APPENDIX B

SOLUTIONS AND HINTS TO SELECTED EXERCISES

Section 1.1

1. a. $x^2 = -1$ (*Or*: the square of x is -1)

b. A real number x

3. a. Between a and b

b. Real numbers a and b; there is a real number c

5. a. r is positive

b. Positive; the reciprocal of r is positive (*Or*: positive; $1/r$ is positive)

c. Is positive; $1/r$ is positive (*Or*: is positive; the reciprocal of r is positive)

7. a. There are real numbers whose sum is less than their difference.

True. For example, $1+(-1)=0$, $1-(-1)=1+1=2$, and $0<2$.

c. The square of any positive integer is greater than the integer.

True. If n is any positive integer, then $n \geq 1$. Multiplying both sides by the positive number n does not change the direction of the inequality (see Appendix A, T20), and so $n^2 \geq n$.

8. a. Have four sides

b. Has four sides

c. Has four sides

d. Is a square; J has four sides

e. J has four sides

10. a. Have a reciprocal

b. A reciprocal

c. s is a reciprocal for r

12. a. Real number; product with every number leaves the number unchanged

b. With every number leaves the number unchanged

c. $rs = s$

Section 1.2

1. $A = C$ and $B = D$

2. a. The set of all positive real numbers x such that 0 is less than x and x is less than 1

c. The set of all integers n such that n is a factor of 6

3. a. No, $\{4\}$ is a set with one element, namely 4, whereas 4 is just a symbol that represents the number 4

b. Three: the elements of the set are 3, 4, and 5.

c. Three: the elements are the symbol 1, the set $\{1\}$, and the set $\{1,\{1\}\}$

5. *Hint:* **R** is the set of all real numbers, **Z** is the set of all integers, and $\mathbf{Z}^+$ is the set of all positive integers

6. *Hint:* T_0 and T_1 do not have the same number of elements as T_2 and T_{-3}.

7. a. $\{1, -1\}$

c. $\emptyset$ (the set has no elements)

d. **Z** (every integer is in the set)

8. a. No, $B \not\subseteq A$ $\because$ $j \in B$ and $j \notin A$

d. Yes, C is a proper subset of A. Both elements of C are in A, but A contains elements (namely c and f) that are not in C.

9. a. Yes

b. No

f. No

i. Yes

10. a. No. Observe that $(-2)^2 = (-2)(-2) = 4$ whereas $-2^2 = -(2^2) = -4$. So $((-2)^2, -2^2) = (4, -4)$, $(-2^2, (-2)^2) = (-4, 4)$, and $(4, -4) \neq (-4, 4)$ because $-4 \neq 4$.

c. Yes. Note that $8-9=-1$ and $\sqrt[3]{-1}=-1$, and so $(8-9, \sqrt[3]{-1}) = (-1, -1)$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

11. a. $\{(w, a), (w, b), (x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}$

b. $\{(a, w), (b, w), (a, x), (b, x), (a, y), (b, y), (a, z), (b, z)\}$

c. $\{(w, w), (w, x), (w, y), (w, z), (x, w), (x, x), (x, y), (x, z), (y, w), (y, x), (y, y), (y, z), (z, w), (z, x), (z, y), (z, z)\}$

d. $\{(a, a), (a, b), (b, a), (b, b)\}$

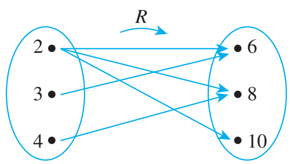
Section 1.3

1. a. No. Yes. No. Yes.

b. $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (4, 8)\}$

c. Domain of $R = A = \{2, 3, 4\}$, co-domain of $R = B = \{6, 8, 10\}$

d.



3. a. $3\ T\ 0$ because $\frac{3-0}{3} = \frac{3}{3} = 1$, which is an integer.

$1 \not{T} (-1)$ because $\frac{1-(-1)}{3} = \frac{2}{3}$, which is not an integer.

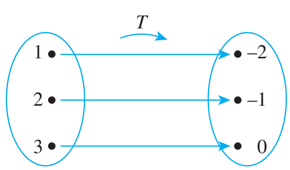
$(2, -1) \in T$ because $\frac{2-(-1)}{3} = \frac{3}{3} = 1$, which is an integer.

$(3, -2) \notin T$ because $\frac{3-(-2)}{3} = \frac{5}{3}$, which is not an integer.

b. $T = \{(1, -2), (2, -1), (3, 0)\}$

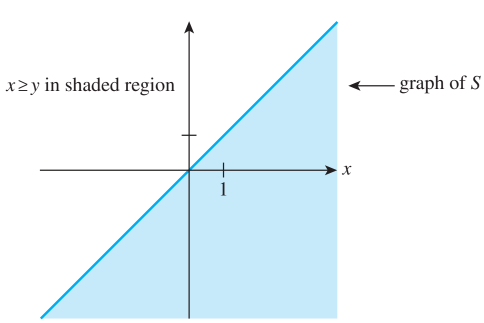
c. Domain of $T = E = \{1, 2, 3\}$, co-domain of $T = F = \{-2, -1, 0\}$

d.

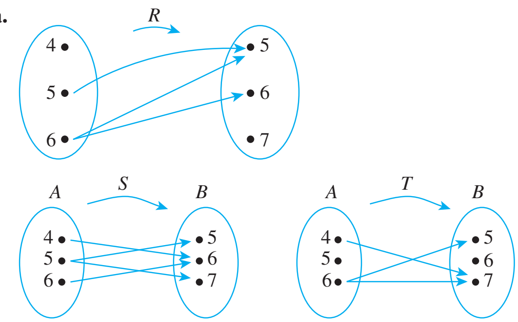


5. a. $(2, 1) \in S$ because $2 \geq 1$. $(2, 2) \in S$ because $2 \geq 2$.
$2 \not{S}\ 3$ because $2 \ngeq 3$. $(-1) \not{S} (-2)$ because $(-1) \ngeq (-2)$.

b.



7. a.



b. R is not a function because it satisfies neither property (1) nor property (2) of the definition. It fails property (1) because $(4, y) \notin R$, for any y in B. It fails property (2) because $(6, 5) \in R$ and $(6, 6) \in R$ and $5 \neq 6$.

S is not a function because $(5, 5) \in S$ and $(5, 7) \in S$ and $5 \neq 7$. So S does not satisfy property (2) of the definition of function.

T is not a function both because $(5, x) \notin T$ for any x in B and because $(6, 5) \in T$ and $(6, 7) \in T$ and $5 \neq 7$. So T does not satisfy either property (1) or property (2) of the definition of function.

9. a. $\emptyset$, $\{(0, 1)\}$, $\{(1, 1)\}$, $\{(0, 1), (1, 1)\}$

b. $\{(0, 1), (1, 1)\}$

c. 1/4

11. No, P is not a function because, for example, $(4, 2) \in P$ and $(4, -2) \in P$ but $2 \neq -2$.

13. a. Domain $= A = \{-1, 0, 1\}$, co-domain $= B = \{t, u, v, w\}$

b. $F(-1) = u$, $F(0) = w$, $F(1) = u$

15. a. This diagram does not determine a function because 2 is related to both 2 and 6.

b. This diagram does not determine a function because 5 is in the domain but it is not related to any element in the co-domain.

16. $f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

19. For all $x \in \mathbf{R}$, $g(x) = \frac{2x^3+2x}{x^2+1} = \frac{2x(x^2+1)}{x^2+1} = 2x = f(x)$. Therefore, by definition of equality of functions, $f = g$.

Section 2.1

1. Common form: If p then q.
p.
Therefore, q.

$(a + 2b)(a^2 - b)$ can be written in prefix notation. All algebraic expressions can be written in prefix notation.

3. Common form: $p \vee q$.
$\sim p$.
Therefore, q.

My mind is shot. Logic is confusing.

5. a. It is a statement because it is a true sentence. 1,024 is a perfect square because $1{,}024 = 32^2$, and the next smaller perfect square is $31^2 = 961$, which has less than four digits.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

6. **a.** $s \wedge i$ **b.** $\sim s \wedge \sim i$

8. **a.** $(h \wedge w) \wedge \sim s$ **d.** $(\sim w \wedge \sim s) \wedge h$

10. **a.** $p \wedge q \wedge r$ **c.** $p \wedge (\sim q \vee \sim r)$

11. Inclusive or. For instance, a team could win the playoff by winning games 1, 3, and 4 and losing game 2. Such an outcome would satisfy both conditions.

12.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

14.

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

16.

p	q	$p \wedge q$	$p \vee (p \wedge q)$	p
T	T	T	T	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

$p \vee (p \wedge q)$ and p always have the same truth values, so they are logically equivalent. (This proves one of the absorption laws.)

18.

p	**t**	$p \vee$ **t**
T	T	T
F	T	T

$p \vee$ **t** and **t** always have the same truth values, so they are logically equivalent. (This proves one of the universal bound laws.)

21.

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

$(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$ always have the same truth values, so they are logically equivalent. (This proves the associative law for $\wedge$.)

23.

p	q	r	$p \wedge q$	$q \vee r$	$(p \wedge q) \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	T	F	F	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

$(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ have different truth values in the fifth and seventh rows, so they are not logically equivalent. (This proves that parentheses are needed with $\wedge$ and $\vee$.)

25. Hal is not a math major or Hal's sister is not a computer science major.

27. The connector is not loose and the machine is not unplugged.

32. $-2 \geq x$ or $x \geq 7$

34. $2 \leq x \leq 5$

36. $1 \leq x$ or $x < -3$

38. This statement's logical form is $(p \wedge q) \vee r$, so its negation has the form $\sim((p \wedge q) \vee r) \equiv \sim(p \wedge q) \wedge \sim r \equiv (\sim p \vee \sim q) \wedge \sim r$. Thus a negation for the statement is ($num\_orders \leq 100$ or $num\_instock > 500$) and $num\_instock \geq 200$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

40.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \vee (p \wedge \sim q)$	$(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	F	T	T
F	F	T	T	F	F	T	T

Its truth values are all T's, so $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$ is a tautology.

41.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Its truth values are all F's, so $(p \wedge \sim q) \wedge (\sim p \vee q)$ is a contradiction.

44. Let p be '$x < 2$', q be '$1 < x$', and r be '$x < 3$'. Then the sentences in (a) and (b) are symbolized as $p \vee \sim(q \wedge r)$ and $\sim q \vee (p \vee \sim r)$, respectively.

p	q	r	$\sim q$	$\sim r$	$q \wedge r$	$\sim(q \wedge r)$	$p \vee \sim r$	$p \vee \sim(q \wedge r)$	$\sim q \vee (p \vee \sim r)$
T	T	T	F	F	T	F	T	T	T
T	T	F	F	T	F	T	T	T	T
T	F	T	T	F	F	T	T	T	T
T	F	F	T	T	F	T	T	T	T
F	T	T	F	F	T	F	F	F	F
F	T	F	F	T	F	T	T	T	T
F	F	T	T	F	F	T	F	T	T
F	F	F	T	T	F	T	T	T	T

The statement forms $p \vee \sim(q \wedge r)$ and $\sim q \vee (p \vee \sim r)$ always have the same truth values, so they are logically equivalent.

Therefore the statements in (a) and (b) are logically equivalent.

46. a. *Solution 1:* Construct a truth table for $p \oplus p$ using the truth values for *exclusive or*.

p	$p \oplus p$
T	F
F	F

because an *exclusive or* statement is false when both components are true and when both components are false.

Since all its truth values are false, $p \oplus p \equiv \mathbf{c}$, a contradiction.

Solution 2: Replace q by p in the logical equivalence $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$, and simplify the result.

$p \oplus p \equiv (p \vee q) \wedge \sim(p \wedge p)$ by defintion of $\oplus$

$\equiv p \wedge \sim p$ by the identity laws

$\equiv \mathbf{c}$ by the negation law for $\wedge$

47. There is a famous story about a philosopher who once gave a talk in which he observed that whereas in English and many other languages a double negative is equivalent to a positive, there is no language in which a double positive is equivalent to a negative. To this, another philosopher, Sidney Morgenbesser, responded sarcastically, "Yeah, yeah."

[Strictly speaking, sarcasm functions like negation. When spoken sarcastically, the words "Yeah, yeah" are not a true double positive; they just mean "no."]

48. a. The distributive law

b. The commutative law for $\vee$

c. The negation law for $\vee$

d. The identity law for $\wedge$

50. $(p \wedge \sim q) \vee p \equiv p \vee (p \wedge \sim q)$ by the commutative law for $\vee$

$\equiv p$ by the absorption law (with $\sim q$ in place of q)

53. $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$

$\equiv \sim[\sim p \wedge (q \vee \sim q)] \vee (p \wedge q)$ by the distributive law

$\equiv \sim(\sim p \wedge \mathbf{t}) \vee (p \wedge q)$ by the negation law for $\vee$

$\equiv \sim(\sim p) \vee (p \wedge q)$ by the identity law for $\wedge$

$\equiv p \vee (p \wedge q)$ by the double negative law

$\equiv p$ by the absorption law

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 2.2

1. If this loop does not contain a **stop** or a **go to**, then it will repeat exactly N times.

3. If you do not freeze, then I'll shoot.

5.

			conclusion	hypothesis	
p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \vee q \to \sim q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

7.

		conclusion		hypothesis	
p	q	r	$\sim q$	$p \wedge \sim q$	$p \wedge \sim q \to r$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

9.

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

12. If $x > 2$ then $x^2 > 4$, and if $x < -2$ then $x^2 > 4$.

13. a.

p	q	$\sim p$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$p \to q$ and $\sim p \vee q$ always have the same truth values, so they are logically equivalent.

14. a. *Hint:* $p \to q \vee r$ is true in all cases except when p is true and both q and r are false.

16. Let p represent "You paid full price" and q represent "You didn't buy it at Crown Books." Thus, "If you paid full price, you didn't buy it at Crown Books" has the form $p \to q$. And "You didn't buy it at Crown Books or you paid full price" has the form $q \vee p$.

p	q	$p \to q$	$q \vee p$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

These two statements are not logically equivalent because their forms have different truth values in rows 2 and 4.

(An alternative representation for the forms of the two statements is $p \to \sim q$ and $\sim q \vee p$. In this case, the truth values differ in rows 1 and 3.)

19. False. The negation of an if-then statement is not an if-then statement. It is an *and* statement.

20. a. P is a square and P is not a rectangle.

d. n is prime and both n is not odd and n is not 2.

Or: n is prime and n is neither odd nor 2.

f. Tom is Ann's father and either Jim is not her uncle or Sue is not her aunt.

21. a. Because $p \to q$ is false, p is true and q is false. Hence $\sim p$ is false, and so $\sim p \to q$ is true.

22. a. If P is not a rectangle, then P is not a square.

d. If n is not odd and n is not 2, then n is not prime.

f. If either Jim is not Ann's uncle or Sue is not her aunt, then Tom is not her father.

23. a. *Converse:* If P is a rectangle, then P is a square.
Inverse: If P is not a square, then P is not a rectangle.

d. *Converse:* If n is odd or n is 2, then n is prime.
Inverse: If n is not prime, then n is not odd and n is not 2.

f. *Converse:* If Jim is Ann's uncle and Sue is her aunt, then Tom is her father.
Inverse: If Tom is not Ann's father, then Jim is not her uncle or Sue is not her aunt.

24.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \to q$ and $q \to p$ have different truth values in the second and third rows, so they are not logically equivalent.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

26.

p	q	$\sim q$	$\sim p$	$\sim q \to \sim p$	$p \to q$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

$\sim q \to \sim p$ and $p \to q$ always have the same truth values, so they are logically equivalent.

28. *Hint:* A person who says "I mean what I say" claims to speak sincerely. A person who says "I say what I mean" claims to speak with precision.

29. $(p \to (q \vee r)) \leftrightarrow ((p \wedge \sim q) \to r)$

p	q	r	$\sim q$	$q \vee r$	$p \wedge \sim q$	$p \to (q \vee r)$	$p \wedge \sim q \to r$	$(p \to (q \vee r)) \leftrightarrow ((p \wedge \sim q) \to r)$
T	T	T	F	T	F	T	T	T
T	T	F	F	T	F	T	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	F	T	F	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	F	F	T	T	T

$(p \to (q \vee r)) \leftrightarrow ((p \wedge \sim q) \to r)$ is a tautology because all of its truth values are T.

32. If this quadratic equation has two distinct real roots, then its discriminant is greater than zero, and if the discriminant of this quadratic equation is greater than zero, then the equation has two real roots.

34. If the Cubs do not win tomorrow's game, then they will not win the pennant.
If the Cubs win the pennant, then they will have won tomorrow's game.

37. If a new hearing is not granted, payment will be made on the fifth.

40. If I catch the 8:05 bus, then I am on time for work.

42. If this number is not divisible by 3, then it is not divisible by 9.
If this number is divisible by 9, then it is divisible by 3.

44. If Jon's team wins the rest of its games, then it will win the championship.

46. **a.** This statement is the converse of the given statement, and so it is not necessarily true. For instance, if the actual boiling point of compound X were 200°C, then the given statement would be true but this statement would be false.

b. This statement must be true. It is the contrapositive of the given statement.

47. **a.** $p \wedge \sim q \to r \equiv \sim(p \wedge \sim q) \vee r$

b. Result of $(a) \equiv \sim[\sim(\sim(p \wedge \sim q)) \wedge \sim r]$
an acceptable answer

$\equiv \sim[(p \wedge \sim q) \wedge \sim r]$
by the double negative law (another acceptable answer)

49. **a.** $(p \to r) \leftrightarrow (q \to r) \equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$

$\equiv \sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$
an acceptable answer

$\equiv [(p \wedge \sim r) \vee (\sim q \vee r)] \wedge [(q \wedge \sim r) \vee (\sim p \vee r)]$
by De Morgan's law (another acceptable answer)

b. Result of (a) $\equiv \sim[\sim(p \wedge \sim r) \wedge \sim(\sim q \vee r)] \wedge$
$\sim[\sim(q \wedge \sim r) \wedge \sim(\sim p \vee r)]$
by De Morgan's law

$\equiv \sim[\sim(p \wedge \sim r) \wedge (q \wedge \sim r)] \wedge$
$\sim[\sim(q \wedge \sim r) \wedge (p \wedge \sim r)]$
by De Morgan's law

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 2.3

1. $\sqrt{2}$ is not rational. **3.** Logic is not easy.

6.

p	q	$p \to q$ (premises)	$q \to p$ (premises)	$p \vee q$ (conclusion)
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	T	F

This row shows that it is possible for an argument of this form to have true premises and a false conclusion. Thus this argument form is invalid.

7.

p	q	r	$\sim q$ (premises)	p (premises)	$p \to q$ (premises)	$\sim q \vee r$ (premises)	r (conclusion)
T	T	T	F	T	T	T	T
T	T	F	F	T	T	F	
T	F	T	T	T	F	T	
T	F	F	T	T	F	T	
F	T	T	F	F	T	T	
F	T	F	F	F	T	F	
F	F	T	T	F	T	T	
F	F	F	T	F	T	T	

This row describes the only situation in which all the premises are true. Because the conclusion is also true here, the argument form is valid.

8.

p	q	r	$\sim q$ (premises)	$p \vee q$ (premises)	$p \to \sim q$ (premises)	$p \to r$ (premises)	r (conclusion)
T	T	T	F	T	F	T	
T	T	F	F	T	F	F	
T	F	T	T	T	T	T	T
T	F	F	T	T	T	F	
F	T	T	F	T	T	T	T
F	T	F	F	T	T	T	F
F	F	T	T	F	T	T	
F	F	F	T	F	T	T	

This row shows that it is possible for an argument of this form to have true premises and a false conclusion. Thus this argument form is invalid.

12. a.

p	q	$p \to q$ (premises)	q (premises)	p (conclusion)
T	T	T	T	T
T	F	F	F	
F	T	T	T	F
F	F	T	F	

This row shows that it is possible for an argument of this form to have true premises and a false conclusion. Thus this argument form is invalid.

14.

p	q	p (premise)	$p \vee q$ (conclusion)
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

These two rows show that in all situations where the premise is true, the conclusion is also true. Thus the argument form is valid.

18.

p	q	$p \vee q$ (premises)	$\sim q$ (premises)	p (conclusion)
T	T	T	F	
T	F	T	T	T
F	T	T	F	
F	F	F	T	

This row represents the only situation in which both premises are true. Because the conclusion is also true here the argument form is valid.

22. Let p represent "Tom is on team A" and q represent "Hua is on team B." Then the argument has the form

$$\sim p \to q$$
$$\sim q \to p$$
$$\therefore \sim p \vee \sim q$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

				premises		conclusion
p q		$\sim p$	$\sim q$	$\sim p \to q$	$\sim q \to p$	$\sim p \vee \sim q$
T	T	F	F	T	T	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	F	

This row shows that it is possible for an argument of this form to have true premises and a false conclusion. Thus this argument form is invalid.

24. $p \to q$
q
$\therefore p$ invalid: converse error

25. $p \vee q$
$\sim p$
$\therefore q$ valid: elimination

26. $p \to q$
$q \to r$
$\therefore p \to r$ valid: transitivity

27. $p \to q$
$\sim p$
$\therefore \sim q$ invalid: inverse error

36. The program contains an undeclared variable.
One explanation:
1. There is not a missing semicolon and there is not a misspelled variable name. *(by (c) and (d) and definition of $\wedge$)*
2. It is not the case that there is a missing semicolon or a misspelled variable name. *(by (1) and De Morgan's laws)*
3. There is not a syntax error in the first five lines. *(by (b) and (2) and modus tollens)*
4. There is an undeclared variable. *(by (a) and (3) and elimination)*

37. The treasure is buried under the flagpole.
One explanation:
1. The treasure is not in the kitchen. *(by (c) and (a) and modus ponens)*
2. The tree in the front yard is not an elm. *(by (b) and (1) and modus tollens)*
3. The treasure is buried under the flagpole. *(by (d) and (2) and elimination)*

38. a. A is a knave and B is a knight.
One explanation:
1. Suppose A is a knight.
2. $\therefore$ What A says is true. *(by definition of knight)*
3. $\therefore$ B is a knight also. *(That's what A said.)*
4. $\therefore$ What B says is true. *(by definition of knight)*
5. $\therefore$ A is a knave. *(That's what B said.)*
6. $\therefore$ We have a contradiction: A is a knight and a knave. *(by (1) and (5))*
7. $\therefore$ The supposition that A is a knight is false. *(by the contradiction rule)*
8. $\therefore$ A is a knave. *(negation of supposition)*
9. $\therefore$ What B says is true. *(B said A was a knave, which we now know to be true.)*
10. $\therefore$ B is a knight. *(by definition of knight)*

d. *Hint:* W and Y are knights; the rest are knaves.

39. The chauffeur killed Lord Hazelton.
One explanation:
1. Suppose the cook was in the kitchen at the time of the murder.
2. $\therefore$ The butler killed Lord Hazelton with strychnine. *(by (c) and (1) and modus ponens)*
3. $\therefore$ We have a contradiction: Lord Hazelton was killed by strychnine and a blow on the head. *(by (2) and (a))*
4. $\therefore$ The supposition that the cook was in the kitchen is false. *(by the contradiction rule)*
5. $\therefore$ The cook was not in the kitchen at the time of the murder. *(negation of supposition)*
6. $\therefore$ Sara was not in the dining room when the murder was committed. *(by (e) and (5) and modus ponens)*
7. $\therefore$ Lady Hazelton was in the dining room when the murder was committed. *(by (b) and (6) and elimination)*
8. $\therefore$ The chauffeur killed Lord Hazelton. *(by (d) and (7) and modus ponens)*

41.

(1)	$p \to t$	by premise (d)
	$\sim t$	by premise (c)
	$\therefore \sim p$	by modus tollens
(2)	$\sim p$	by (1)
	$\therefore \sim p \vee q$	by generalization
(3)	$\sim p \vee q \to r$	by premise (a)
	$\sim p \vee q$	by (2)
	$\therefore r$	by modus ponens
(4)	$\sim p$	by (1)
	r	by (3)
	$\therefore \sim p \wedge r$	by conjunction
(5)	$\sim p \wedge r \to \sim s$	by premise (e)
	$\sim p \wedge r$	by (4)
	$\therefore \sim s$	by modus ponens
(6)	$s \vee \sim q$	by premise (b)
	$\sim s$	by (5)
	$\therefore \sim q$	by elimination

43.

(1)	$\sim w$	by premise (d)
	$u \vee w$	by premise (e)
	$\therefore u$	by elimination
(2)	$u \to \sim p$	by premise (c)
	u	by (1)
	$\therefore \sim p$	by modus ponens
(3)	$\sim p \to r \wedge \sim s$	by premise (a)
	$\sim p$	by (2)
	$\therefore r \wedge \sim s$	by modus ponens
(4)	$r \wedge \sim s$	by (3)
	$\therefore \sim s$	by specialization
(5)	$t \to s$	by premise (b)
	$\sim s$	by (4)
	$\therefore \sim t$	by modus tollens

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 2.4

1. $R = 1$

3. $S = 1$

5.

Input		Output
P	***Q***	***R***
1	1	1
1	0	1
0	1	0
0	0	1

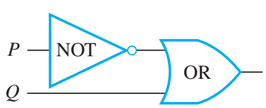
7.

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

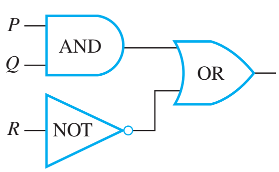
9. $P \vee \sim Q$

11. $(P \wedge \sim Q) \vee R$

13.

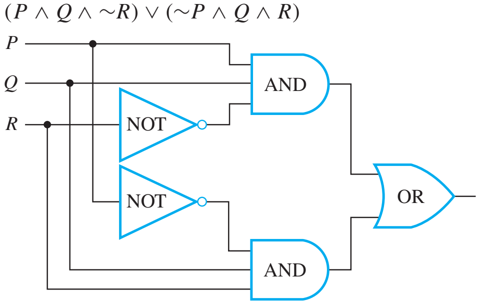


16.



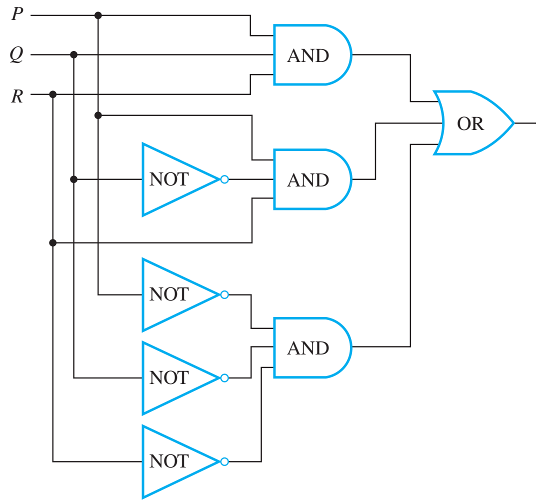
18. a. $(P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R)$

b.



20. a. $(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (\sim P \wedge \sim Q \wedge \sim R)$

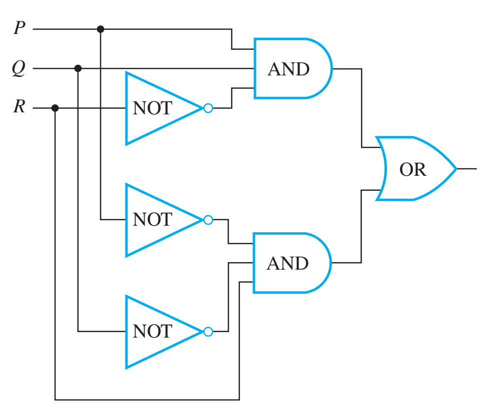
b.



22. The input/output table is

Input			Output
P	***Q***	***R***	***S***
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

One circuit (among many) having this input/output table is shown below.

