}\text{number of}\\ \text{subsets of}\\ \text{size } 0\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{subsets of}\\ \text{size } 1\end{bmatrix} + \cdots + \begin{bmatrix}\text{number of}\\ \text{subsets of}\\ \text{size } n\end{bmatrix}.$$

Now the number of subsets of size k of a set with n elements is $\binom{n}{k}$. Hence the

$$\text{number of subsets of } S = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

But by Theorem 6.3.1, S has 2^n subsets. Hence

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n.$$ ■

Example 9.7.8 Using the Binomial Theorem to Simplify a Sum

Express the following sum in **closed form** (without using a summation symbol and without using an ellipsis $\cdots$):

$$\sum_{k=0}^{n} \binom{n}{k} 9^k$$

Solution When the number 1 is raised to any power, the result is still 1. Thus

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} 9^k &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 9^k \\ &= (1+9)^n \qquad \text{by the binomial theorem with } a = 1 \text{ and } b = 9 \\ &= 10^n.\end{aligned}$$ ■

Test Yourself

1. If n and r are nonnegative integers with $r \leq n$, then the relation between $\binom{n}{r}$ and $\binom{n}{n-r}$ is _____.

2. Pascal's formula says that if n and r are positive integers with $r \leq n$, then _____.

3. The crux of the algebraic proof of Pascal's formula is that to add two fractions you need to express both of them with a _____.

4. The crux of the combinatorial proof of Pascal's formula is that the set of subsets of size r of a set $\{x_1, x_2, \ldots, x_{n+1}\}$ can be partitioned into the set of subsets of size r that contain _____ and those that _____.

5. The binomial theorem says that given any real numbers a and b and any nonnegative integer n, _____.

6. The crux of the algebraic proof of the binomial theorem is that, after making a change of variable so that two

summations have the same lower and upper limits and the exponents of a and b are the same, you use the fact that $\binom{m}{k} + \binom{m}{k-1} =$ _____.

7. The crux of the combinatorial proof of the binomial theorem is that the number of ways to arrange k b's and $(n-k)$ a's in order is _____.

Exercise Set 9.7

In 1–4, use Theorem 9.5.1 to compute the values of the indicated quantities. (Assume n is an integer.)

1. $\binom{n}{0}$, for $n \geq 0$

2. $\binom{n}{1}$, for $n \geq 1$

3. $\binom{n}{2}$, for $n \geq 2$

4. $\binom{n}{3}$, for $n \geq 3$

5. Use Theorem 9.5.1 to prove algebraically that $\binom{n}{r} = \binom{n}{n-r}$, for integers n and r with $0 \leq r \leq n$. (This can be done by direct calculation; it is not necessary to use mathematical induction.)

Justify the equations in 6–9 either by deriving them from formulas in Example 9.7.1 or by direct computation from Theorem 9.5.1. Assume m, n, k, and r are integers.

6. $\binom{m+k}{m+k-1} = m+k$, for $m+k \geq 1$

7. $\binom{n+3}{n+1} = \dfrac{(n+3)(n+2)}{2}$, for $n \geq -1$

8. $\binom{k-r}{k-r} = 1$, for $k-r \geq 0$

9. $\binom{2n}{n}$ for $n \geq 0$

10. a. Use Pascal's triangle given in Table 9.7.1 to compute the values of $\binom{6}{2}$, $\binom{6}{3}$, $\binom{6}{4}$, and $\binom{6}{5}$.

b. Use the result of part (a) and Pascal's formula to compute $\binom{7}{3}$, $\binom{7}{4}$, and $\binom{7}{5}$.

c. Complete the row of Pascal's triangle that corresponds to $n = 7$.

11. The row of Pascal's triangle that corresponds to $n = 8$ is as follows:

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1.$$

What is the row that corresponds to $n = 9$?

12. Use Pascal's formula repeatedly to derive a formula for $\binom{n+3}{r}$ in terms of values of $\binom{n}{k}$ with $k \leq r$. (Assume n and r are integers with $n \geq r \geq 3$.)

13. Use Pascal's formula to prove by mathematical induction that if n is an integer and $n \geq 1$, then

$$\sum_{i=2}^{n+1} \binom{i}{2} = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3}.$$

H **14.** Prove that if n is an integer and $n \geq 1$, then

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = 2\binom{n+2}{3}.$$

15. Prove the following generalization of exercise 13: Let r be a fixed nonnegative integer. For all integers n with $n \geq r$,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

16. Think of a set with $m+n$ elements as composed of two parts, one with m elements and the other with n elements. Give a combinatorial argument to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0},$$

where m and n are positive integers and r is an integer that is less than or equal to both m and n.

This identity gives rise to many useful additional identities involving the quantities $\binom{n}{k}$. Because Alexander Vandermonde published an influential article about it in 1772, it is generally called the *Vandermonde convolution.* However, it was known at least in the 1300s in China by Chu Shih-chieh.