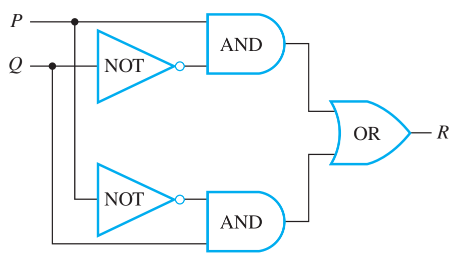


Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

24. Let P and Q represent the positions of the switches in the classroom, with 0 being "down" and 1 being "up." Let R represent the condition of the light, with 0 being "off" and 1 being "on." Initially, $P = Q = 0$ and $R = 0$. If either P or Q (but not both) is changed to 1, the light turns on. So when $P = 1$ and $Q = 0$, then $R = 1$, and when $P = 0$ and $Q = 1$, then $R = 1$. Thus when one switch is up and the other is down the light is on, and hence moving the switch that is down to the up position turns the light off. So when $P = 1$ and $Q = 1$, then $R = 0$. It follows that the input/output table has the following appearance:

Input		Output
P	***Q***	***R***
1	1	0
1	0	1
0	1	1
0	0	0

One circuit (among many) having this input/output table is the following:



26. The Boolean expression for (a) is $(P \wedge Q) \vee Q$, and for (b) it is $(P \vee Q) \wedge Q$. We must show that if these expressions are regarded as statement forms, then they are logically equivalent. But

$(P \wedge Q) \vee Q$

$\equiv Q \vee (P \wedge Q)$ by the commutative law for $\vee$

$\equiv (Q \vee P) \wedge (Q \vee Q)$ by the distributive law

$\equiv (Q \vee P) \wedge Q$ by the idempotent law

$\equiv (P \vee Q) \wedge Q$ by the commutative law for $\wedge$

Alternatively, by the absorption laws, both statement forms are logically equivalent to Q.

28. The Boolean expression for (a) is

$$(P \wedge Q) \vee (P \wedge {\sim}Q) \vee ({\sim}P \wedge {\sim}Q)$$

and for (b) it is $P \vee {\sim}Q$. We must show that if these expressions are regarded as statement forms, then they are logically equivalent. But

$(P \wedge Q) \vee (P \wedge {\sim}Q) \vee ({\sim}P \wedge {\sim}Q)$

$\equiv ((P \wedge Q) \vee (P \wedge {\sim}Q)) \vee ({\sim}P \wedge {\sim}Q)$ by inserting parentheses (which is legal by the associative law)

$\equiv (P \wedge (Q \vee {\sim}Q)) \vee ({\sim}P \wedge {\sim}Q)$ by the distributive law

$\equiv (P \wedge \mathbf{t}) \vee ({\sim}P \wedge {\sim}Q)$ by the negation law for $\vee$

$\equiv P \vee ({\sim}P \wedge {\sim}Q)$ by the identity law for $\wedge$

$\equiv (P \vee {\sim}P) \wedge (P \vee {\sim}Q)$ by the distibutive law

$\equiv \mathbf{t} \wedge (P \vee {\sim}Q)$ by the negation law for $\vee$

$\equiv (P \vee {\sim}Q) \wedge \mathbf{t}$ by the commutative law for $\wedge$

$\equiv P \vee {\sim}Q$ by the identity law for $\wedge$

30. $(P \wedge Q) \vee ({\sim}P \wedge Q) \vee ({\sim}P \wedge {\sim}Q)$

$\equiv (P \wedge Q) \vee (({\sim}P \wedge Q) \vee ({\sim}P \wedge {\sim}Q))$ by inserting parentheses (which is legal by the associative law)

$\equiv (P \wedge Q) \vee ({\sim}P \wedge (Q \vee {\sim}Q))$ by the distributive law

$\equiv (P \wedge Q) \vee ({\sim}P \wedge \mathbf{t})$ by the negation law for $\vee$

$\equiv (P \wedge Q) \vee {\sim}P$ by the identity law for $\wedge$

$\equiv {\sim}P \vee (P \wedge Q)$ by the commutative law for $\vee$

$\equiv ({\sim}P \vee P) \wedge ({\sim}P \vee Q)$ by the distributive law

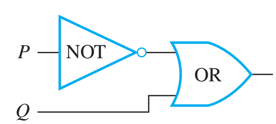
$\equiv (P \vee {\sim}P) \wedge ({\sim}P \vee Q)$ by the commutative law for $\vee$

$\equiv \mathbf{t} \wedge ({\sim}P \vee Q)$ by the negation law for $\vee$

$\equiv ({\sim}P \vee Q) \wedge \mathbf{t}$ by the commutative law for $\wedge$

$\equiv {\sim}P \vee Q$ by the identity law for $\wedge$

The following is, therefore, a circuit with at most two logic gates that has the same input/output table as the circuit corresponding to the given expression.



34. b. $(P \downarrow Q) \downarrow (P \downarrow Q)$

$\equiv {\sim}(P \downarrow Q)$ by part (a)

$\equiv {\sim}[{\sim}(P \vee Q)]$ by definition of $\downarrow$

$\equiv P \vee Q$ by the double negative law

d. *Hint:* Use the results of exercise 13 of Section 2.2 and part (a) and (c) of this exercise.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 2.5

1. $19_{10} = 16 + 2 + 1 = 10011_2$

4. $458_{10} = 256 + 128 + 64 + 8 + 2 = 111001010_2$

7. $1110_2 = 8 + 4 + 2 = 14_{10}$

10. $1100101_2 = 64 + 32 + 4 + 1 = 101_{10}$

13.
$$\begin{array}{cccccc} & & 1 & 1 & 1 & \\ & & 1 & 0 & 1 & 1_2 \\ + & & & 1 & 0 & 1_2 \\ \hline & 1 & 0 & 0 & 0 & 0_2 \end{array}$$

15.
$$\begin{array}{cccccccc} & & 1 & 1 & 1 & & 1 & \\ & & 1 & 0 & 1 & 1 & 0 & 1_2 \\ + & & & 1 & 1 & 1 & 0 & 1_2 \\ \hline & 1 & 0 & 0 & 1 & 0 & 1 & 0_2 \end{array}$$

17.
$$\begin{array}{cccccc} & & & & 1 & \\ & & 1 & 10 & \cancel{10} & 1 \\ & \cancel{1} & \cancel{0} & \cancel{1} & 0 & 0_2 \\ - & & 1 & 1 & 0 & 1_2 \\ \hline & & & 1 & 1 & 1_2 \end{array}$$

19.
$$\begin{array}{ccccccc} & & & & 0 & \cancel{10} & \\ & 1 & 0 & 1 & \cancel{1} & 0 & 1_2 \\ - & & 1 & 0 & 0 & 1 & 1_2 \\ \hline & & 1 & 1 & 0 & 1 & 0_2 \end{array}$$

21. a. $S = 0, T = 1$

23. $23_{10} = (16 + 4 + 2 + 1)_{10} = 00010111_2 \to 11101000 \to 11101001$. So the answer is 11101001.

25. $4_{10} = 00000100_2 \to 11111011 \to 11111100$. So the answer is 11111100.

27. Because the leading bit is 1, this is the 8-bit representation of a negative integer. $11010011 \to 00101100 \to 00101101_2 \leftrightarrow -(32 + 8 + 4 + 1)_{10} = -45_{10}$. So the answer is -45_{10}.

29. Because the leading bit is 1, this is the 8-bit representation of a negative integer. $11110010 \to 00001101 \to 00001110_2 \leftrightarrow -(8 + 4 + 2)_{10} = -14_{10}$. So the answer is -14_{10}.

31. $57_{10} = (32 + 16 + 8 + 1)_{10} = 111001_2 \to 00111001$ $-118_{10} = -(64 + 32 + 16 + 4 + 2)_{10} = -1110110 \to 01110110 \to 10001001 \to 10001010$. So the 8-bit representations of 57 and -118 are 00111001 and 10001010. Adding the 8-bit representations gives

$$\begin{array}{c|c|c|c|c|c|c|c|c|} \cline{2-9} & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ \cline{2-9} + & & & & & & & & \\ \cline{2-9} & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \cline{2-9} \multicolumn{9}{c}{} \\ \hline \cline{2-9} & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \cline{2-9} \end{array}$$

Since the leading bit of this number is a 1, the answer is negative. Converting back to decimal form gives

$$\begin{aligned} 11000011 \to 00111100 \to -00111101_2 \\ = -(32 + 16 + 8 + 4 + 1)_{10} = -61_{10}. \end{aligned}$$

So the answer is -61.

32. $62_{10} = (32 + 16 + 8 + 4 + 2)_{10}$
$= 111110_2 \to 00111110$
$-18_{10} = -(16 + 2)_{10}$
$= -10010_2 \to 00010010 \to 11101101 \to 11101110$

Thus the 8-bit representations of 62 and -18 are 00111110 and 11101110. Adding the 8-bit representations gives

$$\begin{array}{c|c|c|c|c|c|c|c|c|} \cline{2-9} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ \cline{2-9} + & & & & & & & & \\ \cline{2-9} & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ \cline{2-9} \multicolumn{9}{c}{} \\ \hline \cline{2-9} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ \cline{2-9} \end{array}$$

Truncating the 1 in the 2^8th position gives 00101100. Since the leading bit of this number is a 0, the answer is positive. Converting back to decimal form gives

$$00101100 \to 101100_2 = (32 + 8 + 4)_{10} = 44_{10}.$$

So the answer is 44.

33. $-6_{10} = -(4 + 2)_{10}$
$= -110_2 \to 00000110 \to 11111001 \to 11111010$
$-73_{10} = -(64 + 8 + 1)_{10} =$
$-1001001_2 \to 01001001 \to 10110110 \to 10110111$

Thus the 8-bit representations of -6 and -73 are 11111010 and 10110111. Adding the 8-bit representations gives

$$\begin{array}{c|c|c|c|c|c|c|c|c|} \cline{2-9} & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ \cline{2-9} + & & & & & & & & \\ \cline{2-9} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ \cline{2-9} \multicolumn{9}{c}{} \\ \hline \cline{2-9} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \cline{2-9} \end{array}$$

Truncating the 1 in the 2^8th position gives 10110001. Since the leading bit of this number is a 1, the answer is negative. Converting back to decimal form gives

$$\begin{aligned} 10110001 \to 01001110 \to -01001111_2 \\ = -(64 + 8 + 4 + 2 + 1)_{10} = -79_{10}. \end{aligned}$$

So the answer is -79.

38. $A2BC_{16} = 10 \cdot 16^3 + 2 \cdot 16^2 + 11 \cdot 16 + 12 = 41660_{10}$

41. $0001110000001010101011110_2$

44. $2E_{16}$

47. a. $6 \cdot 8^4 + 1 \cdot 8^3 + 5 \cdot 8^2 + 0 \cdot 8 + 2 \cdot 1 = 25{,}410_{10}$

Section 3.1

1. a. False **b.** True

2. a. The statement is true. The integers correspond to certain of the points on a number line, and the real numbers correspond to all the points on the number line.

b. The statement is false; 0 is neither positive nor negative.

c. The statement is false. For instance, let $r = -2$. Then $-r = -(-2) = 2$, which is positive.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

d. The statement is false. For instance, the number $\frac{1}{2}$ is a real number, but it is not an integer.

3. a. $P(2)$ is "$2 > \frac{1}{2}$," which is true.
$P\left(\frac{1}{2}\right)$ is "$\frac{1}{2} > \frac{1}{\frac{1}{2}}$." This is false because $\frac{1}{\frac{1}{2}} = 2$, and $\frac{1}{2} \not> 2$.
$P(-1)$ is "$-1 > \frac{1}{-1}$." This is false because $\frac{1}{-1} = -1$, and $-1 \not> -1$.
$P\left(-\frac{1}{2}\right)$ is "$-\frac{1}{2} > \frac{1}{-\frac{1}{2}}$." This is true because $\frac{1}{-\frac{1}{2}} = -2$ and $-\frac{1}{2} > -2$.
$P(-8)$ is "$-8 > \frac{1}{-8}$." This is false because $\frac{1}{-8} = -\frac{1}{8}$ and $-8 \not> -\frac{1}{8}$.

b. If the domain of $P(x)$ is the set of all real numbers, then its truth set is the set of all real numbers x for which either $x > 1$ or $-1 < x < 0$.

c. If the domain of $P(x)$ is the set of all positive real numbers, then its truth set is the set of all real numbers x for which $x > 1$.

4. b. If the domain of $Q(n)$ is the set of all integers, then its truth set is $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$.

5. a. $Q(-2,1)$ is the statement "If $-2 < 1$ then $(-2)^2 < 1^2$." The hypothesis of this statement is $-2 < 1$, which is true. The conclusion is $(-2)^2 < 1^2$, which is false because $(-2)^2 = 4$ and $1^2 = 1$ and $4 \not< 1$. Thus $Q(-2, 1)$ is a conditional statement with a true hypothesis and a false conclusion. So $Q(-2, 1)$ is false.

c. $Q(3,8)$ is the statement "If $3 < 8$ then $3^2 < 8^2$." The hypothesis of this statement is $3 < 8$, which is true. The conclusion is $3^2 < 8^2$, which is also true because $3^2 = 9$ and $8^2 = 64$ and $9 < 64$. Thus $Q(3, 8)$ is a conditional statement with a true hypothesis and a true conclusion. So $Q(3, 8)$ is true.

7. a. The truth set is the set of all integers d such that $6/d$ is an integer, so the truth set is $\{-6, -3, -2, -1, 1, 2, 3, 6\}$.

c. The truth set is the set of all real numbers x with the property that $1 \le x^2 \le 4$, so the truth set is $\{x \in \mathbf{R} \mid -2 \le x \le -1 \text{ or } 1 \le x \le 2\}$. In other words, the truth set is the set of all real numbers between -2 and -1 inclusive together with those between 1 and 2 inclusive.

8. a. $\{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

9. Counterexample: Let $x = 1 : 1 \not> \frac{1}{1}$. (*This is one counterexample among many.*)

11. Counterexample: Let $m = 1$ and $n = 1$. Then $m \cdot n = 1 \cdot 1 = 1$ and $m + n = 1 + 1 = 2$. But $1 \not\geq 2$, and so $m \cdot n \not\geq m + n$. (*This is one counterexample among many.*)

13. (a), (e), (f) **14.** (b), (c), (e), (f)

15. a. *Partial answer*: Every rectangle is a quadrilateral.
b. *Partial answer*: At least one set has 16 subsets.

16. a. $\forall$ dinosaurs x, x is extinct.
c. $\forall$ irrational numbers x, x is not an integer.
e. $\forall$ integers x, x^2 does not equal 2, 147, 581, 953.

17. a. $\exists$ an exercise x such that x has an answer.

18. a. $\exists s \in D$ such that $E(s)$ and $M(s)$. (Or: $\exists s \in D$ such that $E(s) \wedge M(s)$.)
b. $\forall s \in D$, if $C(s)$ then $E(s)$. (Or: $\forall s \in D$, $C(s) \rightarrow E(s)$.)
e. $(\exists s \in D$ such that $C(s) \wedge E(s)) \wedge (\exists s \in D$ such that $C(s) \wedge \sim E(s))$

19. (b), (d), (e)

20. *Partial answer*: The square root of a positive real number is positive.

21. a. The total degree of G is even, for any graph G.
c. p is even, for some prime number p

22. a. $\forall x$, if x is a Java program, then x has at least 5 lines.

23. a. $\forall x$ if x is an equilateral triangle, then x is isosceles.

24. a. $\exists$ a hatter x such that x is mad.
$\exists x$ such that x is a hatter and x is mad.

25. a. $\forall$ nonzero fractions x, the reciprocal of x is a fraction.
$\forall x$, if x is a nonzero fraction, then the reciprocal of x is a fraction.
c. $\forall$ triangles x, the sum of the angles of x is 180°.
$\forall x$, if x is a triangle, then the sum of the angles of x is 180°.
e. $\forall$ even integers x and y, the sum of x and y is even.
$\forall x$ and y, if x and y are even integers, then the sum of x and y is even.

26. b. $\forall x(\text{Int}(x) \longrightarrow \text{Ratl}(x)) \wedge \exists x(\text{Ratl}(x) \wedge \sim\text{Int}(x))$

27. a. False. Figure b is a circle that is not gray.
b. True. All the gray figures are circles.

28. b. *One answer among many:* If a real number is negative, then when its opposite is computed, the result is a positive real number.
This statement is true because for all real numbers x, $-(-|x|) = |x|$ (and any negative real number can be represented as $-|x|$, for some real number x).
d. *One answer among many:* There is a real number that is not an integer. This statement is true. For instance, $\frac{1}{2}$ is a real number that is not an integer.

30. b. *One answer among many:* If an integer is prime, then it is not a perfect square.
This statement is true because a prime number is an integer greater than 1 that is not a product of two smaller positive integers. So a prime number cannot be a perfect square because if it were, it would be a product of two smaller positive integers.

31. *Hint:* Your answer should have the appearance shown in the following made-up example:
Statement: "If a function is differentiable, then it is continuous."
Formal version: $\forall$ functions f, if f is differentiable, then f is continuous.
Citation: Calculus by D. R. Mathematician, Best Publishing Company, 2004, page 263.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

32. a. True: Any real number that is greater than 2 is greater than 1.

c. False: $(-3)^2 > 4$ but $-3 \not> 2$.

33. a. True. Whenever both a and b are positive, so is their product.

b. False. Let $a = -2$ and $b = -3$. Then $ab = 6$, which is not less than zero.

Section 3.2

1. (a) and (e) are negations.

3. a. $\exists$ a fish x such that x does not have gills.

c. $\forall$ movies m, m is less than or equal to 6 hours long. (*Or:* $\forall$ movies m, m is no more than 6 hours long.)

In 4–6 there are other correct answers in addition to those shown.

4. a. Some dogs are unfriendly. (*Or:* There is at least one unfriendly dog.)

c. All suspicions were unsubstantiated. (*Or:* No suspicions were substantiated.)

5. a. There is a valid argument that does not have a true conclusion. (*Or:* At least one valid argument does not have a true conclusion.)

6. a. Sets A and B have at least one point in common.

7. The statement is not existential.
Informal negation: There is at least one order from store A for item B.
Formal version of statement: $\forall$ orders x, if x is from store A, then x is not for item B.

9. $\exists$ a real number x such that $x > 3$ and $x^2 \le 9$.

11. The proposed negation is not correct. Consider the given statement: "The sum of any two irrational numbers is irrational." For this to be false means that it is possible to find at least one pair of irrational numbers whose sum is rational. On the other hand, the negation proposed in the exercise ("The sum of any two irrational numbers is rational") means that given *any* two irrational numbers, their sum is rational. This is a much stronger statement than the actual negation: The truth of this statement implies the truth of the negation (assuming that there are at least two irrational numbers), but the negation can be true without having this statement be true.
Correct negation: There are at least two irrational numbers whose sum is rational.
Or: The sum of some two irrational numbers is rational.

13. The proposed negation is not correct. There are two mistakes: The negation of a "for all" statement is not a "for all" statement; and the negation of an if-then statement is not an if-then statement.
Correct negation: There exists an integer n such that n^2 is even and n is not even.

15. a. True: All the odd numbers in D are positive.

c. False: $x = 16$, $x = 26$, $x = 32$, and $x = 36$ are all counterexamples.

16. $\exists$ a real number x such that $x^2 \ge 1$ and $x \not> 0$. In other words, $\exists$ a real number x such that $x^2 \ge 1$ and $x \le 0$.

18. $\exists$ a real number x such that $x(x+1) > 0$ and both $x \le 0$ and $x \ge -1$.

20. $\exists$ integers a, b, and c such that $a - b$ is even and $b - c$ is even and $a - c$ is not even.

22. There is an integer such that the square of the integer is odd but the integer is not odd. (*Or:* At least one integer has an odd square but is not itself odd.)

24. a. If a person is a child in Tom's family, then the person is female.
If a person is a female in Tom's family, then the person is a child.
The second statement is the converse of the first.

25. a. *Converse:* If $n + 1$ is an even integer, then n is a prime number that is greater than 2.
Counterexample: Let $n = 15$. Then $n + 1$ is even but n is not a prime number that is greater than 2.

26. *Statement*: $\forall$ real numbers x, if $x^2 \ge 1$ then $x > 0$.
Contrapositive: $\forall$ real numbers x, if $x \le 0$ then $x^2 < 1$.
Converse: $\forall$ real numbers x, if $x > 0$ then $x^2 \ge 1$.
Inverse: $\forall$ real numbers x, if $x^2 < 1$ then $x \le 0$.
The statement and its contrapositive are false. As a counterexample, let $x = -2$. Then $x^2 = (-2)^2 = 4$, and so $x^2 \ge 1$. However $x \not> 0$.
The converse and the inverse are also false. As a counterexample, let $x = 1/2$. Then $x^2 = 1/4$, and so $x > 0$ but $x^2 \not\ge 1$.

28. *Statement*: $\forall x \in \mathbf{R}$, if $x(x+1) > 0$ then $x > 0$ or $x < -1$.
Contrapositive: $\forall x \in \mathbf{R}$, if $x \le 0$ and $x \ge -1$, then $x(x+1) \le 0$.
Converse: $\forall x \in \mathbf{R}$, if $x > 0$ or $x < -1$ then $x(x+1) > 0$.
Inverse: $\forall x \in \mathbf{R}$, if $x(x+1) \le 0$ then $x \le 0$ and $x \ge -1$.
The statement, its contrapositive, its converse, and its inverse are all true.

30. *Statement*: $\forall$ integers a, b, and c, if $a - b$ is even and $b - c$ is even, then $a - c$ is even.
Contrapositive: $\forall$ integers a, b, and c, if $a - c$ is not even, then $a - b$ is not even or $b - c$ is not even.
Converse: $\forall$ integers a, b and c, if $a - c$ is even then $a - b$ is even and $b - c$ is even.
Inverse: $\forall$ integers a, b, and c, if $a - b$ is not even or $b - c$ is not even, then $a - c$ is not even.
The statement is true, but its converse and inverse are false. As a counterexample, let $a = 3$, $b = 2$, and $c = 1$. Then $a - c = 2$, which is even, but $a - b = 1$ and $b - c = 1$, so it is not the case that both $a - b$ and $b - c$ are even.

32. *Statement*: If the square of an integer is odd, then the integer is odd.
Contrapositive: If an integer is not odd, then the square of the integer is not odd.
Converse: If an integer is odd, then the square of the integer is odd.
Inverse: If the square of an integer is not odd, then the integer is not odd.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

The statement, its contrapositive, its converse, and its inverse are all true.

34. a. If n is divisible by some prime number strictly between 1 and $\sqrt{n}$, then n is not prime.

36. a. *One possible answer:* Let $P(x)$ be "$2x \neq 1$." The statement "$\forall x \in \mathbf{Z}$, $2x \neq 1$" is true, but the statements "$\forall x \in \mathbf{Q}$, $2x \neq 1$" and "$\forall x \in \mathbf{R}$, $2x \neq 1$" are both false.

37. The claim is "$\forall x$, if $x = 1$ and x is in the sequence 0204, then x is to the left of all the 0's in the sequence."
The negation is "$\exists x$ such that $x = 1$ and x is in the sequence 0204, and x is not to the left of all the 0's in the sequence." The negation is false because the sequence does not contain the character 1. So the claim is vacuously true (or true by default).

39. If a person earns a grade of C^- in this course, then the course counts toward graduation.

41. If a person is not on time each day, then the person will not keep this job.

43. It is not the case that if a number is divisible by 4, then that number is divisible by 8. In other words, there is a number that is divisible by 4 and is not divisible by 8.

45. It is not the case that if a person has a large income, then that person is happy. In other words, there is a person who has a large income and is not happy.

48. No. Interpreted formally, the statement says, "If carriers do not offer the same lowest fare, then you may not select among them," or, equivalently, "If you may select among carriers, then they offer the same lowest fare."

Section 3.3

1. a. True: Tokyo is the capital of Japan.
b. False: Athens is not the capital of Egypt.

2. a. True: $2^2 > 3$ **b.** False: $1^2 \not> 1$

3. a. $y = \frac{1}{2}$ **b.** $y = -1$

4. a. Let $n = 16$. Then $n > x$ because $16 > 15.83$.

5. The statement says that no matter what circle anyone might give you, you can find a square of the same color. This is true because the only circles are a, c, and b, and given a or c, which are blue, square j is also blue, and given b, which is gray, squares g and h are also gray.

7. This is true because triangle d is above every square.

9. a. There are five elements in D. For each, an element in E must be found so that the sum of the two equals 0. So: if $x = -2$, take $y = 2$; if $x = -1$, take $y = 1$; if $x = 0$, take $y = 0$; if $x = 1$, take $y = -1$; if $x = 2$, take $y = -2$.
Alternatively, note that for each integer x in D, the integer $-x$ is also in D, including 0 (because $-0 = 0$), and for all integers x, $x + (-x) = 0$.

10. a. True. Every student chose at least one dessert: Uta chose pie, Tim chose both pie and cake, and Yuen chose pie.
c. This statement says that some particular dessert was chosen by every student. This is true: Every student chose pie.

11. a. There is a student who has seen *Casablanca*.
c. Every student has seen at least one movie.
d. There is a movie that has been seen by every student. (There are many other acceptable ways to state these answers.)

12. a. Negation: $\exists x$ in D such that $\forall y$ in E, $x + y \neq 1$.
The negation is true. When $x = -2$, the only number y with the property that $x + y = 1$ is $y = 3$, and 3 is not in E.
b. *Negation*: $\forall x$ in D, $\exists y$ in E such that $x + y \neq -y$.
The negation is true and the original statement is false. To see that the original statement is false, take any x in D and choose y to be any number in E with $y \neq -\frac{x}{2}$. Then $2y \neq -x$, and adding x and subtracting y from both sides gives $x + y \neq -y$.

In 13–19 there are other correct answers in addition to those shown.

13. a. *Statement:* For every color, there is an animal of that color.
There are animals of every color.
b. *Negation:* $\exists$ a color C such that $\forall$ animals A, A is not colored C.
For some color, there is no animal of that color.

14. *Statement*: There is a book that all people have read.
Negation: There is no book that all people have read.
(*Or:* $\forall$ books b, $\exists$ a person p such that p has not read b.)

15. a. *Statement:* For every odd integer n, there is an integer k such that $n = 2k + 1$.
Given any odd integer, there is another integer for which the given integer equals twice the other integer plus 1.
Given any odd integer n, we can find another integer k so that $n = 2k + 1$.
An odd integer is equal to twice some other integer plus 1.
Every odd integer has the form $2k + 1$ for some integer k.
b. *Negation:* $\exists$ an odd integer n such that $\forall$ integers k, $n \neq 2k + 1$.
There is an odd integer that is not equal to $2k + 1$ for any integer k.
Some odd integer does not have the form $2k + 1$ for any integer k.

18. a. *Statement:* For every real number x, there is a real number y such that $x + y = 0$.
Given any real number x, there exists a real number y such that $x + y = 0$.
Given any real number, we can find another real number (possibly the same) such that the sum of the given number plus the other number equals 0.
Every real number can be added to some other real number (possibly itself) to obtain 0.
b. *Negation:* $\exists$ a real number x such that $\forall$ real numbers y, $x + y \neq 0$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

There is a real number x for which there is no real number y with $x + y = 0$.
There is a real number x with the property that $x + y \neq 0$ for any real number y.
Some real number has the property that its sum with any other real number is nonzero.

20. Statement (1) says that no matter what square anyone might give you, you can find a triangle of a different color. This is true because the only squares are e, g, h, and j, and given squares g and h, which are gray, you could take triangle d, which is black; given square e, which is black, you could take either triangle f or i, which are gray; and given square j, which is blue, you could take either triangle f or h, which are gray, or triangle d, which is black.

21. a. (1) The statement "$\forall$ real numbers x, $\exists$ a real number y such that $2x + y = 7$" is true.
(2) The statement "$\exists$ a real number x such that $\forall$ real numbers y, $2x + y = 7$" is false.

b. Both statements (1) "$\forall$ real numbers x, $\exists$ a real number y such that $x + y = y + x$" and (2) "$\exists$ a real number x such that $\forall$ real numbers y, $x + y = y + x$" are true.

22. a. Given any real number, you can find a real number so that the sum of the two is zero. In other words, every real number has an additive inverse. This statement is true.

b. There is a real number with the following property: No mattter what real number is added to it, the sum of the two will be zero. In other words, there is one particular real number whose sum with any real number is zero. This statement is false; no one number will work for all numbers. For instance, if $x + 0 = 0$, then $x = 0$, but in that case $x + 1 = 1 \neq 0$.

24. a. $\sim(\forall x \in D(\forall y \in E(P(x, y))))$

$$\equiv \exists x \in D(\sim(\forall y \in E(P(x, y))))$$
$$\equiv \exists x \in D(\exists y \in E(\sim P(x, y)))$$

25. This statement says that all of the circles are above all of the squares. This statement is true because the circles are a, b, and c, and the squares are e, g, h, and j, and all of a, b, and c lie above all of e, g, h, and j.
Negation: There is a circle x and a square y such that x is not above y. In other words, at least one of the circles is not above at least one of the squares.

27. The statement says that there are a circle and a square with the property that the circle is above the square and has a different color from the square. This statement is true. For example, circle a lies above square e and is differently colored from e. (Several other examples could also be given.)

29. a. *Version with interchanged quantifiers:* $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}$, x < y.

b. The given statement says that for any real number x, there is a real number y that is greater than x. This is true: For any real number x, let $y = x + 1$. Then $x < y$. The version with interchanged quantifiers says that there is a real number that is less than every other real number. This is false.

31. $\forall$ people x, $\exists$ a person y such that x is older than y.

32. $\exists$ a person x such that $\forall$ people y, x is older than y.

33. a. *Formal version:* $\forall$ people x, $\exists$ a person y such that x loves y.

b. *Negation:* $\exists$ a person x such that $\forall$ people y, x does not love y. In other words, there is someone who does not love anyone.

34. a. *Formal version:* $\exists$ a person x such that $\forall$ people y, x loves y.

b. *Negation:* $\forall$ people x, $\exists$ a person y such that x does not love y. In other words, everyone has someone whom they do not love.

37. a. *Statement:* $\forall$ even integers n, $\exists$ an integer k such that $n = 2k$.

b. *Negation:* $\exists$ an even integer n such that $\forall$ integers k, $n \neq 2k$.
There is some even integer that is not equal to twice any other integer.

39. a. *Statement:* $\exists$ a program P such that $\forall$ questions Q posed to P, P gives the correct answer to Q.

b. *Negation:* $\forall$ programs P, there is a question Q that can be posed to P such that P does not give the correct answer to Q.

40. a. $\forall$ minutes m, $\exists$ a sucker s such that s was born in minute m.

41. a. This statement says that given any positive integer, there is a positive integer such that the first integer is one more than the second integer. This is false. Given the positive integer $x = 1$, the only integer with the property that $x = y + 1$ is $y = 0$, and 0 is not a positive integer.

b. This statement says that given any integer, there is an integer such that the first integer is one more than the second integer. This is true. Given any integer x, take $y = x - 1$. Then y is an integer, and $y + 1 = (x - 1) + 1 = x$.

e. This statement says that given any real number, there is a real number such that the product of the two is equal to 1. This is false because $0 \cdot y = 0 \neq 1$ for every number y. So when $x = 0$, there is no real number y with the property that $xy = 1$.

42. $\exists \varepsilon > 0$ such that $\forall$ integers N, $\exists$ an integer n such that $n > N$ and either $L - \varepsilon \geq a_n$ or $a_n \geq L + \varepsilon$. In other words, there is a positive number ε such that for all integers N, it is possible to find an integer n that is greater than N and has the property that a_n does not lie between $L - \varepsilon$ and $L + \varepsilon$.

44. a. This statement is true. The unique real number with the given property is 1. Note that

$$1 \cdot y = y \quad \text{for all real numbers } y,$$

and if x is any real number such that for instance, $x \cdot 2 = 2$, then dividing both sides by 2 gives $x = 2/2 = 1$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

46. a. True. Both triangles a and c lie above all the squares.
b. *Formal version:* $\exists x(\text{Triangle}(x) \wedge (\forall y(\text{Square}(y) \rightarrow \text{Above}(x, y))))$
c. *Formal negation:* $\forall x(\sim\text{Triangle}(x) \vee (\exists y\ (\text{Square}\ (y) \wedge \sim\text{Above}(x, y))))$

48. a. False. There is no square to the right of circle k.
b. *Formal version:* $\forall x(\text{Circle}(x) \rightarrow (\exists y(\text{Square}(y) \wedge \text{RightOf}(y, x))))$
c. *Formal negation:* $\exists x(\text{Circle}(x) \wedge (\forall y(\sim\text{Square}(y) \vee \sim\text{RightOf}(y, x))))$

51. a. False. There is no object that has a different color from every other object.
b. *Formal version:* $\exists y(\forall x(x \neq y \rightarrow \sim\text{SameColor}(x, y)))$
c. *Formal negation:* $\forall y(\exists x(x \neq y \wedge \text{SameColor}(x, y)))$

53. a. False
b. *Formal version:* $\exists x(\text{Circle}(x) \wedge (\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$
c. *Formal negation:* $\forall x(\sim\text{Circle}(x) \vee (\forall y(\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y))))$

55. a. No matter what the domain D or the predicates $P(x)$ and $Q(x)$ are, the given statements have the same truth value. If the statement "$\forall x$ in D, $(P(x) \wedge Q(x))$" is true, then $P(x) \wedge Q(x)$ is true for every x in D, which implies that both $P(x)$ and $Q(x)$ are true for every x in D. But then $P(x)$ is true for every x in D, and also $Q(x)$ is true for every x in D. So the statement "$(\forall x$ in $D, P(x)) \wedge (\forall x,$ in $D, Q(x))$" is true. Conversely, if the statement "$(\forall x$ in $D, P(x)) \wedge (\forall x$ in $D, Q(x))$" is true, then $P(x)$ is true for every x in D, and also $Q(x)$ is true for every x in D. This implies that both $P(x)$ and $Q(x)$ are true for every x in D, and so $P(x) \wedge Q(x)$ is true for every x in D. Hence the statement "$\forall x$ in D, $(P(x) \wedge Q(x))$" is true.

59. a. Yes **b.** $X = w_1, X = w_2$ **c.** $X = b_2, X = w_2$

Section 3.4

1. b. $(f_i + f_j)^2 = f_i^2 + 2f_i f_j + f_j^2$
c. $(3u + 5v)^2 = (3u)^2 + 2(3u)(5v) + (5v)^2$ $(= 9u^2 + 30uv + 25v^2)$
d. $(g(r) + g(s))^2 = (g(r))^2 + 2g(r)g(s) + (g(s))^2$

2. 0 is even.

3. $\frac{2}{3} + \frac{4}{5} = \frac{(2 \cdot 5 + 3 \cdot 4)}{(3 \cdot 5)} \left(= \frac{22}{15}\right)$

5. $\frac{1}{0}$ is not an irrational number.

7. Invalid; converse error

8. Valid by universal modus ponens (or universal instantiation)

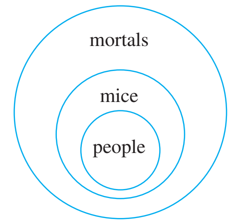
9. Invalid; inverse error

10. Valid by universal modus tollens

16. Invalid; converse error

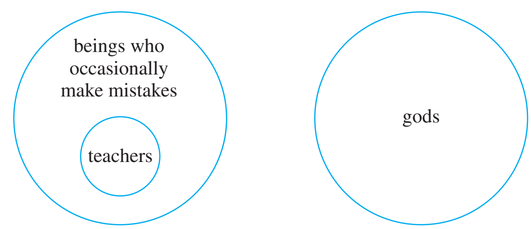
19. $\forall x$, if x is a good car, then x is not cheap.
a. Valid, universal modus ponens (or universal instantiation)
b. Invalid, converse error

21. Valid. (A valid argument can have false premises and a true conclusion!)



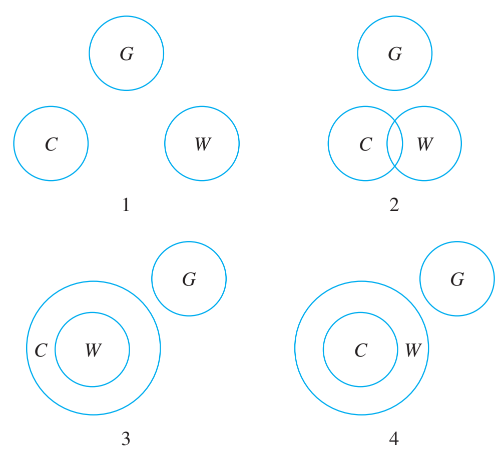
The major premise says the set of people is included in the set of mice. The minor premise says the set of mice is included in the set of mortals. Assuming both of these premises are true, it must follow that the set of people is included in the set of mortals. Since it is impossible for the conclusion to be false if the premises are true, the argument is valid.

23. Valid. The major and minor premises can be diagrammed as follows:



According to the diagram, the set of teachers and the set of gods can have no common elements. Hence, if the premises are true, then the conclusion must also be true, and so the argument is valid.

25. Invalid. Let C represent the set of all college cafeteria food, G the set of all good food, and W the set of all wasted food. Then any one of the following diagrams could represent the given premises.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Only in drawing (1) is the conclusion true. Hence it is possible for the premises to be true while the conclusion is false, and so the argument is invalid.

28. (3) *Contrapositive form:* If an object is gray, then it is a circle.
(2) If an object is a circle, then it is to the right of all the blue objects.
(1) If an object is to right of all the blue objects, then it is above all the triangles.
$\therefore$ If an object is gray, then it is above all the triangles.

31. 4. If an animal is in the yard, then it is mine.
1. If an animal belongs to me, then I trust it.
5. If I trust an animal, then I admit it into my study.
3. If I admit an animal into my study, then it will beg when told to do so.
6. If an animal begs when told to do so, then that animal is a dog.
2. If an animal is a dog, then that animal gnaws bones.
$\therefore$ If an animal is in the yard, then that animal gnaws bones; that is, all the animals in the yard gnaw bones.

33. 2. If a bird is in this aviary, then it belongs to me.
4. If a bird belongs to me, then it is at least 9 feet high.
1. If a bird is at least 9 feet high, then it is an ostrich.
3. If a bird lives on mince pies, then it is not an ostrich.
Contrapositive: If a bird is an ostrich, then it does not live on mince pies.
$\therefore$ If a bird is in this aviary, then it does not live on mince pies; that is, no bird in this aviary lives on mince pies.

Section 4.1

1. a. Yes: $-17 = 2(-9) + 1$
b. Yes: $0 = 2 \cdot 0$
c. Yes: $2k - 1 = 2(k - 1) + 1$ and $k - 1$ is an integer because it is a difference of integers.

2. a. Yes: $6m + 8n = 2(3m + 4n)$ and $(3m + 4n)$ is an integer because 3, 4, m, and n are integers, and products and sums of integers are integers.
b. Yes: $10mn + 7 = 2(5mn + 3) + 1$ and $5mn + 3$ is an integer because 3, 5, m, and n are integers, and products and sums of integers are integers.
c. Not necessarily. For instance, if $m = 3$ and $n = 2$, then $m^2 - n^2 = 9 - 4 = 5$, which is prime. (Note that $m^2 - n^2$ is composite for many values of m and n because of the identity $m^2 - n^2 = (m - n)(m + n)$.)

4. For example, let $m = n = 2$. Then m and n are integers such that $m > 0$ and $n > 0$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{2} = 1$, which is an integer.

7. For example, let $n = 7$. Then n is an integer such that $n > 5$ and $2^n - 1 = 127$, which is prime.

9. For example, 25, 9, and 16 are all perfect squares, because $25 = 5^2$, $9 = 3^2$, and $16 = 4^2$, and $25 = 9 + 16$. Thus 25 is a perfect square that can be written as a sum of two other perfect squares.

11. <u>Counterexample</u>: Let $a = -2$ and $b = -1$. Then $a < b$ because $-2 < -1$, but $a^2 \not< b^2$ because $(-2)^2 = 4$ and $(-1)^2 = 1$ and $4 \not< 1$. *[So the hypothesis of the statement is true but its conclusion is false.]*

14. This property is true for some integers and false for other integers. For instance, if $a = 0$ and $b = 1$, the property is true because $(0 + 1)^2 = 0^2 + 1^2$, but if $a = 1$ and $b = 1$, the property is false because $(1 + 1)^2 = 4$ and $1^2 + 1^2 = 2$ and $4 \neq 2$.

15. *Hint:* This property is true for some integers and false for other integers. To justify this answer you need to find examples of both.

17.
$$2 = 1^2 + 1^2,\ 4 = 2^2,\ 6 = 2^2 + 1^2 + 1^2,$$
$$8 = 2^2 + 2^2,\ 10 = 3^2 + 1^2,\ 12 = 2^2 + 2^2 + 2^2,$$
$$14 = 3^2 + 2^2 + 1^2,\ 16 = 4^2,$$
$$18 = 3^2 + 3^2 = 4^2 + 1^2 + 1^2,\ 20 = 4^2 + 2^2,$$
$$22 = 3^2 + 3^2 + 2^2,\ 24 = 4^2 + 2^2 + 2^2$$

19. a. $\forall$ integers m and n, if m is even and n is odd, then $m + n$ is odd.
$\forall$ even integers m and odd integers n, $m + n$ is odd.
If m is any even integer and n is any odd integer, then $m + n$ is odd.
b. **(a)** any odd integer **(b)** integer r
(c) $2r + (2s + 1)$ **(d)** $m + n$ is odd

20. a. If an integer is greater than 1, then its reciprocal is between 0 and 1.
b. *Start of proof*: Suppose m is any integer such that $m > 1$.
Conclusion to be shown: $0 < 1/m < 1$.

22. a. If the product of two integers is 1, then either both are 1 or both are -1.
b. *Start of proof*: Suppose m and n are any integers with $mn = 1$.
Conclusion to be shown: $m = n = 1$ *or* $m = n = -1$.

24. *Two versions of a correct proof are given below to illustrate some of the variety that is possible.*
<u>Proof 1</u>: Suppose n is any *[particular but arbitrarily chosen]* even integer. *[We must show that $-n$ is even.]* By definition of even, $n = 2k$ for some integer k. Multiplying both side by -1 gives that
$$-n = -(2k) = 2(-k).$$
Let $r = -k$. Then r is an integer because $r = -k = (-1)k$, -1 and k are integers, and a product of two integers is an integer. Hence, $-n = 2r$ for some integer r, and so $-n$ is even *[as was to be shown]*.
<u>Proof 2</u>: Suppose n is any even integer. By definition of even, $n = 2k$ for some integer k. Then
$$-n = -2k = 2(-k).$$
But $-k$ is an integer because it is a product of integers -1 and k. Thus $-n$ equals twice some integer, and so $-n$ is even by definition of even.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

25. Proof: Suppose a is any even integer and b is any odd integer. *[We must show that $a - b$ is odd.]* By definition of even and odd, $a = 2r$ and $b = 2s + 1$ for some integers r and s. By substitution and algebra,

$$a - b = 2r - (2s + 1) = 2r - 2s - 1 = 2(r - s - 1) + 1.$$

Let $t = r - s - 1$. Then t is an integer because differences of integers are integers. Thus $a - b = 2t + 1$, where t is an integer, and so, by definition of odd, $a - b$ is odd *[as was to be shown]*.

26. *Hint:* The conclusion to be shown is that a certain quantity is odd. To show this, you need to show that the quantity equals twice some integer plus one.

29. Proof: Suppose n is any *[particular but arbitrarily chosen]* odd integer. *[We must show that $3n + 5$ is even.]* By definition of odd, there is an integer r such that $n = 2r + 1$. Then

$$\begin{aligned} 3n + 5 &= 3(2r + 1) + 5 && \text{by substitution} \\ &= 6r + 3 + 5 \\ &= 6r + 8 \\ &= 2(3r + 4) && \text{by algebra.} \end{aligned}$$

Let $t = 3r + 4$. Then t is an integer because products and sums of integers are integers. Hence, $3n + 5 = 2t$, where t is an integer, and so, by definition of even, $3n + 5$ is even *[as was to be shown]*.

31. Proof: Suppose k is any *[particular but arbitrarily chosen]* odd integer and m is any even integer. *[We must show that $k^2 + m^2$ is odd.]* By definition of odd and even, $k = 2a + 1$ and $m = 2b$ for some integers a and b. Then

$$\begin{aligned} k^2 + m^2 &= (2a + 1)^2 + (2b)^2 && \text{by substitution} \\ &= 4a^2 + 4a + 1 + 4b^2 \\ &= 4(a^2 + a + b^2) + 1 \\ &= 2(2a^2 + 2a + 2b^2) + 1 && \text{by algebra.} \end{aligned}$$

But $2a^2 + 2a + 2b^2$ is an integer because it is a sum of products of integers. Thus $k^2 + m^2$ is twice an integer plus 1, and so $k^2 + m^2$ is odd *[as was to be shown]*.

33. Proof: Suppose n is any even integer. Then $n = 2k$ for some integer k. Hence

$$(-1)^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1$$

[by the laws of exponents from algebra]. This is what was to be shown.

35. The negation of the statement is "For all integers $m \geq 3$, $m^2 - 1$ is not prime."
Proof of the negation: Suppose m is any integer with $m \geq 3$. By basic algebra, $m^2 - 1 = (m - 1)(m + 1)$. Because $m \geq 3$, both $m - 1$ and $m + 1$ are positive integers greater than 1, and each is smaller than $m^2 - 1$. So $m^2 - 1$ is a product of two smaller positive integers, each greater than 1, and hence $m^2 - 1$ is not prime.

38. The incorrect proof just shows the theorem to be true in the one case where $k = 2$. A real proof must show that it is true for *all* integers $k > 0$.

39. The mistake in the "proof" is that the same symbol, k, is used to represent two different quantities. By setting $m = 2k$ and $n = 2k + 1$, the proof implies that $n = m + 1$, and thus it deduces the conclusion only for this one situation. When $m = 4$ and $n = 17$, for instance, the computations in the proof indicate that $n - m = 1$, but actually $n - m = 13$. In other words, the proof does not deduce the conclusion for an arbitrarily chosen even integer m and odd integer n, and hence it is invalid.

40. This incorrect proof exhibits circular reasoning. The word *since* in the third sentence is completely unjustified. The second sentence tells only what happens *if* $k^2 + 2k + 1$ is composite. But at that point in the proof, it has not been established that $k^2 + 2k + 1$ *is* composite. In fact, that is exactly what is to be proved.

43. True. Proof: Suppose m and n are any odd integers. *[We must show that mn is odd.]* By definition of odd, $n = 2r + 1$ and $m = 2s + 1$ for some integers r and s. Then

$$\begin{aligned} mn &= (2r + 1)(2s + 1) && \text{by subsitution} \\ &= 4rs + 2r + 2s + 1 \\ &= 2(2rs + r + s) + 1 && \text{by algebra.} \end{aligned}$$

Now $2rs + r + s$ is an integer because products and sums of integers are integers and $2, r,$ and s are all integers. Hence $mn = 2\cdot$(some integer) + 1, and so, by definition of odd, mn is odd.

44. True. Proof: Suppose n is any odd integer. *[We must show that $-n$ is odd.]* By definition of odd, $n = 2k + 1$ for some integer k. By substitution and algebra,

$$-n = -(2k + 1) = -2k - 1 = 2(-k - 1) + 1.$$

Let $t = -k - 1$. Then t is an integer because differences of integers are integers. Thus $-n = 2t + 1$, where t is an integer, and so, by definition of odd, $-n$ is odd *[as was to be shown]*.

45. False. Counterexample: Both 3 and 1 are odd, but their difference is $3 - 1 = 2$, which is even.

47. False. Counterexample: Let $m = 1$ and $n = 3$. Then $m + n = 4$ is even, but neither summand m nor summand n is even.

54. Proof: Suppose n is any integer. Then

$$\begin{aligned} 4(n^2 + n + 1) - 3n^2 &= 4n^2 + 4n + 4 - 3n^2 \\ &= n^2 + 4n + 4 = (n + 2)^2 \end{aligned}$$

(by algebra). But $(n + 2)^2$ is a perfect square because $n + 2$ is an integer (being a sum of n and 2). Hence $4(n^2 + n + 1) - 3n^2$ is a perfect square, as was to be shown.

56. *Hint:* This is true.

62. *Hint:* The answer is no.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 4.2

1. $\frac{-35}{6} = \frac{-35}{6}$

3. $\frac{4}{5} + \frac{2}{9} = \frac{4 \cdot 9 + 2 \cdot 5}{45} = \frac{46}{45}$

4. Let $x = 0.3737373737\ldots$.
Then $100x = 37.37373737\ldots$, and so
$100x - x = 37.37373737\ldots - 0.3737373737\ldots$.
Thus $99x = 37$, and hence $x = \frac{37}{99}$.

6. Let $x = 320.5492492492\ldots$.
Then $10000x = 3205492.492492\ldots$, and
$10x = 3205.492492492\ldots$, and so
$10000x - 10x = 3205492 - 3205$.
Thus $9990x = 3202287$, and hence $x = \frac{3202287}{9990}$.

8. b. $\forall$ real numbers x and y, if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

9. Because a and b are integers, $b - a$ and ab^2 are both integers (since differences and products of integers are integers). Also, by the zero product property, $ab^2 \neq 0$ because neither a nor b is zero. Hence $(b-a)/ab^2$ is a quotient of two integers with nonzero denominator, and so it is rational.

11. Proof: Suppose n is any *[particular but arbitrarily chosen]* integer. Then $n = n \cdot 1$, and so $n = n/1$ by by dividing both sides by 1. Now n and 1 are both integers, and $1 \neq 0$. Hence n can be written as a quotient of integers with a nonzero denominator, and so n is rational.

12. **(a)** any *[particular but arbitrarily chosen]* rational number
(b) integers a and b **(c)** $(a/b)^2$ **(d)** b^2
(e) zero product property **(f)** r^2 is rational

13. a. $\forall$ real numbers r, if r is rational then $-r$ is rational.
Or: $\forall r$, if r is a rational number then $-r$ is rational.
Or: $\forall$ rational numbers r, $-r$ is rational.

b. The statement is true. Proof: Suppose r is a *[particular but arbitrarily chosen]* rational number. *[We must show that $-r$ is rational.]* By definition of rational, $r = a/b$ for some integers a and b with $b \neq 0$. Then

$$\begin{aligned} -r &= -\frac{a}{b} && \text{by substitution} \\ &= \frac{-a}{b} && \text{by algebra.} \end{aligned}$$

But since a is an integer, so is $-a$ (being the product of -1 and a). Hence $-r$ is a quotient of integers with a nonzero denominator, and so $-r$ is rational *[as was to be shown]*.

15. Proof: Suppose r and s are rational numbers. By definition of rational, $r = a/b$ and $s = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$. Then

$$\begin{aligned} rs &= \frac{a}{b} \cdot \frac{c}{d} && \text{by substitution} \\ &= \frac{ac}{bd} && \text{by the rules of algebra for multiplying fractions.} \end{aligned}$$

Now ac and bd are both integers (being products of integers) and $bd \neq 0$ (by the zero product property). Hence rs is a quotient of integers with a nonzero denominator, and so, by definition of rational, rs is rational.

16. *Hint:* Counterexample: Let r be any rational number and $s = 0$. Then r and s are both rational, but the quotient of r divided by s is undefined and therefore is not a rational number.
Revised statement to be proved: For all rational numbers r and s, if $s \neq 0$ then r/s is rational.

17. *Hint:* The conclusion to be shown is that a certain quantity (the difference of two rational numbers) is rational. To show this, you need to show that the quantity can be expressed as a ratio of two integers with a nonzero denominator.

18. *Hint:* $\frac{a/b + c/d}{2} = \frac{(ad+bc)/(bd)}{2} = \frac{ad+bc}{2bd}$

19. *Hint:* If $a < b$ then $a + a < a + b$ (by T19 of Appendix A), or equivalently $2a < a + b$. Thus $a < \frac{a+b}{2}$ (by T20 Appendix A).

21. True. Proof: Suppose m is any even integer and n is any odd integer. *[We must show that $m^2 + 3n$ is odd.]* By properties 1 and 3 of Example 4.2.3, m^2 is even (because $m^2 = m \cdot m$) and $3n$ is odd (because both 3 and n are odd). It follows from property 5 *[and the commutative law for addition]* that $m^2 + 3n$ is odd *[as was to be shown]*.

24. Proof: Suppose r and s are any rational numbers. By Theorem 4.2.1, both 2 and 3 are rational, and so, by exercise 15, both $2r$ and $3s$ are rational. Hence, by Theorem 4.2.2, $2r + 3s$ is rational.

27. Let

$$x = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2^{n+1}}}{\frac{1}{2}} = \frac{1 - \frac{1}{2^{n+1}}}{\frac{1}{2}} \cdot \frac{2^{n+1}}{2^{n+1}} = \frac{2^{n+1} - 1}{2^n}.$$

But $2^{n+1} - 1$ and 2^n are both integers (since n is a nonnegative integer) and $2^n \neq 0$ by the zero product property. Therefore, x is rational.

31. Proof: Suppose c is a real number such that

$$r_3c^3 + r_2c^2 + r_1c + r_0 = 0,$$

where r_0, r_1, r_2, and r_3 are rational numbers. By definition of rational, $r_0 = a_0/b_0$, $r_1 = a_1/b_1$, $r_2 = a_2/b_2$, and $r_3 = a_3/b_3$ for some integers, a_0, a_1, a_2, a_3, and nonzero integers b_0, b_1, b_2, and b_3. By substitution,

$$\begin{aligned} &r_3c^3 + r_2c^2 + r_1c + r_0 \\ &= \frac{a_3}{b_3}c^3 + \frac{a_2}{b_2}c^2 + \frac{a_1}{b_1}c + \frac{a_0}{b_0} \\ &= \frac{b_0b_1b_2a_3}{b_0b_1b_2b_3}c^3 + \frac{b_0b_1b_3a_2}{b_0b_1b_2b_3}c^2 + \frac{b_0b_2b_3a_1}{b_0b_1b_2b_3}c + \frac{b_1b_2b_3a_0}{b_0b_1b_2b_3} \\ &= 0. \end{aligned}$$

Multiplying both sides by $b_0b_1b_2b_3$ gives

$$b_0b_1b_2a_3 \cdot c^3 + b_0b_1b_3a_2 \cdot c^2 + b_0b_2b_3a_1 \cdot c + b_1b_2b_3a_0 = 0.$$

Let $n_3 = b_0b_1b_3a_3$, $n_2 = b_0b_1b_3a_2$, $n_1 = b_0b_2b_3a_1$, and $n_0 = b_1b_2b_3a_0$. Then n_0, n_1, n_2, and n_3 are all integers (being products of integers). Hence c satisfies the equation

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$n_3c^3 + n_2c^2 + n_1c + n_0 = 0.$$

where n_0, n_1, n_2, and n_3 are all integers. This is what was to be shown.

33. a. *Hint:* Note that $(x-r)(x-s) = x^2 - (r+s)x + rs$. If both r and s are odd, then $r+s$ is even and rs is odd. So the coefficient of x^2 is 1 (odd), the coefficient of x is even, and the constant coefficient, rs, is odd.