H **17.** Prove that for all integers $n \geq 0$,

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

18. Let m be any nonnegative integer. Use mathematical induction and Pascal's formula to prove that for all integers $n \geq 0$,

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}.$$

Use the binomial theorem to expand the expressions in 19–27.

19. $(1+x)^7$

20. $(p+q)^6$

21. $(1-x)^6$

22. $(u-v)^5$

23. $(p-2q)^4$

24. $(u^2-3v)^4$

25. $\left(x+\dfrac{1}{x}\right)^5$

26. $\left(\dfrac{3}{a}-\dfrac{a}{3}\right)^5$

27. $\left(x^2+\dfrac{1}{x}\right)^5$

28. In Example 9.7.5 it was shown that

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^{35}.$$

Evaluate $(a+b)^6$ by substituting the expression above into the equation

$$(a+b)^6 = (a+b)(a+b)^5$$

and then multiplying out and combining like terms.

In 29–34, find the coefficient of the given term when the expression is expanded by the binomial theorem.

29. x^6y^3 in $(x+y)^9$

30. x^7 in $(2x+3)^{10}$

31. a^5b^7 in $(a-2b)^{12}$

32. $u^{16}v^4$ in $(u^2-v^2)^{10}$

33. $p^{16}q^7$ in $(3p^2-2q)^{15}$

34. x^9y^{10} in $(2x-3y^2)^{14}$

35. As in the proof of the binomial theorem, transform the summation

$$\sum_{k=0}^{n}\binom{m}{k}a^{m-k_b k+1}$$

by making the change of variable $j=k+1$.

Use the binomial theorem to prove each statement in 36–41.

36. For all integers $n \ge 1$,

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0.$$

(*Hint:* Use the fact that $1+(-1)=0$.)

H **37.** For all integers $n \ge 0$,

$$3^n=\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\cdots+2^n\binom{n}{n}.$$

38. For all integers $m \ge 0$, $\sum_{i=0}^{m}(-1)^i\binom{m}{i}2^{m-i}=1$.

39. For all integers $n \ge 0$, $\sum_{i=0}^{n}(-1)^i\binom{n}{i}3^{n-i}=2^n$.

40. For all integers $n \ge 0$ and for all nonnegative real numbers x, $1+nx \le (1+x)^n$.

H **41.** For all integers $n \ge 1$,

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{2^2}\binom{n}{2}-\frac{1}{2^3}\binom{n}{3}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}=\begin{cases}0 & \text{if } n \text{ is even}\\ \frac{1}{2^{n-1}} & \text{if } n \text{ is odd}\end{cases}.$$

42. Use mathematical induction to prove that for all integers $n \ge 1$, if S is a set with n elements, then S has the same number of subsets with an even number of elements as with an odd number of elements. Use this fact to give a combinatorial argument to justify the identity of exercise 36.

Express each of the sums in 43–54 in closed form (without using a summation symbol and without using an ellipsis $\cdots$).

43. $\sum_{k=0}^{n}\binom{n}{k}5^k$

44. $\sum_{i=0}^{m}\binom{m}{i}4^i$

45. $\sum_{i=0}^{n}\binom{n}{i}x^i$

46. $\sum_{k=0}^{m}\binom{m}{k}2^{m-k}x^k$

47. $\sum_{j=0}^{2n}(-1)^j\binom{2n}{j}x^j$

48. $\sum_{r=0}^{n}\binom{n}{r}x^{2r}$

49. $\sum_{i=0}^{m}\binom{m}{i}p^{m-i}q^{2i}$

50. $\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k}$

51. $\sum_{i=0}^{m}(-1)^i\binom{m}{i}\frac{1}{2^i}$

52. $\sum_{k=0}^{n}\binom{n}{k}3^{2n-2k}2^{2k}$

53. $\sum_{i=0}^{n}(-1)^i\binom{n}{i}5^{n-i}2^i$

54. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{2n-2k}2^{2k}$

✶ 55. (For students who have studied calculus)

a. Explain how the equation below follows from the binomial theorem:

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k.$$

b. Write the formula obtained by taking the derivative of both sides of the equation in part (a) with respect to x.

c. Use the result of part (b) to derive the formulas below.

(i) $2^{n-1}=\frac{1}{n}\left[\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}\right]$

(ii) $\sum_{k=1}^{n}k\binom{n}{k}(-1)^k=0$

d. Express $\sum_{k=1}^{n}k\binom{n}{k}3^k$ in closed form (without using a summation sign or ellipsis).

Answers for Test Yourself

1. $\binom{n}{r}=\binom{n}{n-r}$ 2. $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$ 3. common denominator 4. x_{n+1}; do not contain x_{n+1}
5. $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$ 6. $\binom{m+1}{k}$ 7. $\binom{n}{k}$