35. This "proof" assumes what is to be proved.

37. By setting both r and s equal to a/b, this incorrect proof violates the requirement that r and s be arbitrarily chosen rational numbers. If both r and s equal a/b, then $r = s$.

Section 4.3

1. Yes, $52 = 13 \cdot 4$ **2.** Yes, $56 = 7 \cdot 8$

4. Yes, $(3k+1)(3k+2)(3k+3) = 3\,[(3k+1)(3k+2)(k+1)]$, and $(3k+1)(3k+2)(k+1)$ is an integer because k is an integer and sums and products of integers are integers.

6. No, $29/3 \cong 9.67$, which is not an integer.

7. Yes, $66 = (-3)(-22)$.

8. Yes, $6a(a+b) = 3a[2(a+b)]$, and $2(a+b)$ is an integer because a and b are integers and sums and products of integers are integers.

10. No, $34/7 \cong 4.86$, which is not an integer.

12. Yes, $n^2 - 1 = (4k+1)^2 - 1 = (16k^2 + 8k + 1) - 1 = 16k^2 + 8k = 8(2k^2 + k)$, and $2k^2 + k$ is an integer because k is an integer and sums and products of integers are integers.

14. **(a)** $a \mid b$ **(b)** $b = a \cdot r$ **(c)** $-r$ **(d)** $a \mid (-b)$

15. Proof: Suppose a, b, and c are any integers such that $a \mid b$ and $a \mid c$. *[We must show that $a \mid (b+c)$.]* By definition of divides, $b = ar$ and $c = as$ for some integers r and s. Then

$$b + c = ar + as = a(r+s) \quad \text{by algebra.}$$

Let $t = r + s$. Then t is an integer (being a sum of integers), and thus $b + c = at$ where t is an integer. By definition of divides, then, $a \mid (b+c)$ *[as was to be shown].*

16. *Hint:* The conclusion to be shown is that a certain quantity is divisible by a. To show this, you need to show that the quantity equals a times some integer.

17. a. $\forall$ integers n if n is a multiple of 3 then $-n$ is a multiple of 3.

b. The statement is true. Proof: Suppose n is any integer that is a multiple of 3. *[We must show that $-n$ is a multiple of 3.]* By definition of multiple, $n = 3k$ for some integer k. Then

$$\begin{aligned} -n &= -(3k) && \text{by substitution} \\ &= 3(-k) && \text{by algebra.} \end{aligned}$$

Hence, by definition of multiple, $-n$ is a multiple of 3 *[as was to be shown].*

18. Counterexample: Let $a = 2$ and $b = 1$. Then $a + b = 2 + 1 = 3$, and so $3 \mid (a+b)$ because $3 = 3 \cdot 1$. On the other hand, $a - b = 2 - 1 = 1$, and $3 \nmid 1$ because $1/3$ is not an integer. Thus $3 \nmid (a-b)$. *[So the hypothesis of the statement is true but its conclusion is false.]*

19. *Start of proof*: Suppose a, b, and c are any integers such that a divides b. *[We must show that a divides bc.]*

22. *Hint:* The given statement can be rewritten formally as "$\forall$ integers n, if n is divisible by 6, then n is divisible by 2." This statement is true.

24. The statement is true. Proof: Suppose a, b, and c are any integers such that $a \mid b$ and $a \mid c$. *[We must show that $a \mid (2b - 3c)$.]* By definition of divisibility, we know that $b = am$ and $c = an$ for some integers m and n. It follows that $2b - 3c = 2(am) - 3(an)$ (*by substitution*) $= a(2m - 3n)$ (*by basic algebra*). Let $t = 2m - 3n$. Then t is an integer because it is a difference of products of integers. Hence $2b - 3c = at$, where t is an integer, and so $a \mid (2b - 3c)$ by definition of divisibility *[as was to be shown].*

25. The statement is false. Counterexample: Let $a = 2$, $b = 3$, and $c = 8$. Then $a \mid c$ because 2 divides 8, but $ab \nmid c$ because $ab = 6$ and 6 does not divide 8.

26. *Hint:* The statement is true.

27. *Hint*: The statement is false.

32. No. Each of these numbers is divisible by 3, and so their sum is also divisible by 3. But 100 is not divisible by 3. Thus the sum cannot equal \$100.

36. a. The sum of the digits is 54, which is divisible by 9. Therefore, 637,425,403,705,125 is divisible by 9 and hence also divisible by 3 (by transitivity of divisibility). Because the rightmost digit is 5, then 637,425,403,705,125 is not divisible by 5. And because the two rightmost digits are 25, which is not divisible by 4, then 637,425,403,705,125 is not divisible by 4.

37. a. $1176 = 2^3 \cdot 3 \cdot 7^2$

38. a. $p_1^{2e_1} p_2^{2e_2} \ldots p_k^{2e_k}$

b. $n = 42,\ 2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot n = 5880^2$

40. a. Because $12a = 25b$, the unique factorization theorem guarantees that the standard factored forms of $12a$ and $25b$ must be the same. Thus $25b$ contains the factors $2^2 \cdot 3 (= 12)$. But since neither 2 nor 3 divide 25, the factors $2^2 \cdot 3$ must all occur in b, and hence $12 \mid b$. Similarly, $12a$ contains the factors $5^2 = 25$, and since 5 is not a factor of 12, the factors 5^2 must occur in a. So $25 \mid a$.

41. *Hint:* $45^8 \cdot 88^5 = (3^2 \cdot 5)^8 \cdot (2^3 \cdot 11)^5 = 3^{16} \cdot 5^8 \cdot 2^{15} \cdot 11^5$. How many factors of 10 does this number contain?

42. a. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 2 \cdot 3 \cdot 5 \cdot 2 \cdot 2 \cdot 3 \cdot 2 = 2^4 \cdot 3^2 \cdot 5$

44. Proof: Suppose n is a nonnegative integer whose decimal representation ends in 0. Then $n = 10m + 0 = 10m$ for some integer m. Factoring out a 5 yields $n = 10m = 5(2m)$, and $2m$ is an integer since m is an integer. Hence $10m$ is divisible by 5, which is what was to be shown.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

47. *Hint:* You may take it as a fact that for any positive integer k,

$$10^k = \underbrace{99\ldots 9}_{k \text{ of these}} + 1; \text{ that is,}$$

$$10^k = 9\cdot 10^{k-1} + 9\cdot 10^{k-2} + \cdots + 9\cdot 10^1 + 9\cdot 10^0 + 1.$$

Section 4.4

1. $q = 7, r = 7$ **3.** $q = 0, r = 36$

5. $q = -5, r = 10$ **7. a.** 4 **b.** 7

11. a. When today is Saturday, 15 days from today is two weeks (which is Saturday) plus one day (which is Sunday). Hence *DayN* should be 0. According to the formula, when today is Saturday, *DayT = 6*, and so when $N = 15$,

$$\begin{aligned} DayN &= (DayT + N)\ mod\ 7 \\ &= (6 + 15)\ mod\ 7 \\ &= 21\ mod\ 7 = 0, \text{ which agrees.} \end{aligned}$$

13. *Solution 1:* $30 = 4\cdot 7 + 2$. Hence the answer is two days after Monday, which is Wednesday.
Solution 2: By the formula, the answer is $(1 + 30)\ mod\ 7 = 31\ mod\ 7 = 3$, which is Wednesday.

14. *Hint:* There are two ways to solve this problem. One is to find that $1{,}000 = 7\cdot 142 + 6$ and note that if today is Tuesday, then 1,000 days from today is 142 weeks plus 6 days from today. The other way is to use the formula $DayN = (DayT + N)\ mod\ 7$, with $DayT = 2$ (Tuesday) and $N = 1000$.

16. Because $d \mid n$, $n = dq + 0$ for some integer q. Thus the remainder is 0.

18. Proof: Suppose n is any odd integer. By definition of odd, $n = 2q + 1$ for some integer q. Then $n^2 = (2q+1)^2 = 4q^2 + 4q + 1 = 4(q^2 + q) + 1 = 4q(q+1) + 1$. By the result of exercise 17, the product $q(q+1)$ is even, so $q(q+1) = 2m$ for some integer m. Then, by substitution, $n^2 = 4\cdot 2m + 1 = 8m + 1$.

20. Because $a\ mod\ 7 = 4$, the remainder obtained when a is divided by 7 is 4, and so $a = 7q + 4$ for some integer q. Multiplying this equation through by 5 gives that $5a = 35q + 20 = 35q + 14 + 6 = 7(5q + 2) + 6$. Because q is an integer, $5q + 2$ is also an integer, and so $5a = 7\cdot(\text{an integer}) + 6$. Thus, because $0 \le 6 < 7$, the remainder obtained when $5a$ is divided by 7 is 6, and so $5a\ mod\ 7 = 6$.

23. Proof: Suppose n is any *[particular but arbitrarily chosen]* integer such that $n\ mod\ 5 = 3$. Then the remainder obtained when n is divided by 5 is 3, and so $n = 5q + 3$ for some integer q. By substitution,

$$\begin{aligned} n^2 &= (5q+3)^2 = 25q^2 + 30q + 9 \\ &= 25q^2 + 30q + 5 + 4 = 5(5q^2 + 6q + 1) + 4. \end{aligned}$$

Because products and sums of integers are integers, $5q^2 + 6q + 1$ is an integer, and hence $n^2 = 5\cdot(\text{an integer}) + 4$. Thus, since $0 \le 4 < 5$, the remainder obtained when n^2 is divided by 5 is 4, and so $n^2\ mod\ 5 = 4$.

26. *Hint:* You need to show that (1) for all nonnegative integers n and positive integers d, if n is divisible by d then $n\ mod\ d = 0$; and (2) for all nonnegative integers n and positive integers d, if $n\ mod\ d = 0$ then n is divisible by d.

27. Proof: Suppose n is any integer. By the quotient-remainder theorem with $d = 3$, there exist integers q and r such that $n = 3q + r$ and $0 \le r < 3$. But the only nonnegative integers r that are less than 3 are 0, 1, and 2. Therefore, $n = 3q + 0 = 3q$, or $n = 3q + 1$, or $n = 3q + 2$ for some integer q.

28. a. Proof: Suppose n, $n+1$, and $n+2$ are any three consecutive integers. *[We must show that $n(n+1)(n+2)$ is divisible by 3.]* By the quotient-remainder theorem, n can be written in one of the three forms, $3q$, $3q+1$, or $3q+2$ for some integer q. We divide into cases accordingly.

Case 1 ($n = 3q$ for some integer q): In this case,

$$\begin{aligned} &n(n+1)(n+2) \\ &\quad = 3q(3q+1)(3q+2) && \text{by substitution} \\ &\quad = 3\cdot[q(3q+1)(3q+2)] && \text{by factoring out a 3.} \end{aligned}$$

Let $m = q(3q+1)(3q+2)$. Then m is an integer because q is an integer, and sums and products of integers are integers. By substitution,

$$n(n+1)(n+2) = 3m \quad \text{where } m \text{ is an integer.}$$

And so, by definition of divisible, $n(n+1)(n+2)$ is divisible by 3.

Case 2 ($n = 3q+1$ for some integer q): In this case,

$$\begin{aligned} &n(n+1)(n+2) \\ &\quad = (3q+1)((3q+1)+1)((3q+1)+2) && \text{by substitution} \\ &\quad = (3q+1)(3q+2)(3q+3) \\ &\quad = (3q+1)(3q+2)3(q+1) \\ &\quad = 3\cdot[(3q+1)(3q+2)(q+1)] && \text{by algebra.} \end{aligned}$$

Let $m = (3q+1)(3q+2)(q+1)$. Then m is an integer because q is an integer, and sums and products of integers are integers. By substitution,

$$n(n+1)(n+2) = 3m \quad \text{where } m \text{ is an integer.}$$

And so, by definition of divisible, $n(n+1)(n+2)$ is divisible by 3.

Case 3 ($n = 3q+2$ for some integer q): In this case,

$$\begin{aligned} &n(n+1)(n+2) \\ &\quad = (3q+2)((3q+2)+1)((3q+2)+2) && \text{by substitution} \\ &\quad = (3q+2)(3q+3)(3q+4) \\ &\quad = (3q+2)3(q+1)(3q+4) \\ &\quad = 3\cdot[(3q+2)(q+1)(3q+4)] && \text{by algebra} \end{aligned}$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Let $m = (3q+2)(q+1)(3q+4)$. Then m is an integer because q is an integer, and sums and products of integers are integers. By substitution,

$$n(n+1)(n+2) = 3m \quad \text{where } m \text{ is an integer.}$$

And so, by definition of divisible, $n(n+1)(n+2)$ is divisible by 3.

In each of the three cases, $n(n+1)(n+2)$ was seen to be divisible by 3. But by the quotient-remainder theorem, one of these cases must occur. Therefore, the product of *any* three consecutive integers is divisible by 3.

b. For all integers n, $n(n+1)(n+2)\ mod\ 3 = 0$.

29. a. *Hint:* Given any integer n, begin by using the quotient-remainder theorem to say that n can be written in one of the three forms: $n = 3q$, or $n = 3q + 1$, or $n = 3q+2$ for some integer q. Then divide into three cases according to these three possibilities. Show that in each case either $n^2 = 3k$ for some integer k, or $n^2 = 3k+1$ for some integer k. For instance, when $n = 3q+2$, then $n^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1$, and $3q^2 + 4q + 1$ is an integer because it is a sum of products of integers.

31. b. If $m^2 - n^2 = 56$, then $56 = (m+n)(m-n)$. Now $56 = 2^3 \cdot 7$, and by the unique factorization theorem, this factorization is unique. Hence the only representations of 56 as a product of two positive integers are $56 = 7 \cdot 8 = 14 \cdot 4 = 28 \cdot 2 = 56 \cdot 1$. By part (a), m and n must both be odd or both be even. Thus the only solutions are either $m+n = 14$ and $m-n = 4$ or $m+n = 28$ and $m-n = 2$. This gives either $m = 9$ and $n = 5$ or $m = 15$ and $n = 13$ as the only solutions.

32. Under the given conditions, $2a - (b+c)$ is even.

Proof: Suppose a, b, and c are any integers such that $a-b$ is even and $b-c$ is even. *[We must show that* $2a-(b+c)$ *is even.]* Note first that $2a - (b+c) = (a-b) + (a-c)$. Also note that $(a-b) + (b-c)$ is a sum of two even integers and hence is even by Example 4.2.3 #1. But $(a-b) + (b-c) = a-c$, and so $a-c$ is even. Hence $2a - (b+c)$ is a sum of two even integers, and thus it is even *[as was to be shown]*.

34. *Hint:* Express n using the quotient-remainder theorem with $d = 3$.

36. *Hint:* Use the quotient-remainder theorem (as in Example 3.4.5) to say that $n = 4q$, $n = 4q+1$, $n = 4q+2$, or $n = 4q+3$ and divide into cases accordingly.

38. *Hint:* Given any integer n, consider the two cases where n is even and where n is odd.

39. *Hint:* Given any integer n, analyze the sum $n + (n+1) + (n+2) + (n+3)$.

42. *Hint:* Use the quotient-remainder theorem to say that n must have one of the forms $6q, 6q+1, 6q+2, 6q+3, 6q+4$, or $6q+5$ for some integer q.

44. *Hint:* There are three cases: Either x and y are both positive, or they are both negative, or one is positive and the other is negative.

47. a. $7609 + 5 = 7614$

49. *Answer to first question:* No. Counterexample: Let $m = 1$, $n = 3$, and $d = 2$. Then $m\ mod\ d = 1$ and $n\ mod\ d = 1$ but $m \neq n$.

Answer to second question: Yes. Proof: Suppose m, n, and d are integers such that $m\ mod\ d = n\ mod\ d$. Let $r = m\ mod\ d = n\ mod\ d$. By definition of *mod*, $m = dp + r$ and $n = dq + r$ for some integers p and q. Then $m - n = (dp + r) - (dq + r) = d(p-q)$. But $p - q$ is an integer (being a difference of integers), and so $m - n$ is divisible by d by definition of divisible.

Section 4.5

1. $\lfloor 37.999 \rfloor = 37$, $\lceil 37.999 \rceil = 38$

3. $\lfloor -14.00001 \rfloor = -15$, $\lceil -14.00001 \rceil = -14$

8. $\lfloor n/7 \rfloor$. The floor notation is more appropriate. If the ceiling notation is used, two different formulas are needed, depending on whether $n/7$ is an integer or not. (What are they?)

10. a. (i) $\left(2050 + \left\lfloor \frac{2049}{4} \right\rfloor - \left\lfloor \frac{2049}{100} \right\rfloor + \left\lfloor \frac{2049}{400} \right\rfloor\right) mod\ 7$

$$= (2050 + 512 - 20 + 5)\ mod\ 7 = 2547\ mod\ 7$$
$$= 6, \text{ which corresponds to a Saturday}$$

b. *Hint:* One day is added every four years, except that each century the day is not added unless the century is a multiple of 400.

12. Proof: Suppose n is any even integer. By definition of even, $n = 2k$ for some integer k. Then

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = k \qquad \text{because } k \text{ is an integer and } k \le k < k - 1.$$

But $$k = \frac{n}{2} \qquad \text{because } n = 2k.$$

Thus, on the one hand, $\left\lfloor \frac{n}{2} \right\rfloor = k$, and on the other hand, $k = \frac{n}{2}$. It follows that $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$ *[as was to be shown]*.

14. False. Counterexample: Let $x = 2$ and $y = 1.9$. Then $\lfloor x - y \rfloor = \lfloor 2 - 1.9 \rfloor = \lfloor 0.1 \rfloor = 0$, whereas $\lfloor x \rfloor - \lfloor y \rfloor = \lfloor 2 \rfloor - \lfloor 1.9 \rfloor = 2 = 1 = 1$.

15. True. Proof: Suppose x is any real number. Let $m = \lfloor x \rfloor$. By definition of floor, $m \le x < m + 1$. Subtracting 1 from all parts of the inequality gives that

$$m - 1 \le x - 1 < m,$$

and so, by definition of floor, $\lfloor x - 1 \rfloor = m - 1$. It follows by substitution that $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$.

17. *Proof for the case where n mod* $3 = 2$*:*

In the case where $n\ mod\ 3 = 2$, then $n = 3q + 2$ for some integer q by definition of *mod*. By substitution,

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$\left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{3q+2}{3} \right\rfloor$$
$$= \left\lfloor \frac{3q}{3} + \frac{2}{3} \right\rfloor$$
$$= \left\lfloor q + \frac{2}{3} \right\rfloor = q \quad \text{because } q \text{ is an integer and } q \le q + 2/3 < q+1.$$

But

$$q = \frac{n-2}{3} \quad \text{by solving } n = 3q+2 \text{ for } q.$$

Thus, on the one hand, $\left\lfloor \frac{n}{3} \right\rfloor = q$, and on the other hand, $q = \frac{n-2}{3}$. It follows that $\left\lfloor \frac{n}{3} \right\rfloor = \frac{n-2}{3}$.

18. *Hint:* This is false.

19. *Hint:* This is true.

23. Proof: Suppose x is a real number that is not an integer. Let $\lfloor x \rfloor = n$. Then, by definition of floor and because n is not an integer, $n < x < n+1$. Multiplying both sides by -1 gives $-n > -x > -n-1$, or equivalently, $-n-1 < -x < -n$. Since $-n-1$ is an integer, it follows by definition of floor that $\lfloor -x \rfloor = -n-1$. Hence

$$\lfloor x \rfloor + \lfloor -x \rfloor = n + (-n-1) = n - n - 1 = -1,$$

as was to be shown.

25. *Hint:* Let $n = \left\lfloor \frac{x}{2} \right\rfloor$ and consider the two cases: n is even and n odd.

26. Proof: Suppose x is any real number such that $x - \lfloor x \rfloor < \frac{1}{2}$. Multiplying both sides by 2 gives

$$2x - 2\lfloor x \rfloor < 1, \text{ or } 2x < 2\lfloor x \rfloor + 1.$$

Now by definition of floor, $\lfloor x \rfloor \le x$. Hence, $2\lfloor x \rfloor \le 2x$. Putting the two inequalities involving $2x$ together gives

$$2\lfloor x \rfloor \le 2x < 2\lfloor x \rfloor + 1.$$

Thus, by definition of floor (and because $2\lfloor x \rfloor$ is an integer), $\lfloor 2x \rfloor = 2\lfloor x \rfloor$. This is what was to be shown.

30. This incorrect proof exhibits circular reasoning. The equality $\left\lfloor \frac{n}{2} \right\rfloor = \frac{(n-1)}{2}$ is what is to be shown. By substituting $2k+1$ for n into both sides of the equality and working from the result as though it were known to be true, the proof assumes the truth of the conclusion to be proved.

Section 4.6

1. **(a)** A contradiction
(b) A positive real number
(c) x
(d) Both sides by 2
(e) Contradiction

3. Proof: Suppose not. That is, suppose there is an integer n such that $3n+2$ is divisible by 3. *[We must derive a contradiction.]* By definition of divisibility, $3n+2 = 3k$ for some integer k. Subtracting $3n$ from both sides gives that $2 = 3k - 3n = 3(k-n)$. So, by definition of divisibility, $3 \mid 2$. But by Theorem 4.3.1 this implies that $3 \le 2$, which contradicts the fact that $3 > 2$. *[Thus for all integers n, $3n+2$ is not divisible by 3.]*

5. *Negation of statement*: There is a greatest even integer.
Proof of statement: Suppose not. That is, suppose there is a greatest even integer; call it N. Then N is an even integer, and $N \ge n$ for every even integer n. *[We must deduce a contradiction.]* Let $M = N + 2$. Then M is an even integer since it is a sum of even integers, and $M > N$ since $M = N + 2$. This contradicts the supposition that $N \ge n$ for *every* even integer n. *[Hence the supposition is false and the statement is true.]*

8. **(a)** a rational number
(b) an irrational number
(c) $\frac{a}{b}$
(d) $\frac{c}{d}$
(e) $\frac{a}{b} - \frac{c}{d}$
(f) integers
(g) integers
(h) zero product property
(i) rational

9. **a.** The mistake in this proof occurs in the second sentence where the negation written by the student is incorrect: Instead of being existential, it is universal. The problem is that if the student proceeds in a logically correct manner, all that is needed to reach a contradiction is one example of a rational and an irrational number whose sum is irrational. To prove the given statement, however, it is necessary to show that there is *no* rational number and *no* irrational number whose sum is rational.

10. Proof by contradiction: Suppose not. That is, suppose there is an irrational number x such that the square root of x is rational. *[We must derive a contradiction.]* By definition of rational, $\sqrt{x} = \frac{a}{b}$ for some integers a and b with $b \ne 0$. By substitution,

$$(\sqrt{x})^2 = \left(\frac{a}{b}\right)^2,$$

and so, by algebra,

$$x = \frac{a^2}{b^2}.$$

But a^2 and b^2 are both products of integers and thus are integers, and b^2 is nonzero by the zero product property. Thus $\frac{a^2}{b^2}$ is rational. It follows that x is both irrational and rational, which is a contradiction. *[This is what was to be shown.]*

11. Proof: Suppose not. That is, suppose $\exists$ a nonzero rational number x and an irrational number y such that xy is rational. *[We must derive a contradiction.]* By definition of rational, $x = a/b$ and $xy = c/d$ for some integers $a, b, c,$ and d with $b \ne 0$ and $d \ne 0$. Also $a \ne 0$ because x is nonzero. By substitution, $xy = (a/b)y = c/d$. Solving for y gives $y = bc/ad$. Now bc and ad are integers (being products of integers) and $ad \ne 0$ (by the zero product property). Thus,

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

by definition of rational, y is rational, which contradicts the supposition that y is irrational. *[Hence the supposition is false and the statement is true.]*

13. *Hint:* Suppose $n^2 - 2$ is divisible by 4, and consider the two cases where n is even and n is odd. (An alternative solution uses Proposition 4.6.4.)

14. *Hint:* $a^2 = c^2 - b^2 = (c - b)(c + b)$

15. *Hint:* (1) For any integer c, if 2 divides c, then 4 divides c^2. (2) The result of exercise 13 may be helpful.

16. *Hint:* Suppose a, b, and c are odd integers, z is a solution to $ax^2 + bx + c = 0$, and z is rational. Then $z = p/q$ for some integers p and q with $q \neq 0$. We may assume p and q have no common factor. (Why? If p and q do have a common factor, we can divide out their greatest common factor to obtain two integers p' and q' that (1) have no common factor and (2) satisfy the equation $z = p'/q'$. Then we can redefine $q = q'$ and $p = p'$.) Note that because p and q have no common factor, they are not both even. Substitute p/q into $ax^2 + bx + c = 0$, and multiply through by q^2. Show that (1) the assumption that p is even leads to a contradiction, (2) the assumption that q is even leads to a contradiction, and (3) the assumption that both p and q are odd leads to a contradiction. The only remaining possibility is that both p and q are even, which has been ruled out.

18. **a.** $5 \mid n$ **b.** $5 \mid n^2$ **c.** $5k$ **d.** $(5k)^2$ **e.** $5 \mid n^2$

19. Proof (by contraposition): *[To go by contraposition, we must prove that ∀ positive real numbers, r and s, if $r \leq 10$ and $s \leq 10$, then $rs \leq 100$.]* Suppose r and s are positive real numbers and $r \leq 10$ and $s \leq 10$. By the algebra of inequalities, $rs \leq 100$. *[To derive this fact, multiply both sides of $r \leq 10$ by s to obtatin $rs \leq 10s$. And multiply both sides of $s \leq 10$ by 10 to obtain $10s \leq 10 \cdot 10 = 100$. By transitivity of $\leq$, then, $rs \leq 100$.]* But this is what was to be shown.

21. **a.** Proof by contradiction: Suppose not. That is, suppose there is an integer n such that n^2 is odd and n is even. Show that this supposition leads logically to a contradiction.

b. Proof by contraposition: Suppose n is any integer such that n is not odd. Show that n^2 is not odd.

23. **a.** The contrapositive is the statement "∀ real numbers x, if $-x$ is not irrational, then x is not irrational." Equivalently (because $-(-x) = x$), "∀ real numbers x, if x is rational then $-x$ is rational."

Proof by contraposition: Suppose x is any rational number. *[We must show that $-x$ is also rational.]* By definition of rational, $x = a/b$ for some integers a and b with $b \neq 0$. Then $x = -(a/b) = (-a)/b$. Since both $-a$ and b are integers and $b \neq 0$, $-x$ is rational *[as was to be shown]*.

b. Proof by contradiction: Suppose not. *[We take the negation and suppose it to be true.]* That is, suppose ∃ an irrational number x such that $-x$ is rational. *[We must derive a contradiction.]* By definition of rational, $-x = a/b$ for some integers a and b with $b \neq 0$. Multiplying both sides by -1 gives $x = -(a/b) = -a/b$. But $-a$ and b are integers (since a and b are) and $b \neq 0$. Thus x is a ratio of the two integers $-a$ and b with $b \neq 0$. Hence x is rational (by definition of rational), which is a contradiction. *[This contradiction shows that the supposition is false, and so the given statement is true.]*

25. *Hints:* See the answer to exercise 21 and look carefully at the two proofs for Proposition 4.6.4.

26. **a.** Proof by contraposition: Suppose a, b, and c are any *[particular but arbitrarily chosen]* integers such that $a \mid b$. *[We must show that $a \mid bc$.]* By definition of divides, $b = ak$ for some integer k. Then $bc = (ak)c = a(kc)$. But kc is an integer (because it is a product of the integers k and c). Hence $a \mid bc$ by definition of divisibility *[as was to be shown]*.

b. Proof by contradiction: Suppose not. *[We take the negation and suppose it to be true.]* Suppose ∃ integers a, b, and c such that $a \nmid bc$ and $a \mid b$. Since $a \mid b$, there exists an integer k such that $b = ak$ by definition of divides. Then $bc = (ak)c = a(kc)$ *[by the associative law of algebra]*. But kc is an integer (being a product of integers), and so $a \mid bc$ by definition of divides. Thus $a \nmid bc$ and $a \mid bc$, which is a contradiction. *[This contradiction shows that the supposition is false, and hence the given statement is true.]*

27. **a.** *Hint:* The contrapositive is "For all integers m and n, if m and n are not both even and m and n are not both odd, then $m + n$ is not even." Equivalently: "For all integers m and n, if one of m and n is even and the other is odd, then $m + n$ is odd."

b. *Hint:* The negation of the given statement is the following: ∃ integers m and n such that $m + n$ is even, and either m is even and n is odd, or m is odd and n is even.

30. The negation of "Every integer is rational" is "There is at least one integer that is irrational" not "Every integer is irrational." Deriving a contradiction from an incorrect negation of a statement does not prove the statement is true.

31. **a.** Proof: Suppose r, s, and n are integers and $r > \sqrt{n}$ and $s > \sqrt{n}$. Note that r and s are both positive because $\sqrt{n}$ cannot be negative. By multiplying both sides of the first inequality by s and both sides of the second inequality by $\sqrt{n}$ (Appendix A, T20), we have that $rs > \sqrt{ns}$ and $\sqrt{ns} > \sqrt{n}\sqrt{n} = n$. Thus, by the transitive law for inequality (Appendix A, T18), $rs > n$.

32. **a.** $\sqrt{667} \cong 25.8$, and so the possible prime factors to be checked are 2, 3, 5, 7, 11, 13, 17, 19, and 23. Testing each in turn shows that 667 is not prime because $667 = 23 \cdot 29$.

b. $\sqrt{557} \cong 23.6$, and so the possible prime factors to be checked are 2, 3, 5, 7, 11, 13, 17, 19, and 23. Testing each in turn shows that none divides 557. Therefore, 557 is prime.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

34. a. $\sqrt{9269} \cong 96.3$, and so the possible prime factors to be checked are all among those you found for exercise 33. Testing each in turn shows that 9,269 is not prime because $9{,}269 = 13 \cdot 713$.

b. $\sqrt{9103} \cong 95.4$, and so the possible prime factors to be checked are all among those you found for exercise 33. Testing each in turn shows that none divides 9,103. Therefore, 9,103 is prime.

35. *Hint:* Is it possible for all three of $n-4$, $n-6$, and $n-8$ to be prime?

Section 4.7

1. The value of $\sqrt{2}$ given by a calculator is an approximation. Calculators can give exact values only for numbers that can be represented using at most the number of decimal digits in the calculator display. In particular, every number in a calculator display is rational, but even many rational numbers cannot be represented exactly. For instance, consider the number formed by writing a decimal point and following it with the first 1 million digits of $\sqrt{2}$. By the discussion in Section 4.2, this number is rational, but you could not infer this from the calculator display.

3. Proof by contradiction: Suppose not. That is, suppose $6 - 7\sqrt{2}$ is rational. *[We must prove a contradiction.]* By definition of rational, there exist integers a and $b \neq 0$ with

$$6 - 7\sqrt{2} = \frac{a}{b}.$$

Then $\sqrt{2} = \dfrac{1}{-7}\left(\dfrac{a}{b} - 6\right)$ by subtracting 6 from both sides and dividing both sides by -7

and so $\sqrt{2} = \dfrac{a-6b}{-7b}$ by the rules of algebra.

But $a - 6b$ and $-7b$ are both integers (since a and b are integers and products and difference of integers are integers), and $-7b \neq 0$ by the zero product property. Hence $\sqrt{2}$ is a ratio of the two integers $a-6b$ and $-7b$ with $-7b \neq 0$, so $\sqrt{2}$ is a rational number (by definition of rational). This contradicts the fact that $\sqrt{2}$ is irrational, and so the supposition is false and $6 - 7\sqrt{2}$ is irrational.

5. This is false. $\sqrt{4} = 2 = 2/1$, which is rational.

7. Counterexample: Let $x = \sqrt{2}$ and let $y = -\sqrt{2}$. Then x and y are irrational, but $x + y = 0 = 0/1$, which is rational.

9. True.

Formal version of the statement: $\forall$ positive real numbers r, if r is irrational, then $\sqrt{r}$ is irrational.

Proof by contraposition: Suppose r is any positive real number such that $\sqrt{r}$ is rational. *[We must show that r is rational.]* By definition of rational, $\sqrt{r} = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Then $r = \left(\sqrt{r}\right)^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. But both a^2 and b^2 are integers because they are products of integers, and $b^2 \neq 0$ by the zero product property. Thus r is rational *[as was to be shown]*.

(The statement may also be proved by contradiction.)

13. *Hint:* Can you think of any "nice" integers x and y that are greater than 1 and have the property that $x^2 = y^3$?

16. a. Proof by contradiction: Suppose not. That is, suppose there is an integer n such that $n = 3q_1 + r_1 = 3q_2 + r_2$, where q_1, q_2, r_1, and r_2 are integers, $0 \le r_1 < 3$, $0 \le r_2 < 3$, and $r_1 \neq r_2$. By interchanging the labels for r_1 and r_2 if necessary, we may assume that $r_2 > r_1$. Then $3(q_1 - q_2) = r_2 - r_1 > 0$, and because both r_1 and r_2 are less than 3, either $r_2 - r_1 = 1$ or $r_2 - r_1 = 2$. So either $3(q_1 - q_2) = 1$ or $3(q_1 - q_2) = 2$. The first case implies that $3 \mid 1$, and hence, by Theorem 4.3.1, that $3 \le 1$, and the second case implies that $3 \mid 2$, and hence, by Theorem 4.3.1, that $3 \le 2$. These results contradict the fact that 3 is greater than both 1 and 2. Thus in either case we have reached a contradiction, which shows that the supposition is false and the given statement is true.

b. Proof by contradiction: Suppose not. That is, suppose there is an integer n such that n^2 is divisible by 3 and n is not divisible by 3. *[We must deduce a contradiction.]* By definition of divisible, $n^2 = 3q$ for some integer q, and by the quotient-remainder theorem and part (a), $n = 3k+1$ or $n = 3k+2$ some integer k.

Case 1 (n = 3k + 1 for some integer k): In this case

$$n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

Let $s = 3k^2 + 2k$. Then $n^2 = 3s + 1$, and s is an integer because it is a sum of products of integers. It follows that $n^2 = 3q = 3s + 1$ for some integers q and s, which contradicts the result of part (a).

Case 2 (n = 3k + 2 for some integer k): In this case

$$n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 6k + 1) + 1.$$

Let $t = 3k^2 + 6k + 1$. Then $n^2 = 3t + 1$, and t is an integer because it is a sum of products of integers. It follows that $n^2 = 3q = 3t + 1$ for some integers q and t, which contradicts the result of part (a).

Thus in either case, a contradiction is reached, which shows that the supposition is false and the given statement is true.

c. Proof by contradiction: Suppose not. That is, suppose $\sqrt{3}$ is rational. By definition of rational, $\sqrt{3} = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Without loss of generality, assume that a and b have no common factor. (If not, divide both a and b by their greatest common factor to obtain integers a' and b' with the property that a' and b' have no common factor and $\sqrt{3} = \frac{a'}{b'}$. Then redefine $a = a'$ and $b = b'$.) Squaring both sides of $\sqrt{3} = \frac{a}{b}$ gives $3 = \frac{a^2}{b^2}$, and multiplying both sides by b^2 gives

$$3b^2 = a^2 (*).$$

Thus a^2 is divisible by 3, and so, by part (b), a is also divisible by 3. By definition of divisibility, then, $a = 3k$ for some integer k, and so

$$a^2 = 9k^2 (**).$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Substituting equation (**) into equation (*) gives $3b^2 = 9k^2$, and dividing both sides by 3 yields

$$b^2 = 3k^2.$$

Hence b^2 is divisible by 3, and so, by part (b), b is also divisible by 3. Consequently, both a and b are divisible by 3, which contradicts the assumption that a and b have no common factor. Thus the supposition is false, and so $\sqrt{3}$ is irrational.

18. *Hint:* The proof is a generalization of the one given in the solution for exercise 16(a).

19. *Hint:* (1) The parts of the proof are similar to those in exercise 16(b) and 16(c). (2) Use the result of exercise 18.

20. *Hint:* This statement is true. If $a^2 - 3 = 9b$, then $a^2 = 9b + 3 = 3(3b + 1)$, and so a^2 is divisible by 3. Hence, by exercise 16(b), a is divisible by 3. Thus $a^2 = (3c)^2$ for some integer c.

21. Proof by contradiction: Suppose not. That is, suppose $\sqrt{2}$ is rational. *[We will show that this supposition leads to a contradiction.]* By definition of rational, we may write $\sqrt{2} = a/b$ for some integers a and b with $b \neq 0$. Then $2 = a^2/b^2$, and so $a^2 = 2b^2$. Consider the prime factorizations for a^2 and for $2b^2$. By the unique factorization of integers theorem, these factorizations are unique except for the order in which the factors are written. Now because every prime factor of a occurs twice in the prime factorization of a^2, the prime factorization of a^2 contains an even number of 2's. (If 2 is a factor of a, then this even number is positive, and if 2 is not a factor of a, then this even number is 0.) On the other hand, because every prime factor of b occurs twice in the prime factorization of b^2, the prime factorization of $2b^2$ contains an odd number of 2's. Therefore, the equation $a^2 = 2b^2$ cannot be true. So the supposition is false, and hence $\sqrt{2}$ is irrational.

23. *Hint:* By the result of exercise 22, $\sqrt{6}$ is irrational.

25. *Hint:* $\dfrac{2\cdot 3\cdot 5\cdot 7 + 1}{2} = 3\cdot 5\cdot 7 + \dfrac{1}{2}$ and $\dfrac{2\cdot 3\cdot 5\cdot 7 + 1}{3} = 2\cdot 5\cdot 7 + \dfrac{1}{3}$.

26. *Hint:* You can deduce that $p = 3$.

27. a. *Hint:* For example, $N_4 = 2\cdot 3\cdot 5\cdot 7 + 1 = 211$.

29. *Hint:* By Theorem 4.3.4 (divisibility by a prime) there is a prime number p such that $p \mid (n! - 1)$. Show that the supposition that $p \leq n$ leads to a contradiction. It will then follow that $n < p < n!$.

30. *Hint:* Every odd integer can be written as $4k + 1$ or as $4k + 3$ for some integer k. (Why?) If $p_1 p_2 \ldots p_n + 1 = 4k + 1$, then $4 \mid p_1 p_2 \ldots p_n$. Is this possible?

31. a. *Hint:* Prove the contrapositive: If for some integer $n > 2$ that is not a power of 2, $x^n + y^n = z^n$ has a positive integer solution, then for some prime number $p > 2$, $x^p + y^p = z^p$ has a positive integer solution. Note that if $n = kp$, then $x^n = x^{kp} = (x^k)^p$.

32. Existence proof: When $n = 2$, then $n^2 - 1 = 3$, which is prime. Hence there exists a prime number of the form $n^2 - 1$, where n is an integer and $n \geq 2$.

Uniqueness proof (by contradiction): Suppose to the contrary that m is another integer satisfying the given conditions. That is, $m > 2$ and $m^2 - 1$ is prime. *[We must derive a contradiction.]* Factor $m^2 - 1$ to obtain $m^2 - 1 = (m - 1)(m + 1))$. But $m > 2$, and so $m - 1 > 1$ and $m + 1 > 1$. Hence $m^2 - 1$ is not prime, which is a contradiction. *[This contradiction shows that the supposition is false, and so there is no other integer $m > 2$ such that $n^2 - 1$ is prime.]*

Uniqueness proof (direct): Suppose m is any integer such that $m \geq 2$ and $m^2 - 1$ is prime. *[We must show that $m = 2$.]* By factoring, $m^2 - 1 = (m - 1)(m + 1)$. Since $m^2 - 1$ is prime, either $m - 1 = 1$ or $m + 1 = 1$. But $m + 1 \geq 2 + 1 = 3$. Hence, by elimination, $m - 1 = 1$, and so $m = 2$.

34. Proof (by contradiction): Suppose not. That is, suppose there are two distinct real numbers a_1 and a_2 such that for all real numbers r,

$$(1)\ a_1 + r = r \quad \text{and} \quad (2)\ a_2 + r = r$$

Then

$$a_1 + a_2 = a_2 \quad \text{by (1) with} \quad r = a_2$$

and

$$a_2 + a_1 = a_1 \quad \text{by (2) with} \quad r = a_1.$$

It follows that

$$a_2 = a_1 + a_2 = a_2 + a_1 = a_1$$

which implies that $a_2 = a_1$. But this contradicts the supposition that a_1 and a_2 are distinct. *[Thus the supposition is false and there is at most one real number a such that $a + r = r$ for all real numbers r.]*

Proof (direct): Suppose a_1 and a_2 are real numbers such that for all real numbers r,

$$(1)\ a_1 + r = r \quad \text{and} \quad (2)\ a_2 + r = r$$

Then

$$a_1 + a_2 = a_2 \quad \text{by (1) with} \quad r = a_2$$

and

$$a_2 + a_1 = a_1 \quad \text{by (2) with} \quad r = a_1.$$

It follows that

$$a_2 = a_1 + a_2 = a_2 + a_1 = a_1.$$

Hence $a_2 = a_1$. *[Thus there is at most one real number a such that $a + r = r$ for all real numbers r.]*

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 4.8

1. $z = 0$ **3. a.** $z = 18$ **4.** $a = \frac{17}{12}$

6.

Iteration Number

	0	1	2	3
a	26			
d	7			
q	0	1	2	3
r	26	19	12	5

8. a.

A	69	19	9	
q	2			
d		1		
n			1	
p				4

9. $\gcd(27, 72) = 9$ **10.** $\gcd(5, 9) = 1$

13. Divide the larger number, 1,188, by the smaller, 385, to obtain a quotient of 3 and a remainder of 33. Next divide 385 by 33 to obtain a quotient of 11 and a remainder of 22. Then divide 33 by 22 to obtain a quotient of 1 and a remainder of 11. Finally, divide 22 by 11 to obtain a quotient of 2 and a remainder of 0. Thus, by Lemma 4.8.2, $\gcd(1188, 385) = \gcd(385, 33) = \gcd(33, 22) = \gcd(22, 11) = \gcd(11, 0)$, and by Lemma 4.8.1, $\gcd(11, 0) = 11$. So gcd $(1188, 385) = 11$.

14. Divide the larger number, 1,177, by the smaller, 509, to obtain a quotient of 2 and a remainder of 159. Next divide 509 by 159 to obtain a quotient of 3 and a remainder of 32. Next divide 159 by 32 to obtain a quotient of 4 and a remainder of 31. Then divide 32 by 31 to obtain a quotient of 1 and a remainder of 1. Finally, divide 31 by 1 to obtain a quotient of 31 and a remainder of 0. Thus, by Lemma 4.8.2, $\gcd(1177, 509) = \gcd(509, 159) = \gcd(159, 32) = \gcd(32, 31) = \gcd(31, 1) = \gcd(1, 0)$, and by Lemma 4.8.1, $\gcd(1, 0) = 1$. So $\gcd(1177, 509) = 1$.

17.

A	1,001					
B	871					
r		130	91	39	13	0
b	871	130	91	39	13	0
a	1,001	871	130	91	39	13
gcd						13

19. *Hint:* Divide the proof into two parts. In part 1 suppose a and b are any positive integers such that $a \mid b$, and derive the conclusion that $\gcd(a, b) = a$. To do this, note that because it is also the case that $a \mid a$, a is a common divisor of a and b. Thus, by definition of greatest common divisor, a is less than or equal to the greatest common divisor of a and b. In symbols, $a \le \gcd(a, b)$. Then show that $a \ge \gcd(a, b)$ by using Theorem 4.3.1. In part 2 of the proof, suppose a and b are any positive integers such that $\gcd(a, b) = a$, and deduce that $a \mid b$.

22. a. *Hint 1:* If $a = dq - r$, then $-a = -dq + r = -dq - d + d - r = d(-q - 1) + (d - r)$.

Hint 2: If $0 \le r < d$, then $0 \ge -r > -d$. Add d to all parts of this inequality and see what results.

23. a. Proof: Suppose a, d, q, and r are integers such that $a = dq + r$ and $0 \le r < d$. *[We must show that* $q = \left\lfloor \frac{a}{d} \right\rfloor$ *and* $r = a - d\left\lfloor \frac{a}{d} \right\rfloor$*.]* Solving $a = dq + r$ for r gives $r = a - dq$, and substituting into $0 \le r < d$ gives $0 \le a - dq < d$. Add dq to both sides to obtain $dq \le a < d + dq = d(q + 1)$. Then divide through by d to obtain $q \le \frac{a}{d} < q + 1$. Therefore, by definition of floor, $\left\lfloor \frac{a}{d} \right\rfloor = q$. Finally, substitution into $a = dq + r$ gives $a = d\left\lfloor \frac{a}{d} \right\rfloor + r$, and subtracting $d\left\lfloor \frac{a}{d} \right\rfloor$ from both sides yields $r = a - d\left\lfloor \frac{a}{d} \right\rfloor$ *[as was to be shown].*

24. b.

Iteration Number

	0	1	2	3	4
a	630	294	294	252	210
b	336	336	42	42	42
gcd					

Iteration Number

	5	6	7	8	9
a	168	126	84	42	0
b	42	42	42	42	42
gcd					42

25. a. $\operatorname{lcm}(12, 18) = 36$

26. Proof: *Part 1:* Let a and b be positive integers, and suppose $d = \gcd(a, b) = \operatorname{lcm}(a, b)$. By definition of greatest common divisor and least common multiple, $d > 0$, $d \mid a$, $d \mid b$, $a \mid d$, and $b \mid d$. Thus, in particular, $a = dm$ and $d = an$ for some integers m and n. By substitution, $a = dm = (an)m = anm$. Dividing both sides by a gives $1 = nm$. But the only divisors of 1 are 1 and -1 (Theorem 4.3.2), and so $m = n = \pm 1$. Since both a and d are positive, $m = n = 1$, and hence $a = d$. Similar reasoning shows that $b = d$ also, and so $a = b$.

Part 2: Given any positive integers a and b such that $a = b$, we have $\gcd(a, b) = \gcd(a, a) = a$ and $\operatorname{lcm}(a, b) = \operatorname{lcm}(a, a) = a$, and hence $\gcd(a, b) = \operatorname{lcm}(a, b)$.

29. *Hint:* Divide the proof into two parts. In part 1, suppose a and b are any positive integers, and deduce that

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) \le ab.$$

Derive this result by showing that lcm $(a, b) \le \frac{ab}{\gcd(a,b)}$. To do this, show that $\frac{ab}{\gcd(a,b)}$ is a multiple of both

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

a and b. For instance, to see that $\frac{ab}{\gcd(a,b)}$ is a multiple of b, note that because $\gcd(a, b)$ divides a, $a = \gcd(a, b) \cdot k$ for some integer k, and thus $ab = \gcd(a, b) \cdot kb$. Divide both sides by $\gcd(a, b)$ to obtain $\frac{ab}{\gcd(a,b)} = kb$. But since k is an integer, this equation implies that $\frac{ab}{\gcd(a,b)}$ is a multiple of b. The argument that $\frac{ab}{\gcd(a,b)}$ is a multiple of a is almost identical. In part 2 of the proof, use the definition of least common multiple to show that $\frac{ab}{\text{lcm}(a,b)} \mid a$ and $\frac{ab}{\text{lcm}(a,b)} \mid b$. Conclude that $\frac{ab}{\text{lcm}(a,b)} \le \gcd(a, b)$ and hence that $ab \le \gcd(a, b) \cdot \text{lcm}(a, b)$.

Section 5.1

1. $\frac{1}{11}, \frac{2}{12}, \frac{3}{13}, \frac{4}{14}$ **3.** $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}$ **5.** $0, 0, 2, 2$

8. $g_1 = \lfloor \log_2 1 \rfloor = 0$

$g_2 = \lfloor \log_2 2 \rfloor = 1, \quad g_3 = \lfloor \log_2 3 \rfloor = 1$
$g_4 = \lfloor \log_2 4 \rfloor = 2, \quad g_5 = \lfloor \log_2 5 \rfloor = 2$
$g_6 = \lfloor \log_2 6 \rfloor = 2, \quad g_7 = \lfloor \log_2 7 \rfloor = 2$
$g_8 = \lfloor \log_2 8 \rfloor = 3, \quad g_9 = \lfloor \log_2 9 \rfloor = 3$
$g_{10} = \lfloor \log_2 10 \rfloor = 3, \quad g_{11} = \lfloor \log_2 11 \rfloor = 3$
$g_{12} = \lfloor \log_2 12 \rfloor = 3, \quad g_{13} = \lfloor \log_2 13 \rfloor = 3$
$g_{14} = \lfloor \log_2 14 \rfloor = 3, \quad g_{15} = \lfloor \log_2 15 \rfloor = 3$

When n is an integral power of 2, g_n is the exponent of that power. For instance, $8 = 2^3$ and $g_8 = 3$. More generally, if $n = 2^k$, where k is an integer, then $g_n = k$. All terms of the sequence from g_n up to g_m, where $m = 2^{k+1}$ is the next integral power of 2, have the same value as g_n, namely k. For instance, all terms of the sequence from g_8 through g_{15} have the value 3.

Exercises 10–16 have more than one correct answer.

10. $a_n = (-1)^n$, where n is an integer and $n \ge 1$.

11. $a_n = (n-1)(-1)^n$, where n is an integer and $n \ge 1$.

12. $a_n = \frac{n}{(n+1)^2}$, where n is an integer and $n \ge 1$

14. $a_n = \frac{n^2}{3^n}$, where n is an integer and $n \ge 1$

18. **a.** $2 + 3 + (-2) + 1 + 0 + (-1) + (-2) = 1$
b. $a_0 = 2$
c. $a_2 + a_4 + a_6 = -2 + 0 + (-2) = -4$
d. $2 \cdot 3 \cdot (-2) \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$

19. $2 + 3 + 4 + 5 + 6 = 20$ **20.** $2^2 \cdot 3^2 \cdot 4^2 = 576$

23. $1(1 + 1) = 2$

27.
$$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right)$$
$$+ \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{8}} - \cancel{\frac{1}{9}}\right)$$
$$+ \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \frac{1}{11}\right) = 1 - \frac{1}{11} = \frac{10}{11}$$

29. $(-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^n$
$= -2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n$

31. $\sum_{k=0}^{n+1} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n+1)!}$

33. $\frac{1}{1!} = 1$

35. $\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right) = \left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{4}\right) = \frac{1}{4}$

37. $\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)!$

40. $\sum_{i=1}^{k} i^3 + (k+1)^3 = \sum_{i=1}^{k+1} i^3$

Exercises 43–52 have more than one correct answer.

43. $\sum_{k=1}^{7} (-1)^{k+1} k^2$ or $\sum_{k=0}^{6} (-1)^k (k+1)^2$

46. $\sum_{j=2}^{6} \frac{(-1)^j j}{(j+1)(j+2)}$ or $\sum_{k=3}^{7} \frac{(-1)^{k+1}(k-1)}{k(k+1)}$

47. $\sum_{i=0}^{5} (-1)^i r^i$ **49.** $\sum_{k=1}^{n} k^3$

51. $\sum_{i=0}^{n-1} (n - i)$

53. When $k = 0$, then $i = 1$. When $k = 5$, then $i = 6$. Since $i = k + 1$, then $k = i - 1$. Thus,
$$k(k-1) = (i-1)((i-1)-1) = (i-1)(i-2),$$
and so
$$\sum_{k=0}^{5} k(k-1) = \sum_{i=1}^{6} (i-1)(i-2)$$

55. When $i = 1$, then $j = 0$. When $i = n + 1$, then $j = n$. Since $j = i - 1$, then $i = j + 1$. Thus,
$$\frac{(i-1)^2}{i \cdot n} = \frac{((j+1)-1)^2}{(j+1) \cdot n} = \frac{j^2}{jn + n}.$$
(Note that n is constant as far as the sum is concerned.)
So $\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} = \sum_{j=0}^{n} \frac{j^2}{jn + n}$.

56. When $i = 3$, then $j = 2$. When $i = n$ then $j = n - 1$. Since $j = i - 1$, then $i = j + 1$. Thus,
$$\sum_{i=3}^{n} \frac{i}{i + n - 1} = \sum_{j=2}^{n-1} \frac{j+1}{(j+1) + n - 1}$$
$$= \sum_{j=2}^{n-1} \frac{j+1}{j + n}.$$

59. $\sum_{k=1}^{n} [3(2k-3) + (4-5k)]$
$$= \sum_{k=1}^{n} [(6k - 9) + (4 - 5k)] = \sum_{k=1}^{n} (k - 5)$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

62. $\dfrac{4\cdot 3\cdot \cancel{2\cdot 1}}{3\cdot \cancel{2\cdot 1}} = 4$

65. $\dfrac{n\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}} = n$

66. $\dfrac{\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{(n+1)n\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}} = \dfrac{1}{n(n+1)}$

68. $\dfrac{[(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1]^2}{[n(n-1)(n-2)\cdots 3\cdot 2\cdot 1]^2} = (n+1)^2$

69.

$$\frac{n(n-1)(n-2)\cdots(n-k+1)\cancel{(n-k)(n-k-1)\cdots 2\cdot 1}}{\cancel{(n-k)(n-k-1)\cdots 2\cdot 1}}$$
$$= n(n-1)(n-2)\cdots(n-k+1)$$

71. $\dbinom{5}{3} = \dfrac{5!}{(3!)(5-3)!} = \dfrac{5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{(\cancel{3\cdot 2\cdot 1})(2\cdot 1)} = 10$

73. $\dbinom{3}{0} = \dfrac{3!}{(0!)(3-0)!} = \dfrac{\cancel{3!}}{(1)(\cancel{3!})} = 1$

75. $\dbinom{n}{n-1} = \dfrac{n!}{(n-1)!(n-(n-1))!} = \dfrac{n\cancel{(n-1)!}}{\cancel{(n-1)!}(n-n+1)!} = \dfrac{n}{1} = n$

77. a. <u>Proof</u>: Let n be an integer such that $n \geq 2$. By definition of factorial,

$$n! = \begin{cases} 2\cdot 1 & \text{if } n = 2 \\ 3\cdot 2\cdot 1 & \text{if } n = 3 \\ n\cdot(n-1)\cdots 2\cdot 1 & \text{if } n > 3. \end{cases}$$

In each case, $n!$ has a factor of 2, and so $n! = 2k$ for some integer k. Then

$$\begin{aligned} n! + 2 &= 2k + 2 && \text{by substitution} \\ &= 2(k+1) && \text{by factoring out the 2.} \end{aligned}$$

Since $k+1$ is an integer, $n! + 2$ is divisible by 2 *[as was to be shown].*

c. *Hint:* Consider the sequence $m!+2, m!+3, m!+4, \ldots, m!+m$.

78. <u>Proof</u>: Suppose n and r are nonnegative integers with $r+1 \leq n$. The right-hand side of the equation to be shown is

$$\begin{aligned} \frac{n-r}{r+1}\cdot\binom{n}{r} &= \frac{n-r}{r+1}\cdot\frac{n!}{r!(n-r)!} \\ &= \frac{n-r}{r+1}\cdot\frac{n!}{r!(n-r)\cdot(n-r-1)!} \\ &= \frac{n!}{(r+1)!\cdot(n-r-1)!} \\ &= \frac{n!}{(r+1)!\cdot(n-(r+1))!} \\ &= \binom{n}{r+1}, \end{aligned}$$

which is the left-hand side of the equation to be shown.

80. a. $m-1$, *sum* $+a[i+1]$

81.

$$\begin{array}{rl|l}
 & 0 & \text{remainder} = r[6] = 1 \\
2 & \underline{1} & \text{remainder} = r[5] = 0 \\
2 & \underline{2} & \text{remainder} = r[4] = 1 \\
2 & \underline{5} & \text{remainder} = r[3] = 1 \\
2 & \underline{11} & \text{remainder} = r[2] = 0 \\
2 & \underline{22} & \text{remainder} = r[1] = 1 \\
2 & \underline{45} & \text{remainder} = r[0] = 0 \\
2 & 90 &
\end{array}$$

Hence $90_{10} = 1011010_2$.

84.

a	23					
i	0	1	2	3	4	5
q	23	11	5	2	1	0
***r*[0]**		1				
***r*[1]**			1			
***r*[2]**				1		
***r*[3]**					0	
***r*[4]**						1

88.

$$\begin{array}{rl|l}
 & 0 & \text{remainder } 1 \;\; = r[2] = 1_{16} \\
16 & \underline{1} & \text{remainder } 1 \;\; = r[1] = 1_{16} \\
16 & \underline{17} & \text{remainder } 15 = r[0] = \mathrm{F}_{16} \\
16 & 287 &
\end{array}$$

Hence $287_{10} = 11\mathrm{F}_{16}$.

Section 5.2

1. <u>Proof</u>: Let $P(n)$ be the property "n cents can be obtained by using 3-cent and 8-cent coins."

Show that P(14) is true:

Fourteen cents can be obtained by using two 3-cent coins and one 8-cent coin.

Show that for all integers $k \geq 14$, if P(k) is true, then P(k + 1) is true:

Suppose k cents (where $k \geq 14$) can be obtained using 3-cent and 8-cent coins. *[Inductive hypothesis]* We must show that $k+1$ cents can be obtained using 3-cent and 8-cent coins. If the k cents includes an 8-cent coin, replace it by three 3-cent coins to obtain a total of $k+1$ cents. Otherwise the k cents consists of 3-cent coins exclusively, and so there must be least five 3-cent coins (since the total amount is at least 14 cents). In this case, replace five of the 3-cent coins by two 8-cent coins to obtain a total of $k+1$ cents. Thus, in either case, $k+1$ cents can be obtained using 3-cent and 8-cent coins. *[This is what we needed to show.]*

[Since we have proved the basis step and the inductive step, we conclude that the given statement is true for all integers $n \geq 14$.]

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

3. a. $P(1)$ is "$1^2 = \frac{1\cdot(1+1)\cdot(2\cdot1+1)}{6}$." $P(1)$ is true because $1^2 = 1$ and $\frac{1\cdot(1+1)\cdot(2+1)}{6} = \frac{2\cdot3}{6} = 1$ also.

b. $P(k)$ is "$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$."

c. $P(k+1)$ is "$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2\cdot(k+1)+1)}{6}$."

d. *Must show:* If for some integer $k \geq 1$,

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}, \text{ then}$$

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)[(k+1)+1][(2(k+1)+1)]}{6}.$$

5. a. 1^2 **b.** k^2

c. $1 + 3 + 5 + \cdots + [2(k+1) - 1]$

d. $(k+1)^2$

e. the odd integer just before $2k+1$ is $2k-1$

f. inductive hypothesis

6. Proof: For the given statement, the property $P(n)$ is the equation

$$2 + 4 + 6 + \cdots + 2n = n^2 + n. \quad \leftarrow P(n)$$

Show that P(1) is true:

To prove $P(1)$, we must show that when 1 is substituted into the equation in place of n, the left-hand side equals the right-hand side. But when 1 is substituted for n, the left-hand side is the sum of all the even integers from 2 to $2\cdot1$, which is just 2, and the right-hand side is $1^2 + 1$, which also equals 2. Thus $P(1)$ is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k+1) is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$2 + 4 + 6 + \cdots + 2k = k^2 + k. \quad \leftarrow P(k) \text{ inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$2 + 4 + 6 + \cdots + 2(k+1) = (k+1)^2 + (k+1).$$

Because $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + 3k + 2$, this is equivalent to showing that

$$2 + 4 + 6 + \cdots + 2(k+1) = k^2 + 3k + 2. \quad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} &2 + 4 + 6 + \cdots + 2(k+1) \\ &\quad = 2 + 4 + 6 + \cdots + 2k + 2(k+1) && \text{by making the next-to-last term explicit} \\ &\quad = (k^2 + k) + 2(k+1) && \text{by substitution from the inductive hypothesis} \\ &\quad = k^2 + 3k + 2, && \text{by algebra,} \end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true.

[Since both the basis step and the inductive step have been proved, $P(n)$ is true for all integers $n \geq 1$.]

8. Proof: For the given statement, the property $P(n)$ is the equation

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1. \quad \leftarrow P(n)$$

Show that P(0) is true:

The left-hand side of $P(0)$ is 1, and the right-hand side is $2^{0+1} - 1 = 2 - 1 = 1$ also. Thus $P(0)$ is true.

Show that for all integers k ≥ 0, if P(k) is true then P(k+1) is true:

Let k be any integer with $k \geq 0$, and suppose $P(k)$ is true. That is, suppose

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1. \quad \leftarrow P(k) \text{ inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1,$$

or, equivalently,

$$1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1. \quad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} &1 + 2 + 2^2 + \cdots + 2^{k+1} \\ &\quad = 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} && \text{by making the next-to-last term explicit} \\ &\quad = (2^{k+1} - 1) + 2^{k+1} && \text{by substitution from the inductive hypothesis} \\ &\quad = 2\cdot2^{k+1} - 1 && \text{by combining like terms} \\ &\quad = 2^{k+2} - 1, && \text{by the laws of exponents,} \end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence the property is true for $n = k+1$.

[Since both the basis step and the inductive step have been proved, $P(n)$ is true for all integers $n \geq 0$.]

10. Proof: For the given statement, the property is the equation

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \quad \leftarrow P(n)$$

Show that P(1) is true:

The left-hand side of $P(1)$ is $1^2 = 1$, and the right-hand side is $\frac{1(1+1)(2\cdot1+1)}{6} = \frac{2\cdot3}{6} = 1$ also. Thus $P(1)$ is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k+1) is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \quad \leftarrow P(k) \text{ inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},$$

or, equivalently,

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}. \quad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 && \text{by making the next-to-last term explicit} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by substitution from the inductive hypothesis} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} && \text{because } \tfrac{6}{6} = 1 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{by adding fractions} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{by factoring out } (k+1) \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{by multiplying out and combining like terms} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{because } (k+2)(2k+3) = 2k^2 + 7k + 6,
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence the property is true for $n = k+1$.

[Since both the basis step and the inductive step have been proved, $P(n)$ is true for all integers $n \ge 1$.]

13. <u>Proof</u>: For the given statement, the property $P(n)$ is the equation

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}. \quad \leftarrow P(n)$$

Show that P(2) is true:

The left-hand side of $P(2)$ is $\sum_{i=1}^{1} i(i+1) = 1 \cdot (1+1) = 2$, and the right-hand side is $\frac{2(2-1)(2+1)}{3} = \frac{6}{3} = 2$ also. Thus $P(2)$ is true.

Show that for all integers $k \ge 2$, if P(k) is true then P(k + 1) is true:

Let k be any integer with $k \ge 2$, and suppose $P(k)$ is true. That is, suppose

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3} \quad \leftarrow P(k) \text{ inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\sum_{i=1}^{(k+1)-1} i(i+1) = \frac{(k+1)((k+1)-1)((k+1)+1)}{3},$$

or, equivalently,

$$\sum_{i=1}^{k} i(i+1) = \frac{(k+1)k(k+2)}{3}. \quad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\sum_{i=1}^{k} i(i+1) \\
&= \sum_{i=1}^{k-1} i(i+1) + k(k+1) && \text{by writing the last term separately} \\
&= \frac{k(k-1)(k+1)}{3} + k(k+1) && \text{by substitution from the inductive hypothesis} \\
&= \frac{k(k-1)(k+1)}{3} + \frac{3k(k+1)}{3} && \text{because } \tfrac{3}{3} = 1 \\
&= \frac{k(k-1)(k+1) + 3k(k+1)}{3} && \text{by adding the fractions} \\
&= \frac{k(k+1)[(k-1)+3]}{3} && \text{by factoring out } k(k+1) \\
&= \frac{k(k+1)(k+2)}{3}, && \text{by algebra,}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true.

[Since both the basis step and the inductive step have been proved, $P(n)$ is true for all integers $n \ge 0$.]

15. *Hint:* To prove the basis step, show that $\sum_{i=1}^{1} i(i!) = (1+1)! - 1$. To prove the inductive step, suppose that $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$ for some integer $k \ge 1$ and show that $\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$. Note that $[(k+1)! - 1] + (k+1)[(k+1)!] = (k+1)![1 + (k+1)] - 1$.

18. *Hints:* $\sin^2 x + \cos^2 x = 1$, $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, $\sin(2x) = 2 \sin x \cos x$, $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

20. $4 + 8 + 12 + 16 + \cdots + 200 = 4(1 + 2 + 3 + \cdots + 50) = 4\left(\frac{50 \cdot 51}{2}\right) = 5100$

22. $3 + 4 + 5 + 6 + \cdots + 1000 = (1 + 2 + 3 + 4 + \cdots + 1000) - (1+2) = \left(\frac{1000 \cdot 1001}{2}\right) - 3 = 500{,}497$

24. $\frac{(k-1)((k-1)+1)}{2} = \frac{k(k-1)}{2}$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

25. **a.** $\dfrac{2^{26}-1}{2-1} = 2^{26} - 1 = 67{,}108{,}863$

b. $2 + 2^2 + 2^3 + \cdots + 2^{26}$
$= 2(1 + 2 + 2^2 + \cdots + 2^{25})$
$= 2 \cdot (67{,}108{,}863)$ by part (a)
$= 134{,}217{,}726$

28. $$\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{2^{n+1}} - 1}{-\frac{1}{2}} = \left(\frac{1}{2^{n+1}} - 1\right)(-2)$$
$$= -\frac{2}{2^{n+1}} + 2 = 2 - \frac{1}{2^n}$$

30. *Hint:* $c + (c + d) + (c + 2d) + \cdots + (c + nd)$
$= (n + 1)\, c + d \cdot \frac{n(n+1)}{2}$.

33. In the inductive step, both the inductive hypothesis and what is to be shown are wrong. The inductive hypothesis should be

Suppose that for some integer $k \geq 1$,
$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$
And what is to be shown should be
$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

34. *Hint:* See the Caution note for Example 5.1.8.

35. *Hint:* See the subsection Proving an Equality on page 254.

37. *Hint:* Form the sum $n^2 + (n+1)^2 + (n+2)^2 + \cdots + (n + (p-1))^2$, and show that it equals
$$pn^2 + 2n(1 + 2 + 3 + \cdots + (p-1)) + (1 + 4 + 9 + 16 + \cdots + (p-1)^2).$$

Section 5.3

1. *General formula:* $\prod_{i=2}^{n}\left(1 - \frac{1}{i}\right) = \frac{1}{n}$ for all integers $n \geq 2$.

Proof (by mathematical induction): Let the property $P(n)$ be the equation
$$\prod_{i=2}^{n}\left(1 - \frac{1}{i}\right) = \frac{1}{n}. \qquad \leftarrow P(n)$$

Show that P(2) is true:

The left-hand side of $P(2)$ is $\prod_{i=2}^{2}\left(1 - \frac{1}{i}\right) = 1 - \frac{1}{2} = \frac{1}{2}$, which equals the right-hand side.

Show that for all integers $k \geq 2$, if P(k) is true then P(k + 1) is also true:

Suppose that k is any integer with $k \geq 2$ such that
$$\prod_{i=2}^{k}\left(1 - \frac{1}{i}\right) = \frac{1}{k}. \qquad \leftarrow P(k) \text{ Inductive hypothesis}$$

We must show that
$$\prod_{i=2}^{k+1}\left(1 - \frac{1}{i}\right) = \frac{1}{k+1}. \qquad \leftarrow P(k+1)$$

But by the laws of algebra and substitution from the inductive hypothesis, the left-hand side of $P(k+1)$ is
$$\prod_{i=2}^{k+1}\left(1 - \frac{1}{i}\right)$$
$$= \prod_{i=2}^{k}\left(1 - \frac{1}{i}\right)\left(1 - \frac{1}{k+1}\right)$$
$$= \left(\frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) = \left(\frac{1}{k}\right)\left(\frac{(k+1)-1}{k+1}\right)$$
$$= \frac{1}{k+1} \text{ which is the right-hand side of } P(k+1)$$

[as was to be shown].

3. *General formula:* $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ for all integers $n \geq 1$.

Proof (by mathematical induction): Let the property $P(n)$ be the equation
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Show that P(1) is true:

The left-hand side of $P(1)$ equals $\frac{1}{1\cdot 3}$, and the right-hand side equals $\frac{1}{2\cdot 1+1}$. But both of these equal $\frac{1}{3}$, so $P(1)$ is true.

Show that for any integer $k \geq 1$, if P(k) is true then P(k + 1) is true:

Suppose that k is any integer with $k \geq 1$, and
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$
$\uparrow P(k)$ inductive hypothesis

We must show that
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k+1}{2(k+1)+1}.$$

or, equivalently,
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}.$$
$\uparrow P(k+1)$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

But the left-hand side of $P(k+1)$ is

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \text{by inductive hypothesis}$$
$$=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$
$$=\frac{k+1}{2k+3} \quad \text{by algebra,}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown].*

4. *Hint 1:* The general formula is

$$1-4+9-16+\cdots+(-1)^{n-1}n^2$$
$$=(-1)^{n-1}(1+2+3+\cdots+n) \quad \text{in expanded form}$$

Or: $$\sum_{i=1}^{n}(-1)^{i-1}i^2=(-1)^{n-1}\left(\sum_{i=1}^{n} i\right) \quad \text{in summation notation.}$$

Hint 2: In the proof, use the fact that

$$1+2+3+\cdots+n=\sum_{i=1}^{n} i=\frac{n(n+1)}{2}.$$

6. a. $P(0)$ is "5^0-1 is divisible by 4." $P(0)$ is true because $5^0-1=0$, which is divisible by 4.
b. $P(k)$ is "5^k-1 is divisible by 4."
c. $P(k+1)$ is "$5^{k+1}-1$ is divisible by 4."
d. *Must show:* If for some integer $k\ge 0$, 5^k-1 is divisible by 4, then $5^{k+1}-1$ is divisible by 4.

8. Proof (by mathematical induction): For the given statement, the property is the sentence "5^n-1 is divisible by 4."

Show that $P(0)$ is true:

$P(0)$ is the sentence "5^0-1 is divisible by 4." But $5^0-1=1-1=0$, and 0 is divisible by 4 because $0=4\cdot 0$. Thus $P(0)$ is true.

Show that for all integers $k\ge 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k\ge 0$, and suppose $P(k)$ is true. That is, suppose 5^k-1 is divisible by 4. *[This is the inductive hypothesis.]* We must show that $P(k+1)$ is true. That is, we must show that $5^{k+1}-1$ is divisible by 4. Now

$$5^{k+1}-1=5^k\cdot 5-1$$
$$=5^k\cdot(4+1)-1=5^k\cdot 4+(5^k-1). \quad (*)$$

By the inductive hypothesis 5^k-1 is divisible by 4, and so $5^k-1=4r$ for some integer r. By substitution into equation (*),

$$5^{k+1}-1=5^k\cdot 4+4r=4(5^k+r).$$

But 5^k+r is an integer because k and r are integers. Hence, by definition of divisibility, $5^{k+1}-1$ is divisible by 4 *[as was to be shown].*

An alternative proof of the inductive step goes as follows: Suppose that for some integer $k\ge 0$, 5^k-1 is divisible by 4. Then $5^k-1=4r$ for some integer r, and hence $5^k=4r+1$.
It follows that $5^{k+1}=5^k\cdot 5=(4r+1)\cdot 5=20r+5$. Subtracting 1 from both sides gives that $5^{k+1}-1=20r+4=4(5r+1)$. But $5r+1$ is an integer, and so, by definition of divisibility, $5^{k+1}-1$ is divisible by 4.

11. Proof (by mathematical induction): For the given statement, the property $P(n)$ is the sentence "$3^{2n}-1$ is divisible by 8."

Show that $P(0)$ is true:

$P(0)$ is the sentence "$3^{2\cdot 0}-1$ is divisible by 8." But $3^{2\cdot 0}-1=1-1=0$, and 0 is divisible by 8 because $0=8\cdot 0$. Thus $P(0)$ is true.

Show that for all integers $k\ge 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k\ge 0$, and suppose $P(k)$ is true. That is, suppose $3^{2k}-1$ is divisible by 8. *[This is the inductive hypothesis.]* We must show that $P(k+1)$ is true. That is, we must show that $3^{2(k+1)}-1$ is divisible by 8, or equivalently, $3^{2k+2}-1$ is divisible by 8. Now

$$3^{2k+2}-1=3^{2k}\cdot 3^2-1=3^{2k}\cdot 9-1$$
$$=3^{2k}\cdot(8+1)-1=3^{2k}\cdot 8+(3^{2k}-1)\cdot \quad (*)$$

By the inductive hypothesis $3^{2k}-1$ is divisible by 8, and so $3^{2k}-1=8r$ for some integer r. By substitution into equation (*),

$$3^{2k+2}-1=3^{2k}\cdot 8+8r=8(3^{2k}+r).$$

But $3^{2k}+r$ is an integer because k and r are integers. Hence, by definition of divisibility, $3^{2k+2}-1$ is divisible by 8 *[as was to be shown].*

13. *Hint:* $$x^{k+1}-y^{k+1}=x^{k+1}-x\cdot y^k+x\cdot y^k-y^{k+1}$$
$$=x\cdot(x^k-y^k)+y^k\cdot(x-y)$$

14. *Hint 1:* $$(k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1$$
$$=(k^3-k)+3k^2+3k$$
$$=(k^3-k)+3k(k+1)$$

Hint 2: $k(k+1)$ is a product of two consecutive integers. By Theorem 4.4.3, one of these must be even.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

16. Proof (by mathematical induction): For the given statement, let the property $P(n)$ be the inequality $2^n < (n+1)!$.

Show that P(2) is true:

$P(2)$ says that $2^2 < (2+1)!$. The left-hand side is $2^2 = 4$ and the right-hand side is $3! = 6$. So, because $4 < 6$, $P(2)$ is true.

Show that for all integers $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 2$, and suppose $P(k)$ is true. That is, suppose $2^k < (k+1)!$. *[This is the inductive hypothesis.]* We must show that $P(k+1)$ is true. That is, we must show that $2^{k+1} < ((k+1)+1)$, or, equivalently, $2^{k+1} < (k+2)!$. By the laws of exponents and the inductive hypothesis,

$$2^{k+1} = 2 \cdot 2^k < 2(k+1)!. \qquad (*)$$

Since $k \geq 2$, then $2 < k+2$, and so

$$2(k+1)! < (k+2)(k+1)! = (k+2)!. \qquad (**)$$

Combining inequalities (*) and (**) gives

$$2^{k+1} < (k+2)!$$

[as was to be shown].

19. Proof (by mathematical induction): For the given statement, let the property $P(n)$ be the inequality $n^2 < 2^n$.

Show that P(5) is true:

$P(5)$ says that $5^2 < 2^5$. But $5^2 = 25$ and $2^5 = 32$, and $25 < 32$. Hence $P(5)$ is true.

Show that for any integer $k \geq 5$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 5$, and suppose $P(k)$ is true. That is, suppose $k^2 < 2^k$. *[This is the inductive hypothesis.]* We must show that $P(k+1)$ is true. That is, we must show that $(k+1)^2 < 2^{k+1}$. But

$$(k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1$$

by inductive hypothesis
Also, by Proposition 5.3.2,

$$2k + 1 < 2^k \qquad \text{Prop. 5.3.2 applies since } k \geq 5 \geq 3.$$

Putting these inequalities together gives

$$(k+1)^2 < 2^k + 2k + 1 < 2^k + 2^k = 2^{k+1}$$

[as was to be shown].

24. Proof (by mathematical induction): For the given statement, let the property $P(n)$ be the equation $a_n = 3 \cdot 7^{n-1}$.

Show that P(1) is true:

The left-hand side of $P(1)$ is a_1, which equals 3 by definition of the sequence. The right-hand side is $3 \cdot 7^{1-1} = 3$ also. Thus $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose $a_k = 3 \cdot 7^{k-1}$. *[This is the inductive hypothesis.]* We must show that $P(k+1)$ is true. That is, we must show that $a_{k+1} = 3 \cdot 7^{(k+1)-1}$, or, equivalently, $a_{k+1} = 3 \cdot 7^k$. But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= 7a_k && \text{by definition of the sequence } a_1, a_2, a_3, \ldots \\ &= 7(3 \cdot 7^{k-1}) && \text{by inductive hypothesis} \\ &= 3 \cdot 7^k && \text{by the laws of exponents,} \end{aligned}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown].*

30. The inductive step fails for going from $n = 1$ to $n = 2$, because when $k = 1$,

$$A = \{a_1, a_2\} \quad \text{and} \quad B = \{a_1\},$$

and no set C can be defined to have the properties claimed for the C in the proof. The reason is that $C = \{a_1\} = B$, and so an element of A, namely a_2, is not in either B or C.

Since the inductive step fails for going from $n = 1$ to $n = 2$, the truth of the following statement is never proved: "All the numbers in a set of two numbers are equal to each other." This breaks the sequence of inductive steps, and so none of the statements for $n > 2$ is proved true either.

Here is an explanation for what happens in terms of the domino analogy. The first domino *is* tipped backward (the basis step is proved). Also, if any domino from the second onward tips backward, then it tips the one behind it backward (the inductive step works for $n \geq 2$). However, when the first domino is tipped backward, it does *not* tip the second one backward. So only the first domino falls down; the rest remain standing.

31. *Hint:* Is the basis step true?

32. *Hint:* Consider the problem of trying to cover a 3×3 checkerboard with trominoes. Place a checkmark in certain squares as shown in the following figure.

✓		✓
✓		✓

Observe that no two squares containing checkmarks can be covered by the same tromino. Since there are four checkmarks, four tromiones would be needed to cover these squares. But, since each tromino covers three squares, four trominoes would cover twelve squares, not the nine squares in this checkerboard. It follows that such a covering is impossible.

34. **a**. *Hint:* For the inductive step, note that a $3 \times (2(k+1))$ checkerboard can be split into a $3 \times 2k$ checkerboard and a 3×2 checkerboard.

35. **b**. *Hint:* Consider a 3×5 checkerboard, and refer to the hint for exercise 32. Figure out a way to place six checkmarks in squares so that no two of the squares that contain checkmarks can be covered by the same tromino.

37. *Hint:* Use proof by contradiction. If the statement is false, then there exists some ordering of the integers from 1 to 30, say $x_1, x_2, \ldots, x_{30}$, such that $x_1 + x_2 + x_3 < 45$,

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$x_2 + x_3 + x_4 < 45, \ldots,$ and $x_{30} + x_1 + x_2 < 45$. Evaluate the sum of all these inequalities using the fact that $\sum_{i=1}^{30} x_i = \sum_{i=1}^{30} i$ and Theorem 4.2.2.

38. *Hint:* Given $k+1$ a's and $k+1$ b's arrayed around the outside of the circle, there has to be at least one location where an a is followed by a b as one travels in the clockwise direction. In the inductive step, temporarily remove such an a and the b that follows it, and apply the inductive hypothesis.

Section 5.4

1. Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence "a_n is odd."

Show that $P(1)$ and $P(2)$ are true:

Observe that $a_1 = 1$ and $a_2 = 3$ and both 1 and 3 are odd. Thus $P(1)$ and $P(2)$ are true.

Show that for any integer $k \geq 2$, if $P(i)$ is true for all integers i with $1 \leq i \leq k$, then $P(k+1)$ is true:

Let $k \geq 2$ be any integer, and suppose a_i is odd for all integers i with $1 \leq i \leq k$. *[This is the inductive hypothesis.]* We must show that a_{k+1} is odd. We know that $a_{k+1} = a_{k-1} + 2a_k$ by definition of $a_1, a_2, a_3, \ldots$. Moreover, $k-1$ is less than $k+1$ and is greater than or equal to 1 (because $k \geq 2$). Thus, by inductive hypothesis, a_{k-1} is odd. Also, every term of the sequence is an integer (being a sum of products of integers), and so $2a_k$ is even by definition of even. Hence a_{k+1} is the sum of an odd integer and an even integer and hence is odd (by exercise 19, in Section 4.1). *[This is what was to be shown.]*

4. Proof (by strong mathematical induction): Let the property $P(n)$ be the inequality $d_n \leq 1$.

Show that $P(1)$ and $P(2)$ are true:

Observe that $d_1 = \frac{9}{10}$ and $d_2 = \frac{10}{11}$ and both $\frac{9}{10} \leq 1$ and $\frac{10}{11} \leq 1$. Thus $P(1)$ and $P(2)$ are true.

Show that for any integer $k \geq 2$, if $P(i)$ is true for all integers i with $1 \leq i \leq k$, then $P(k+1)$ is true:

Let $k \geq 2$ be any integer, and suppose $d_i \leq 1$ for all integers i with $1 \leq i \leq k$. *[This is the inductive hypothesis.]* We must show that $d_{k+1} \leq 1$. But, by definition of $d_1, d_2, d_3, \ldots, d_{k+1} = d_k \cdot d_{k-1}$. Now $d_k \leq 1$ and $d_{k-1} \leq 1$ by inductive hypothesis *[since $1 \leq k < k+1$ and $1 \leq k-1 < k+1$ because $k \geq 2$.]*. Consequently, $d_{k+1} = d_k \cdot d_{k-1} \leq 1$ because if two positive numbers are each less than or equal to 1, then their product is less than or equal to 1. *[If $0 < a \leq 1$ and $0 < b \leq 1$, then multiplying $a \leq 1$ by b gives $ab \leq b$, and since $b \leq 1$, then by transitivity of order, $ab \leq 1$.]* This is what was to be shown. *[Since we have proved both the basis step and the inductive step, we conclude that $d_n \leq 1$ for all integers $n \geq 1$.]*

5. Proof (by strong mathematical induction): Let the property $P(n)$ be the equation $e_n = 5 \cdot 3^n + 7 \cdot 2^n$.

Show that $P(0)$ and ***$P(1)$ are true***.

We must show that $e_0 = 5 \cdot 3^0 + 7 \cdot 2^0$ and $e_1 = 5 \cdot 3^1 + 7 \cdot 2^1$. The left-hand side of the first equation is 12 (by definition of $e_0, e_1, e_2, \ldots$), and its right-hand side is $5 \cdot 1 + 7 \cdot 1 = 12$ also. The left-hand side of the second equation is 29 (by definition of $e_0, e_1, e_2, \ldots$), and its right-hand side is $5 \cdot 3 + 7 \cdot 2 = 29$ also. Thus $P(0)$ and $P(1)$ are true.

Show that for any integer $k \geq 1$, if $P(i)$ is true for all integers i with $0 \leq i \leq k$, then $P(k+1)$ is true:

Let $k \geq 1$ be an integer, and suppose $e_i = 5 \cdot 3^i + 7 \cdot 2^i$ for all integers i with $0 \leq i \leq k$. *[Inductive hypothesis]* We must show that $e_{k+1} = 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}$.
But

$$\begin{aligned}
e_{k+1} &= 5e_k - 6e_{k-1} && \text{by definition of } e_0, e_1, e_2, \ldots \\
&= 5(5 \cdot 3^k + 7 \cdot 2^k) - 6(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) \\
& && \text{by inductive hypothesis} \\
&= 25 \cdot 3^k + 35 \cdot 2^k - 30 \cdot 3^{k-1} - 42 \cdot 2^{k-1} \\
&= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3 \cdot 3^{k-1} - 21 \cdot 2 \cdot 2^{k-1} \\
&= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3^k - 21 \cdot 2^k \\
&= (25 - 10) \cdot 3^k + (35 - 21) \cdot 2^k \\
&= 15 \cdot 3^k + 14 \cdot 2^k \\
&= 5 \cdot 3 \cdot 3^k + 7 \cdot 2 \cdot 2^k \\
&= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1} && \text{by algebra.}
\end{aligned}$$

[This is what was to be shown.]

10. *Hint:* In the basis step, show that $P(14)$, $P(15)$, and $P(16)$ are all true. For the inductive step, note that $k+1 = [(k+1) - 3] + 3$, and if $k \geq 16$, then $(k+1) - 3 \geq 14$.

11. Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

"A jigsaw puzzle consisting of n pieces takes $n-1$ steps to put together."

Show that $P(1)$ is true:

A jigsaw puzzle consisting of just one piece does not take any steps to put together. Hence it is correct to say that it takes zero steps to put together.

Show that for any integer $k \geq 1$, if $P(i)$ is true for all integers i with $1 \leq i \leq k$ then $P(k+1)$ is true:

Let $k \geq 1$ be an integer and suppose that for all integers i with $1 \leq i \leq k$, a jigsaw puzzle consisting of i pieces takes $i-1$ steps to put together. *[This is the inductive hypothesis.]* We must show that a jigsaw puzzle consisting of $k+1$ pieces takes k steps to put together. Consider assembling a jigsaw puzzle consisting of $k+1$ pieces. The last step involves fitting together two blocks. Suppose one of the blocks consists of r pieces and the other consists of s pieces. Then $r + s = k + 1$, and $1 \leq r \leq k$ and $1 \leq s \leq k$. Thus by inductive hypothesis, the numbers of steps required to assemble the blocks are $r-1$ and $s-1$, respectively. Then the total number of steps required to assemble the puzzle is $(r-1) + (s-1) + 1 = (r+s) - 1 = (k+1) - 1 = k$ *[as was to be shown]*.

12. *Hint:* For any collection of cans, at least one must contain enough gasoline to enable the car to get to the next can. (Why?) Imagine taking all the gasoline from that can and

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

pouring it into the can that immediately precedes it in the direction of travel around the track.

13. *Sketch of proof:* Given any integer $k > 1$, either k is prime or k is a product of two smaller positive integers, each greater than 1. In the former case, the property is true. In the latter case, the inductive hypothesis ensures that both factors of k are products of primes and hence that k is also a product of primes.

14. Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence "Any product of n odd integers is odd."

Show that P(2) is true:

We must show that any product of two odd integers is odd. But this was established in Chapter 4 (exercise 43 of Section 4.1).

Show that for any integer $k \geq 2$, if $P(i)$ is true for all integers i with $2 \leq i \leq k$ then $P(k+1)$ is true:

Let k be any integer with $k \geq 2$, and suppose that for all integers i with $2 \leq i \leq k$, any product of i odd integers is odd. *[Inductive hypothesis]* Consider any product M of $k+1$ odd integers. Some multiplication is the final one that is used to obtain M. Thus there are integers A and B such that $M = AB$, and each of A and B is a product of between 1 and k odd integers. (For instance, if $M = ((a_1a_2)a_3)a_4$, then $A = (a_1a_2)a_3$ and $B = a_4$.) By inductive hypothesis, each of A and B is odd, and, as in the basis step, we know that any product of two odd integers is odd. Hence $M = AB$ is odd.

16. *Hint:* Let the property $P(n)$ be the sentence "If n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd." For the inductive step, consider any sum S of $k+1$ odd integers. Some addition is the final one that is used to obtain S. Thus there are integers A and B such that $S = A + B$, and A is a sum of r odd integers and B is a sum of $(k+1) - r$ odd integers. Consider the two cases where $k+1$ is even and $k+1$ is odd, and for each case consider the two subcases where r is even and where r is odd.

17. $4^1 = 4, 4^2 = 16, 4^3 = 64, 4^4 = 256, 4^5 = 1024,$
$4^6 = 4096, 4^7 = 16384,$ and $4^8 = 65536.$

Conjecture: The units digit of 4^n equals 4 if n is odd and equals 6 if n is even.

Proof by strong mathematical induction: Let the property $P(n)$ be the sentence "The units digit of 4^n equals 4 if n is odd and equals 6 if n is even."

Show that P(1) and P(2) are true:

When $n = 1, 4^n = 4^1 = 4$, and the units digit is 4. When $n = 2$, then $4^n = 4^2 = 16$, and the units digits is 6. Thus $P(1)$ and $P(2)$ are true.

Show that for any integer $k \geq 2$, if the property is true for all integers i with $1 \leq i \leq k$ then it is true for k+1:

Let k by any integer with $k \geq 2$, and suppose that for all integers i with $0 \leq i \leq k$, the units digit of 4^i equals 4 if i is odd and equals 6 if i is even. *[Inductive hypothesis]* We must show that the units digit of 4^{k+1} equals 4 if $k+1$ is odd and equals 6 if $k+1$ is even.

Case 1 ($k+1$ is odd): In this case, k is even, and so, by inductive hypothesis, the units digits of 4^k is 6. Thus $4^k = 10q + 6$ for some nonnegative integer q. It follows that $4^{k+1} = 4^k \cdot 4 = (10q+6) \cdot 4 = 40q + 24 = 10(4q + 2) + 4$. Thus the units digit of 4^{k+1} is 4 *[as was to be shown].*

Case 2 ($k+1$ is even): In this case, k is odd, and so, by inductive hypothesis, the units digit of 4^k is 4. Thus $4^k = 10q + 4$ for some nonnegative integer q. It follows that $4^{k+1} = 4^k \cdot 4 = (10q+4) \cdot 4 = 40q + 16 = 10(4q + 1) + 6$. Thus the units digit of 4^{k+1} is 6 *[as was to be shown].*

20. Proof: Let n be any integer greater than 1. Consider the set S of all positive integers other than 1 that divide n. Since $n \mid n$ and $n > 1$, there is at least one element in S. Hence, by the well-ordering principle for the integers, S has a smallest element; call it p. We claim that p is prime. For suppose p is not prime. Then there are integers a and b with $1 < a < p$, $1 < b < p$, and $p = ab$. By definition of divides, $a \mid p$. Also $p \mid n$ because p is in S and every element in S divides n. Therefore, $a \mid p$ and $p \mid n$, and so, by transitivity of divisibility, $a \mid n$. Consequently, $a \in S$. But this contradicts the fact that $a < p$, and p is the smallest element of S. *[This contradiction shows that the supposition that p is not prime is false.]* Hence p is prime, and we have shown the existence of a prime number that divides n.

22. a. Proof: Suppose r is any rational number. *[We need to show that there is an integer n such that $r < n$.]*

Case 1 ($r \leq 0$): In this case, take $n = 1$. Then $r < n$.

Case 2 ($r > 0$): In this case, $r = \frac{a}{b}$ for some positive integers a and b (by definition of rational and because r is positive). Note that $r = \frac{a}{b} < n$ if, and only if, $a < nb$. Let $n = 2a$. Multiply both sides of the inequality $1 < 2$ by a to obtain $a < 2a$, and multiply both sides of the inequality $1 < b$ by $2a$ to obtain $2a < 2ab = nb$. Thus $a < 2a < nb$, and so, by transitivity of order, $a < nb$. Dividing both sides by b gives that $\frac{a}{b} < n$, or, equivalently, that $r < n$.

Hence, in both cases, $r < n$ *[as was to be shown].*

23. *Hint:* If r is any rational number, let S be the set of all integers n such that $r < n$. Use the results of exercises 22(a), 22(c), and the well-ordering principle for the integers to show that S has a least element, say v, and then show that $v - 1 \leq r < v$.

24. Proof: Let S be the set of all integers r such that $n = 2^i \cdot r$ for some integer i. Then $n \in S$ because $n = 2^0 \cdot n$, and so $S \neq \emptyset$. Also, since $n \geq 1$, each r in S is positive, and so, by the well-ordering principle, S has a least element m. This means that $n = 2^k \cdot m$ (*) for some nonnegative integer k and $m \leq r$ for every r in S. We claim that m is odd. The reason is that if m were even, then $m = 2p$ for some integer p. Substituting into equation (*) gives

$$n = 2^k \cdot m = 2^k \cdot 2p = (2^k \cdot 2)p = 2^{k+1} \cdot p.$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

It follows that $p \in S$ and $p < m$, which contradicts the fact that m is the *least* element of S. Hence m is odd, and so $n = m \cdot 2^k$ for some odd integer m and nonnegative integer k.

29. *Hint:* In the inductive step, divide into cases depending upon whether k can be written as $k = 3x$ or $k = 3x + 1$ or $k = 3x + 2$ for some integer x.

30. *Hint:* In the inductive step, let an integer $k \geq 0$ be given and suppose that there exist integers q' and r' such that $k = dq' + r'$ and $0 \leq r' < d$. You must show that there exist integers q and r such that

$$k + 1 = dq + r \quad \text{and} \quad 0 \leq r < d.$$

To do this, consider the two cases $r' < d - 1$ and $r' = d - 1$.

31. *Hint:* Given a predicate $P(n)$ that satisfies conditions (1) and (2) of the principle of mathematical induction, let S be the set of all integers greater than or equal to a for which $P(n)$ is false. Suppose that S has one or more elements, and use the well-ordering principle to derive a contradiction.

32. *Hint:* Suppose S is a set containing one or more integers, all of which are greater than or equal to some integer a, and suppose that S does not have a least element. Let the property $P(n)$ be the sentence "$i \notin S$ for any integer i with $a \leq i \leq n$." Use mathematical induction to prove that $P(n)$ is true for all integers $n \geq a$, and explain how this result contradicts the supposition that S does not have a least element.

Section 5.5

1. Proof: Suppose the predicate $m + n = 100$ is true before entry to the loop. Then

$$m_{\text{old}} + n_{\text{old}} = 100.$$

After execution of the loop,

$$m_{\text{new}} = m_{\text{old}} + 1 \quad \text{and} \quad n_{\text{new}} = n_{\text{old}} - 1,$$

so

$$\begin{aligned} m_{\text{new}} + n_{\text{new}} &= (m_{\text{old}} + 1) + (n_{\text{old}} - 1) \\ &= m_{\text{old}} + n_{\text{old}} = 100. \end{aligned}$$

3. Proof: Suppose the predicate $m^3 > n^2$ is true before entry to the loop. Then

$$m_{\text{old}}^3 > n_{\text{old}}^2.$$

After execution of the loop,

$$m_{\text{new}} = 3 \cdot m_{\text{old}} \quad \text{and} \quad n_{\text{new}} = 5 \cdot n_{\text{old}},$$

so

$$m_{\text{new}}^3 = (3 \cdot m_{\text{old}})^3 = 27 \cdot m_{\text{old}}^3 > 27 \cdot n_{\text{old}}^2.$$

But since $n_{\text{new}} = 5 \cdot n_{\text{old}}$, then $n_{\text{old}} = \frac{1}{5} n_{\text{new}}$. Hence

$$\begin{aligned} m_{\text{new}}^3 > 27 \cdot n_{\text{old}}^2 &= 27 \cdot \left(\frac{1}{5} n_{\text{new}}\right)^2 = 27 \cdot \frac{1}{25} n_{\text{new}}^2 \\ &= \frac{27}{25} \cdot n_{\text{new}}^2 > n_{\text{new}}^2. \end{aligned}$$

6. Proof: *[The wording of this proof is almost the same as that of Example 5.5.2.]*

I. Basis Property: *[$I(0)$ is true before the first iteration of the loop.]*
$I(0)$ is "$exp = x^0$ and $i = 0$." According to the pre-condition, before the first iteration of the loop $exp = 1$ and $i = 0$. Since $x^0 = 1$, $I(0)$ is evidently true.

II. Inductive Property: *[If $G \wedge I(k)$ is true before a loop iteration (where $k \geq 0$), then $I(k + 1)$ is true after the loop iteration.]*

Suppose k is a nonnegative integer such that $G \wedge I(k)$ is true before an iteration of the loop. Then as execution reaches the top of the loop, $i \neq m$, $exp = x^k$, and $i = k$. Since $i \neq m$, the guard is passed and statement 1 is executed. Now before execution of statement 1,

$$exp_{\text{old}} = x^k,$$

so execution of statement 1 has the following effect:

$$exp_{\text{new}} = exp_{\text{old}} \cdot x = x^k \cdot x = x^{k+1}.$$

Similarly, before statement 2 is executed,

$$i_{\text{old}} = k,$$

so after execution of statement 2,

$$i_{\text{new}} = i_{\text{old}} + 1 = k + 1.$$

Hence after the loop iteration, the two statements $exp = x^{k+1}$ and $i = k + 1$ are true, and so $I(k + 1)$ is true.

III. Eventual Falsity of Guard: *[After a finite number of iterations of the loop, G becomes false.]*

The guard G is the condition $i \neq m$, and m is a nonnegative integer. By I and II, it is known that

> for *all* integers $n \geq 0$, if the loop is iterated n times, then $exp = x^n$ and $i = n$.

So after m iterations of the loop, $i = m$. Thus G becomes false after m iterations of the loop.

IV. Correctness of the Post-Condition: *[If N is the least number of iterations after which G is false and $I(N)$ is true, then the value of the algorithm variables will be as specified in the post-condition of the loop.]*

According to the post-condition, the value of exp after execution of the loop should be x^m. But when G is false, $i = m$. And when $I(N)$ is true, $i = N$ and $exp = x^N$. Since *both* conditions (G false and $I(N)$ true) are satisfied, $m = i = N$ and $exp = x^m$, as required.

8. Proof:

I. Basis Property: $I(0)$ is "$i = 1$ and $sum = A[1]$." According to the pre-condition, this statement is true.

II. Inductive Property: Suppose k is a nonnegative integer such that $G \wedge I(k)$ is true before an iteration of the loop. Then as execution reaches the top of the loop,

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$i \neq m, i = k+1$, and $sum = A[1] + A[2] + \cdots + A[k+1]$. Since $i \neq m$, the guard is passed and statement 1 is executed. Now before execution of statement, 1, $i_{\text{old}} = k+1$. So after execution of statement 1, $i_{\text{new}} = i_{\text{old}} + 1 = (k+1) + 1 = k+2$. Also before statement 2 is executed, $sum_{\text{old}} = A[1] + A[2] + \cdots + A[k+1]$. Execution of statement 2 adds $A[k+2]$ to this sum, and so after statement 2 is executed, $sum_{\text{new}} = A[1] + A[2] + \cdots + A[k+1] + A[k+2]$. Thus after the loop iteration, $I(k+1)$ is true.

III. Eventual Falsity of Guard: The guard G is the condition $i \neq m$. By I and II, it is known that for all integers $n \geq 1$, after n iterations of the loop, $I(n)$ is true. Hence, after $m-1$ iterations of the loop, $I(m)$ is true, which implies that $i = m$ and G is false.

IV. Correctness of the Post-Condition: Suppose that N is the least number of iterations after which G is false and $I(N)$ is true. Then (since G is false) $i = m$ and (since $I(N)$ is true) $i = N+1$ and $sum = A[1] + A[2] + \cdots + A[N+1]$. Putting these together gives $m = N+1$, and so $sum = A[1] + A[2] + \cdots + A[m]$, which is the post-condition.

10. *Hint:* Assume $G \wedge I(k)$ is true for a nonnegative integer k. Then $a_{\text{old}} \neq 0$ and $b_{\text{old}} \neq 0$ and
(1) a_{old} and b_{old} are nonnegative integers with $\gcd(a_{\text{old}}, b_{\text{old}}) = \gcd(A, B)$.
(2) At most one of a_{old} and b_{old} equals 0.
(3) $0 \leq a_{\text{old}} + b_{\text{old}} \leq A + B - k$.

It must be shown that $I(k+1)$ is true after the loop iteration. That means it is necessary to show that
(1) a_{new} and b_{new} are nonnegative integers with $\gcd(a_{\text{new}}, b_{\text{new}}) = \gcd(A, B)$.
(2) At most one of a_{new} and b_{new} equals 0.
(3) $0 \leq a_{\text{new}} + b_{\text{new}} \leq A + B - (k+1)$.

To show (3), observe that

$$a_{\text{new}} + b_{\text{new}} = \begin{cases} a_{\text{old}} - b_{\text{old}} + b_{\text{old}} & \text{if } a_{\text{old}} \geq b_{\text{old}} \\ b_{\text{old}} - a_{\text{old}} + a_{\text{old}} & \text{if } a_{\text{old}} < b_{\text{old}} \end{cases}$$

[The reason for this is that when $a_{old} \geq b_{old}$, then $a_{new} = a_{old} - b_{old}$ and $b_{new} = b_{old}$, and when $a_{old} < b_{old}$, then $b_{new} = b_{old} - a_{old}$ and $a_{new} = a_{old}$.]
Thus

$$a_{\text{new}} + b_{\text{new}} = \begin{cases} a_{\text{old}} & \text{if } a_{\text{old}} \geq b_{\text{old}} \\ b_{\text{old}} & \text{if } a_{\text{old}} < b_{\text{old}} \end{cases}$$

But since $a_{\text{old}} \neq 0$ and $b_{\text{old}} \neq 0$ and a_{old} and b_{old} are nonnegative integers, then $a_{\text{old}} \geq 1$ and $b_{\text{old}} \geq 1$. Hence $a_{\text{old}} - 1 \geq 0$ and $b_{\text{old}} - 1 \geq 0$ and $a_{\text{old}} \leq a_{\text{old}} + b_{\text{old}} - 1$ and $b_{\text{old}} \leq b_{\text{old}} + a_{\text{old}} - 1$. It follows that $a_{\text{new}} + b_{\text{new}} \leq a_{\text{old}} + b_{\text{old}} - 1 \leq (A + B - k) - 1$ by the truth of (3) going into the kth iteration. Hence $a_{\text{new}} + b_{\text{new}} < A + B - (k+1)$ by algebraic simplification.

Section 5.6

1. $a_1 = 1, a_2 = 2a_1 + 2 = 2 \cdot 1 + 2 = 4,$
$a_3 = 2a_2 + 3 = 2 \cdot 4 + 3 = 11,$
$a_4 = 2a_3 + 4 = 2 \cdot 11 + 4 = 26$

3. $c_0 = 1, c_1 = 1 \cdot (c_0)^2 = 1 \cdot (1)^2 = 1,$
$c_2 = 2(c_1)^2 = 2 \cdot (1)^2 = 2,$
$c_3 = 3(c_2)^2 = 3 \cdot (2)^2 = 12$

5. $s_0 = 1, s_1 = 1, s_2 = s_1 + 2s_0 = 1 + 2 \cdot 1 = 3,$
$s_3 = s_2 + 2s_1 = 3 + 2 \cdot 1 = 5$

7. $u_1 = 1, u_2 = 1, u_3 = 3u_2 - u_1 = 3 \cdot 1 - 1 = 2,$
$u_4 = 4u_3 - u_2 = 4 \cdot 2 - 1 = 7$

9. By definition of $a_0, a_1, a_2, \ldots$, for each integer $k \geq 1$,

(*) $\qquad a_k = 3k + 1$ and
(**) $\qquad a_{k-1} = 3(k-1) + 1.$

Then $a_{k-1} + 3$

$$\begin{aligned} &= 3(k-1) + 1 + 3 \\ &= 3k - 3 + 1 + 3 \\ &= 3k + 1 \\ &= a_k \end{aligned}$$

11. By definition of $c_0, c_1, c_2, \ldots, c_n = 2^n - 1$, for each integer $n \geq 0$. Substitute k and $k-1$ in place of n to get

(*) $\qquad c_k = 2^k - 1$ and
(**) $\qquad c_{k-1} = 2^{k-1} - 1$

for all integers $k \geq 1$. Then

$$\begin{aligned} 2c_{k-1} + 1 &= 2(2^{k-1} - 1) + 1 && \text{by substitution from (**)} \\ &= 2^k - 2 + 1 \\ &= 2^k - 1 && \text{by basic algebra} \\ &= c_k && \text{by substitution from (*)} \end{aligned}$$

13. By definition of $t_0, t_1, t_2, \ldots, t_n = 2 + n$, for each integer $n \geq 0$. Substitute k, $k-1$, and $k-2$ in place of n to get

(*) $\qquad t_k = 2 + k,$
(**) $\qquad t_{k-1} = 2 + (k-1),$ and
(***) $\qquad t_{k-2} = 2 + (k-2)$

for each integer $k \geq 2$. Then

$$\begin{aligned} &2t_{k-1} - t_{k-2} \\ &\quad = 2(2 + (k-1) - (2 + (k-2)) && \text{by substitution from (**) and (***)} \\ &\quad = 2(k+1) - k \\ &\quad = 2 + k && \text{by basic algebra} \\ &\quad = t_k && \text{by substitution from (*).} \end{aligned}$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

15. *Hint:* Mathematical induction is not needed for the proof. Start with the right-hand side of the equation and use algebra to transform it into the left-hand side of the equation.

17. a. $a_1 = 2$

$a_2 = 2$ (moves to move the top disk from pole A to pole C)
$+ 1$ (move to move the bottom disk from pole A to pole B)
$+ 2$ (moves to move the top disk from pole C to pole A)
$+ 1$ (move to move the bottom disk from pole B to pole C)
$+ 2$ (moves to move top disk from pole A to pole C)
$= 8$

$a_3 = 8 + 1 + 8 + 1 + 8 = 26$

c. For all integers $k \geq 2$.

$a_k = a_{k-1}$ (moves to move the top $k - 1$ disks from pole A to pole C)
$+ 1$ (move to move the bottom disk from pole A to pole B)
$+ a_{k-1}$ (moves to move the top disk from pole C to pole A)
$+ 1$ (move to move the bottom disks from pole B to pole C)
$+ a_{k-1}$ (moves to move the top disks from pole A to pole C)
$= 3a_{k-1} + 2$.

18. b. $b_4 = 40$

e. *Hint:* One solution is to use mathematical induction and apply the formula from part (c). Another solution is to prove by mathematical induction that when a most efficient transfer of n disks from one end pole to the other end pole is performed, at some point all the disks are on the middle pole.

19. a. $s_1 = 1,\ s_2 = 1 + 1 + 1 = 3$,
$s_3 = s_1 + (1 + 1 + 1) + s_1 = 5$

b. $s_4 = s_2 + (1 + 1 + 1) + s_2 = 9$

20. b. Call the poles A, B, and C. Compute c_2 by using the following sequence of steps to transfer two disks from A to B:

1 (move to move the top disk for A to B)
$+1$ (move to move the top disk from B to C)
$+1$ (move to move the bottom disk from A to B)
$+1$ (move to move the top disk from C to A)
$+1$ (move to move the top disk from A to B)

This sequence of steps is the least possible, and so $c_2 = 5$.

A tower of 3 disks can be transferred from A to B by using the following sequence of steps:

1 (move to move the top disk from A to B)
$+1$ (move to move the top disk from B to C)
$+1$ (move to move the middle disk from A to B)
$+1$ (move to move the top disk from C to A)
$+1$ (move to move the middle disk from B to C)
$+1$ (move to move the top disk from A to B)
$+1$ (move to move the top disk from B to C).

After these 7 steps have been completed, the bottom disk can be moved from A to B. At that point the top two disks are on C, and a modified version of the initial seven steps can be used to move them from C to B. Thus the total number of steps is $7 + 1 + 7 = 15$, and $15 < 21 = 4c_2 + 1$.

21. b. $t_3 = 14$

22. b. $r_0 = 1,\ r_1 = 1,\ r_2 = 1 + 4\cdot 1 = 5,\ r_3 = 5 + 4\cdot 1 = 9$,
$r_4 = 9 + 4\cdot 5 = 29,\ r_5 = 29 + 4\cdot 9 = 65$,
$r_6 = 65 + 4\cdot 29 = 181$

23. c. There are 904 rabbit pairs, or 1,808 rabbits, after 12 months.

25. a. Each term of the Fibonacci sequence beyond the second equals the sum of the previous two. For any integer $k \geq 1$, the two terms previous to F_{k+1} are F_k and F_{k-1}. Hence, for all integers $k \geq 1$, $F_{k+1} = F_k + F_{k-1}$.

26. By repeated use of definition of the Fibonacci sequence, for all integers $k \geq 4$,

$$\begin{aligned} F_k &= F_{k-1} + F_{k-2} = (F_{k-2} + F_{k-3}) + (F_{k-3} + F_{k-4}) \\ &= ((F_{k-3} + F_{k-4}) + F_{k-3}) + (F_{k-3} + F_{k-4}) \\ &= 3F_{k-3} + 2F_{k-4}. \end{aligned}$$

27. For all integers $k \geq 1$,

$F_k^2 - F_{k-1}^2$

$= (F_k - F_{k-1})(F_k + F_{k-1})$ by basic algebra (difference of two squares)

$= (F_k - F_{k-1})F_{k+1}$ by definition of the Fibonacci sequence

$= F_k F_{k+1} - F_{k-1} F_{k+1}$

32. *Hint:* Use mathematical induction. In the inductive step, use Lemma 4.8.2 and the fact that $F_{k+2} = F_{k+1} + F_k$ to deduce that

$$\gcd(F_{k+2}, F_{k+1}) = \gcd(F_{k+1}, F_k).$$

34. *Hint:* Let $L = \lim_{n\to\infty} \dfrac{F_{n+1}}{F_n}$ and show that $L = \dfrac{1}{L} + 1$. Deduce that $L = \frac{1+\sqrt{5}}{2}$.

35. *Hint:* Use the result of exercise 30 to prove that the infinite sequence $\frac{F_0}{F_1}, \frac{F_2}{F_3}, \frac{F_4}{F_5}, \ldots$ is strictly decreasing and that the infinite sequence $\frac{F_1}{F_2}, \frac{F_3}{F_4}, \frac{F_5}{F_6}, \ldots$ is strictly increasing. The first sequence is bounded below by 0, and the second sequence is bounded above by 1. Deduce that the limits of both sequences exist, and show that they are equal.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

37. a. Because the 4% annual interest is compounded quarterly, the quarterly interest rate is $(4\%)/4 = 1\%$. Then $R_k = R_{k-1} + 0.01R_{k-1} = 1.01R_{k-1}$.

b. Because one year equals four quarters, the amount on deposit at the end of one year is $R_4 = \$5203.02$ (rounded to the nearest cent).

c. The annual percentage rate (APR) for the account is $\frac{\$5203.02-\$5000.00}{\$5000.00} = 4.0604\%$.

39. When one is climbing a staircase consisting of n stairs, the last step taken is either a single stair or two stairs together. The number of ways to climb the staircase and have the final step be a single stair is c_{n-1}; the number of ways to climb the staircase and have the final step be two stairs is c_{n-2}. Therefore, $c_n = c_{n-1} + c_{n-2}$. Note also that $c_1 = 1$ and $c_2 = 2$ *[because either the two stairs can be climbed one by one or they can be climbed as a unit].*

41. Proof (by mathematical induction): Let the property, $P(n)$, be the equation $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, where $a_1, a_2, a_3, \ldots, a_n$ and c are any real numbers.

Show that P(1) is true:

Let a_1 and c be any real numbers. By the recursive definition of sum, $\sum_{i=1}^{1}(ca_i) = ca_1$ and $\sum_{i=1}^{1} a_i = a_1$. Therefore, $\sum_{i=1}^{1}(ca_i) = c\sum_{i=1}^{1} a_i$, and so $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$. Suppose that for any real numbers $a_1, a_2, a_3, \ldots, a_k$ and c, $\sum_{i=1}^{k}(ca_i) = c\sum_{i=1}^{k} a_i$. *[This is the inductive hypothesis]. [We must show that for any real numbers $a_1, a_2, a_3, \ldots a_{k+1}$ and c, $\sum_{i=1}^{k+1}(ca_i) = c\sum_{i=1}^{k+1} a_i$.]*

Let $a_1, a_2, a_3, \ldots, a_{k+1}$ and c be any real numbers. Then

$$\begin{aligned}\sum_{i=1}^{k+1} ca_i &= \sum_{i=1}^{k} ca_i + ca_{k+1} && \text{by the recursive definition of } \Sigma\\ &= c\sum_{i=1}^{k} a_i + ca_{k+1} && \text{by inductive hypothesis}\\ &= c\left(\sum_{i=1}^{k} a_i + a_{k+1}\right) && \text{by the distributive law for the real numbers}\\ &= c\sum_{i=1}^{k+1} a_i && \text{by the recursive definition of } \Sigma.\end{aligned}$$

44. *Hint:* Let the property be the inequality

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

To prove the inductive step, note that because $\left|\sum_{i=1}^{k+1} a_i\right| = \left|\sum_{i=1}^{k} a_i + a_{k+1}\right|$, you can use the triangle inequality for absolute value (Theorem 4.4.6) to deduce $\left|\sum_{i=1}^{k} a_i + a_{k+1}\right| \leq \left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}|$.

Section 5.7

1. a. $1 + 2 + 3 + \cdots + (k-1)$

$$= \frac{(k-1)((k-1)+1)}{2} = \frac{(k-1)k}{2}$$

b. $3 + 2 + 4 + 6 + 8 + \cdots + 2n$

$$\begin{aligned}&= 3 + 2(1 + 2 + 3 + \cdots + n)\\ &= 3 + 2\frac{n(n+1)}{2} = 3 + n(n+1)\\ &= n^2 + n + 3\end{aligned}$$

2. a. $1 + 2 + 2^2 + \cdots + 2^{i-1} = \dfrac{2^{(i-1)+1} - 1}{2-1} = 2^i - 1$

c. $2^n + 2^{n-2}\cdot 3 + 2^{n-3}\cdot 3 + \cdots + 2^2\cdot 3 + 2\cdot 3 + 3$

$$\begin{aligned}&= 2^n + 3(2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1)\\ &= 2^n + 3(1 + 2 + 2^2 + \cdots + 2^{n-3} + 2^{n-2})\\ &= 2^n + 3\left(\frac{2^{(n-2)+1} - 1}{2-1}\right)\\ &= 2^n + 3(2^{n-1} - 1)\\ &= 2\cdot 2^{n-1} + 3\cdot 2^{n-1} - 3\\ &= 5\cdot 2^{n-1} - 3\end{aligned}$$

3. $a_0 = 1$

$a_1 = 1\cdot a_0 = 1\cdot 1 = 1$

$a_2 = 2a_1 = 2\cdot 1$

$a_3 = 3a_2 = 3\cdot 2\cdot 1$

$a_4 = 4a_3 = 4\cdot 3\cdot 2\cdot 1$

$\vdots$

Guess:

$a_n = n(n-1)\cdots 3\cdot 2\cdot 1 = n!$

5. $c_1 = 1$

$c_2 = 3c_1 + 1 = 3\cdot 1 + 1 = 3 + 1$

$c_3 = 3c_2 + 1 = 3\cdot(3+1) + 1 = 3^2 + 3 + 1$

$c_4 = 3c_3 + 1 = 3\cdot(3^2 + 3 + 1) + 1$

$= 3^3 + 3^2 + 3 + 1$

$\vdots$

Guess:

$$\begin{aligned}c_n &= 3^{n-1} + 3^{n-2} + \cdots + 3^3 + 3^2 + 3 + 1\\ &= \frac{3^n - 1}{3-1} \qquad \text{by Theorem 5.2.3 with } r = 3\\ &= \frac{3^n - 1}{2}\end{aligned}$$

6. *Hint:*

$d_n = 2^n + 2^{n-2}\cdot 3 + 2^{n-3}\cdot 3 + \cdots + 2^2\cdot 3 + 2\cdot 3 + 3$

$= 5\cdot 2^{n-1} - 3$ for all integers $n \geq 1$

9. *Hint:* For any positive real numbers a and b,

$$\frac{\frac{a}{b}}{\frac{a}{b} + 2} = \frac{\frac{a}{b}}{\frac{a}{b} + 2}\cdot\frac{b}{b} = \frac{a}{a + 2b}.$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

10. $h_0 = 1$

$h_1 = 2^1 - h_0 = 2^1 - 1$

$h_2 = 2^2 - h_1 = 2^2 - (2^1 - 1) = 2^2 - 2^1 + 1$

$h_3 = 2^3 - h_2 = 2^3 - (2^2 - 2^1 + 1)$

$= 2^3 - 2^2 + 2^1 - 1$

$h_4 = 2^4 - h_3 = 2^4 - (2^3 - 2^2 + 2^2 - 1)$

$= 2^4 - 2^3 + 2^2 - 2^1 + 1$

$\vdots$

Guess:

$$
\begin{aligned}
h_n &= 2^n - 2^{n-1} + \cdots + (-1)^n \cdot 1 \\
&= (-1)^n[1 - 2 + 2^2 - \cdots + (-1)^n \cdot 2^n] \\
&= (-1)^n[1 + (-2) \\
&\quad + (-2)^2 - \cdots + (-2)^n] && \text{by basic algebra} \\
&= (-1)^n\left[\frac{(-2)^{n+1} - 1}{(-2) - 1}\right] && \text{by Theorem 5.2.3} \\
&= \frac{(-1)^{n+1} \cdot [(-2)^{n+1} - 1]}{(-1) \cdot (-3)} \\
&= \frac{2^{n+1} - (-1)^{n+1}}{3} && \text{by basic algebra}
\end{aligned}
$$

12. $s_0 = 3$

$s_1 = s_0 + 2 \cdot 1 = 3 + 2 \cdot 1$

$s_2 = s_1 + 2 \cdot 2 = [3 + 2 \cdot 1] + 2 \cdot 2$

$= 3 + 2 \cdot (1 + 2)$

$s_3 = s_2 + 2 \cdot 3 = [3 + 2 \cdot (1 + 2)] + 2 \cdot 3$

$= 3 + 2 \cdot (1 + 2 + 3)$

$s_4 = s_3 + 2 \cdot 4 = [3 + 2 \cdot (1 + 2 + 3)] + 2 \cdot 4$

$= 3 + 2 \cdot (1 + 2 + 3 + 4)$

$\vdots$

Guess:

$$
\begin{aligned}
s_n &= 3 + 2 \cdot (1 + 2 + 3 + \cdots + (n-1) + n) \\
&= 3 + 2 \cdot \frac{n(n+1)}{2} && \text{by Theorem 5.2.2} \\
&= 3 + n(n+1) && \text{by basic algebra}
\end{aligned}
$$

14. $x_1 = 1$

$x_2 = 3x_1 + 2 = 3 + 2$

$x_3 = 3x_2 + 3 = 3(3 + 2) + 3 = 3^2 + 3 \cdot 2 + 3$

$x_4 = 3x_3 + 4 = 3(3^2 + 3 \cdot 2 + 3) + 4$

$= 3^3 + 3^2 \cdot 2 + 3 \cdot 3 + 4$

$x_5 = 3x_4 + 5 = 3(3^3 + 3^2 \cdot 2 + 3 \cdot 3 + 4) + 5$

$= 3^4 + 3^3 \cdot 2 + 3^2 \cdot 3 + 3 \cdot 4 + 5$

$x_6 = 3x_5 + 6$

$= 3(3^4 + 3^3 \cdot 2 + 3^2 \cdot 3 + 4 \cdot 3 + 5) + 6$

$= 3^5 + 3^4 \cdot 2 + 3^3 \cdot 3 + 3^2 \cdot 4 + 3 \cdot 5 + 6$

$\vdots$

Guess:

$$
\begin{aligned}
x_n &= 3^{n-1} + 3^{n-2} \cdot 2 + 3^{n-3} \cdot 3 + \cdots + 3(n-1) + n \\
&= 3^{n-1} + \underbrace{3^{n-2} + 3^{n-2}}_{\text{2 times}} + \underbrace{3^{n-3} + 3^{n-3} + 3^{n-3}}_{\text{3 times}} + \\
&\qquad + \underbrace{3 + 3 + \cdots + 3}_{(n-1)\text{ times}} + \underbrace{1 + 1 + \cdots + 1}_{n\text{ times}} \\
&= (3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1) \\
&\qquad + (3^{n-2} + 3^{n-3} + \cdots + 3^2 + 3 + 1) + \cdots \\
&\qquad + (3^2 + 3 + 1) + (3 + 1) + 1 \\
&= \frac{3^n - 1}{2} + \frac{3^{n-1} - 1}{2} + \cdots + \frac{3^3 - 1}{2} \\
&\qquad + \frac{3^2 - 1}{2} + \frac{3 - 1}{2} \\
&= \frac{1}{2}[(3^n + 3^{n-1} + \cdots + 3^2 + 3) - n] \\
&= \frac{1}{2}[3(3^{n-1} + 3^{n-2} + \cdots + 3 + 1) - n] \\
&= \frac{1}{2}\left(3\left(\frac{3^n - 1}{3 - 1}\right) - n\right) \\
&= \frac{1}{4}(3^{n+1} - 3 - 2n)
\end{aligned}
$$

18. Proof: Let d be any fixed constant, and let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively by $a_k = a_{k-1} + d$ for all integers $k \geq 1$. The property $P(n)$ is the equation $a_n = a_0 + nd$. We show by mathematical induction that $P(n)$ is true for all integers $n \geq 0$.

Show that P(0) is true:

When $n = 0$, the left-hand side of the equation is a_0, and the right-hand side is $a_0 + 0 \cdot d = a_0$, which equals the left-hand side. Thus $P(0)$ is true.

Show that for all integers $k \geq 0$, if $P(k)$ is true, then $P(k+1)$ is true:

Suppose

$a_k = a_0 + kd$, for some integer $k \geq 0$.

[This is the inductive hypothesis.]

We must show that $a_{k+1} = a_0 + (k+1)d$. But

$$
\begin{aligned}
a_{k+1} &= a_k + d && \text{by definition of } a_0, a_1, a_2, \ldots \\
&= [a_0 + kd] + d && \text{by substitution from the inductive hypothesis} \\
&= a_0 + (k+1)d && \text{by basic algebra}
\end{aligned}
$$

[as was to be shown].

19. Let U_n = the number of units produced on day n. Then

$U_k = U_{k-1} + 2$ for all integers $k \geq 1$,

$U_0 = 170$.

Hence $U_0, U_1, U_2, \ldots$ is an arithmetic sequence with fixed constant 2. It follows that when $n = 30$,

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$U_n = U_0 + n \cdot 2 = 170 + 2n = 170 + 2 \cdot 30$$
$$= 230 \text{ units.}$$

Thus the worker must produce 230 units on day 30.

24. $\displaystyle\sum_{k=0}^{20} 5^k = \frac{5^{21}-1}{4} \cong 1.192 \times 10^{14} \cong$ 119,200,000,000,000 $\cong$ 119 trillion people (This is about 20,000 times the current population of the earth!)

26. b. *Hint:* Before simplification,
$A_n = 1000(1.0025)^n + 200[(1.0025)^{n-1} + (1.0025)^{n-1} + \cdots + (1.0025)^2 + 1.0025 + 1]$.

d. $A_{240} \cong \$67{,}481.15$, $A_{480} \cong \$188{,}527.05$

e. *Hint:* Use logarithms to solve the equation $A_n = 10{,}000$, where A_n is the expression found (after simplification) in part (b).

27. a. *Hint:* APR $\cong$ 19.6%

c. *Hint:* approximately two years

28. Proof: Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively by $a_0 = 1$ and $a_k = ka_{k-1}$ for all integers $k \geq 1$. Let the property $P(n)$ be the equation $a_n = n!$. We show by mathematical induction that $P(n)$ is true for all integers $n \geq 0$.

Show that P(0) is true:

When $n = 0$, the right-hand side of the equation is $0! = 1$, and by definition of $a_0, a_1, a_2, \ldots$, the left-hand side of the equation, a_0, is also 1. Thus the property is true for $n = 0$.

Show that for all integers k ≥ 0, if P(k) is true, then P(k + 1) is true:

Suppose

$a_k = k!$ for some integer $k \geq 0$.

[This is the inductive hypothesis.]

We must show that $a_{k+1} = (k+1)!$. But

$$\begin{aligned} a_{k+1} &= (k+1) \cdot a_k && \text{by definition of } a_0, a_1, a_2, \ldots \\ &= (k+1) \cdot k! && \text{by substitution from the inductive hypotheses} \\ &= (k+1)! && \text{by definition of factorial.} \end{aligned}$$

[Hence if P(k) is true, then P(k + 1) is true.]

30. Proof: Let $c_1, c_2, c_3, \ldots$ be the sequence defined recursively by $c_1 = 1$ and $c_k = 3c_{k-1} + 1$ for all integers $k \geq 2$. Let the property $P(n)$ be the equation $c_n = \frac{3^n-1}{2}$. We show by mathematical induction that $P(n)$ is true for all integers $n \geq 1$.

Show that P(1) is true:

When $n = 1$, the right-hand side of the equation is $\frac{3^1-1}{2} = \frac{3-1}{2} = 1$, and by definition of $c_1, c_2, c_3, \ldots$, the left-hand side of the equation, c_1, is also 1. Thus the property is true for $n = 1$.

Show that for all integers k ≥ 1, if P(k) is true, then P(k + 1) is true:

Suppose that

$$c_k = \frac{3^k - 1}{2} \quad \text{for some integer } k \geq 1.$$

[This is the inductive hypothesis.]

We must show that $c_{k+1} = \dfrac{3^{k+1}-1}{2}$. But

$$\begin{aligned} c_{k+1} &= 3c_k + 1 && \text{by definition of } c_1, c_2, c_3, \ldots \\ &= 3\left(\frac{3^k-1}{2}\right) + 1 && \text{by substitution from the inductive hypothesis} \\ &= \frac{3^{k+1}-3}{2} + \frac{2}{2} \\ &= \frac{3^{k+1}-1}{2} && \text{by basic algebra.} \end{aligned}$$

35. *Hint:*
$$\begin{aligned} &2^{k+1} - \frac{2^{k+1} - (-1)^{k+1}}{3} \\ &= \frac{3 \cdot 2^{k+1}}{3} - \frac{2^{k+1} - (-1)^{k+1}}{3} \\ &= \frac{2 \cdot 2^{k+1} + (-1)^{k+1}}{3} = \frac{2^{k+2} - (-1)^{k+2}}{3} \end{aligned}$$

37. *Hint:*
$$\begin{aligned} &[3 + k(k+1)] + 2(k+1) \\ &= 3 + k^2 + k + 2k + 2 = 3 + [k^2 + 3k + 2] \\ &= 3 + (k+1)(k+2) \\ &= 3 + (k+1)[(k+1)+1] \end{aligned}$$

39. Proof: Let $x_1, x_2, x_3, \ldots$ be the sequence defined recursively by $x_1 = 1$ and $x_k = 3x_{k-1} + k$ for all integers $k \geq 2$. Let the property, $P(n)$, be the equation $x_n = \frac{3^{n+1}-2n-3}{4}$. We show by mathematical induction that $P(n)$ is true for all integers $n \geq 1$.

Show that P(1) is true:

When $n = 1$, the right-hand side of the equation is $\frac{3^{1+1}-2\cdot1-3}{4} = \frac{3^2-2-3}{4} = 1$, and by definition of $x_1, x_2, x_3, \ldots$, the left-hand side of the equation, x_1, is also 1. Thus $P(1)$ is true.

Show that for all integers k ≥ 1, if P(k) is true for, then P(k + 1) is true.

Suppose that for some integer $k \geq 0$, $x_k = \frac{3^{k+1}-2k-3}{4}$. *[Inductive hypothesis]* We must show that

$x_{k+1} = \dfrac{3^{(k+1)+1} - 2(k+1) - 3}{4}$, or, equivalently,

$x_{k+1} = \dfrac{3^{k+2} - 2k - 5}{4}$. But

$$\begin{aligned} x_{k+1} &= 3x_k + k && \text{by definition of } x_1, x_2, x_3, \\ &= 3\left(\frac{3^{k+1} - 2k - 3}{4}\right) + k + 1 && \text{by inductive hypothesis} \\ &= \frac{3 \cdot 3^{k+1} - 3 \cdot 2k - 3 \cdot 3}{4} + \frac{4(k+1)}{4} \end{aligned}$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$= \frac{3^{k+2} - 6k - 9 + 4k + 4}{4}$$

$$= \frac{3^{k+2} - 2k - 5}{4} \quad \text{by algebra.}$$

[This is what was to be shown.]

43. a. $a_0 = 2$

$$a_1 = \frac{a_0}{2a_0 - 1} = \frac{2}{2 \cdot 2 - 1} = \frac{2}{3}$$

$$a_2 = \frac{a_1}{2a_1 - 1} = \frac{\frac{2}{3}}{2 \cdot \frac{2}{3} - \frac{3}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

$$a_3 = \frac{a_2}{2a_2 - 1} = \frac{2}{2 \cdot 2 - 1} = \frac{2}{3}$$

$$a_4 = \frac{a_3}{2a_3 - 1} = \frac{\frac{2}{3}}{2 \cdot \frac{2}{3} - \frac{3}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

Guess: $a_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2}{3} & \text{if } n \text{ is odd} \end{cases}$.

b. Proof: Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively by $x_0 = 2$ and $a_k = \frac{a_{k-1}}{2a_{k-1} - 1}$ for all integers $k \geq 1$. Let the property, $P(n)$, be the equation

$$a_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2}{3} & \text{if } n \text{ is odd} \end{cases}.$$

We show by strong mathematical induction that $P(n)$ is true for all integers $n \geq 1$.

Show that P(0) and P(1) are true:

The results of part (a) show that $P(0)$ and $P(1)$ are true.

Show that for all integers k ≥ 0, if P(k) is true for all integers i with 0 ≤ i ≤ k, then P(k + 1) is true:

Let k be any integer with $k \geq 0$, and suppose that for all integers i with $0 \leq i \leq k$,

$$a_i = \begin{cases} 2 & \text{if } i \text{ is even} \\ \frac{2}{3} & \text{if } i \text{ is odd} \end{cases}. \quad \textit{[Inductive hypothesis]}$$

We must show that

$$a_{k+1} = \begin{cases} 2 & \text{if } k \text{ is even} \\ \frac{2}{3} & \text{if } k \text{ is odd} \end{cases}.$$

But

$$a_{k+1} = \frac{a_k}{2a_k - 1} \quad \text{by definition of } a_0, a_1, a_2, \ldots$$

$$= \begin{cases} \frac{2}{2 \cdot 2 - 1} & \text{if } k \text{ is even} \\ \frac{\frac{2}{3}}{2 \cdot \frac{2}{3} - 1} & \text{if } k \text{ is odd} \end{cases} \quad \text{by inductive hypothesis}$$

$$= \begin{cases} \frac{2}{3} & \text{if } k \text{ is even} \\ \frac{\frac{2}{3}}{\frac{1}{3}} & \text{if } k \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{2}{3} & \text{if } k+1 \text{ is odd} \\ 2 & \text{if } k+1 \text{ is even} \end{cases} \quad \text{because } k+1 \text{ is odd when } k \text{ is even and } k+1 \text{ is even when } k \text{ is odd.}$$

[This is what was to be shown.]

45. $v_1 = 1$

$v_2 = v_{\lfloor 2/2 \rfloor} + v_{\lfloor 3/2 \rfloor} + 2 = v_1 + v_1 + 2$
$= 1 + 1 + 2$

$v_3 = v_{\lfloor 3/2 \rfloor} + v_{\lfloor 4/2 \rfloor} + 2 = v_1 + v_2 + 2$
$= 1 + (1 + 1 + 2) + 2 = 3 + 2 \cdot 2$

$v_4 = v_{\lfloor 4/2 \rfloor} + v_{\lfloor 5/2 \rfloor} + 2 = v_2 + v_2 + 2$
$= (1 + 1 + 2) + (1 + 1 + 2) + 2$
$= 4 + 3 \cdot 2$

$v_5 = v_{\lfloor 5/2 \rfloor} + v_{\lfloor 6/2 \rfloor} + 2 = v_2 + v_3 + 2$
$= (3 + 2 \cdot 2) + (1 + 1 + 2) + 2$
$= 5 + 4 \cdot 2$

$v_6 = v_{\lfloor 6/2 \rfloor} + v_{\lfloor 7/2 \rfloor} + 2 = v_3 + v_3 + 2$
$= (3 + 2 \cdot 2) + (3 + 2 \cdot 2) + 2$
$= 6 + 5 \cdot 2$

$\vdots$

Guess:

$v_n = n + 2(n-1) = 3n - 2$ for all integers $n \geq 1$

b. Proof: Let $v_1, v_2, v_3, \ldots$ be the sequence defined recursively by $v_1 = 1$ and $v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2$ for all integers $k \geq 1$. Let the property, $P(n)$, be the equation

$$v_n = 3n - 2.$$

We show by strong mathematical induction that $P(n)$ is true for all integers $n \geq 1$.

Show that P(1) is true:

When $n = 1$, the right-hand side of the equation is $3 \cdot 1 - 2 = 1$, which equals v_1 by definition of $v_1, v_2, v_3, \ldots$. Thus $P(1)$ is true.

Show that for all integers k ≥ 1, if P(i) is true for all integers i with 0 ≤ i ≤ k, then P(k + 1) is true:

Let k be any integer with $k \geq 1$, and suppose that for all integers i with $1 \leq i \leq k$, $v_i = 3i - 2$.
[This is the inductive hypothesis.] We must show that $v_{k+1} = 3(k+1) - 2 = 3k + 1$.

$$v_{k+1} = v_{\lfloor (k+1)/2 \rfloor} + v_{\lfloor (k+2)/2 \rfloor} + 2 \quad \text{by definition of } v_1, v_2, v_3, \ldots$$

$$= \left(3\left\lfloor \frac{k+1}{2} \right\rfloor - 2\right) + \left(3\left\lfloor \frac{k+2}{2} \right\rfloor - 2\right) + 2$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$= 3\left(\left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor\right) - 2$$

$$= \begin{cases} 3\left(\frac{k}{2} + \frac{k+2}{2}\right) - 2 & \text{if } k \text{ is even} \\ 3\left(\frac{k+1}{2} + \frac{k+1}{2}\right) - 2 & \text{if } k \text{ is odd} \end{cases}$$

$$= 3\left(\frac{2k+2}{2}\right) - 2$$

$$= 3(k+1) - 2$$

$$= 3k + 1 \qquad \text{by the laws of algebra.}$$

[This is what was to be shown.]

46. *Hint:* Show that for all integers $n \geq 0$, $s_{2n} = 2^n$ and $s_{2n+1} = 2^{n+1}$. Then combine these formulas using the ceiling function to obtain $s_n = 2^{\lceil n/2 \rceil}$.

48. a. *Hint:* $w_n = \begin{cases} \left(\frac{n+1}{2}\right)^2 & \text{if } n \text{ is odd} \\ \frac{n}{2}\left(\frac{n}{2} + 1\right) & \text{if } n \text{ is even} \end{cases}$

49. a. *Hint:* Express the answer using the Fibonacci sequence.

50. The sequence does not satisfy the formula. According to the formula, $a_4 = (4-1)^2 = 9$. But by definition of the sequence, $a_1 = 0$, $a_2 = 2 \cdot 0 + (2+1) = 1$, $a_3 = 2 \cdot 1 + (3-1) = 4$, and so $a_4 = 2 \cdot 4 + (4-1) = 11$. Hence the sequence does not satisfy the formula for $n = 4$.

52. a. *Hint:* The maximum number of regions is obtained when each additional line crosses all the previous lines, but not at any point that is already the intersection of two lines. When a new line is added, it divides each region through which it passes into two pieces. The number of regions a newly added line passes through is one more than the number of lines it crosses.

53. *Hint:* The answer involves the Fibonacci numbers!

Section 5.8

1. (a), (d), and (f)

3. a. $\left.\begin{aligned} a_0 &= C \cdot 2^0 + D = C + D = 1 \\ a_1 &= C \cdot 2^1 + D = 2C + D = 3 \end{aligned}\right\}$

$$\Leftrightarrow \begin{Bmatrix} D = 1 - C \\ 2C + (1-C) = 3 \end{Bmatrix} \Leftrightarrow \begin{cases} C = 2 \\ D = -1 \end{cases}$$

$a_2 = 2 \cdot 2^2 + (-1) = 7$

4. a. $\left.\begin{aligned} b_0 &= C \cdot 3^0 + D \cdot (-2)^0 = C + D = 0 \\ b_1 &= C \cdot 3^1 + D \cdot (-2)^1 = 3C - 2D = 5 \end{aligned}\right\}$

$$\Leftrightarrow \begin{Bmatrix} D = -C \\ 3C - 2(-C) = 5 \end{Bmatrix} \Leftrightarrow \begin{cases} C = 1 \\ D = -1 \end{cases}$$

$b_2 = 3^2 + (-1)(-2)^2 = 9 - 4 = 5$

5. <u>Proof</u>: Given that $a_n = C \cdot 2^n + D$, then for any choice of C and D and integer $k > 2$,

$$\begin{aligned} a_k &= C \cdot 2^k + D, \\ a_{k-1} &= C \cdot 2^{k-1} + D, \\ a_{k-2} &= C \cdot 2^{k-2} + D. \end{aligned}$$

Hence

$$\begin{aligned} 3a_{k-1} - 2a_{k-2} &= 3(C \cdot 2^{k-1} + D) - 2(C \cdot 2^{k-2} + D) \\ &= 3C \cdot 2^{k-1} + 3D - 2C \cdot 2^{k-2} - 2D \\ &= 3C \cdot 2^{k-1} - C \cdot 2^{k-1} + D \\ &= 2C \cdot 2^{k-1} + D \\ &= C \cdot 2^k + D = a_k. \end{aligned}$$

8. a. If for all $k > 2$, $t^k = 2t^{k-1} + 3t^{k-2}$ and $t \neq 0$ then $t^2 = 2t + 3$ *[by dividing by t^{k-2}]*, and so $t^2 - 2t - 3 = 0$. But $t^2 - 2t - 3 = (t-3)(t+1)$; hence $t = 3$ or $t = -1$.

b. It follows from (a) and the distinct roots theorem that for some constants C and D, $a_0, a_1, a_2, \ldots$ satisfies the equation

$$a_n = C \cdot 3^n + D \cdot (-1)^n \quad \text{for all integers } n \geq 0.$$

Since $a_0 = 1$ and $a_1 = 2$, then

$$\left.\begin{aligned} a_0 &= C \cdot 3^0 + D \cdot (-1)^0 = C + D = 1 \\ a_1 &= C \cdot 3^1 + D \cdot (-1)^1 = 3C - D = 2 \end{aligned}\right\}$$

$$\Leftrightarrow \begin{Bmatrix} D = 1 - C \\ 3C - (1-C) = 2 \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} D = 1 - C \\ 4C - 1 = 2 \end{Bmatrix}$$

$$\Leftrightarrow \begin{cases} C = 3/4 \\ D = 1/4 \end{cases}$$

Thus $a_n = \frac{3}{4}(3^n) + \frac{1}{4}(-1)^n$ for all integers $n \geq 0$.

11. *Characteristic equation:* $t^2 - 4 = 0$. Since $t^2 - 4 = (t-2)(t+2)$, $t = 2$ and $t = -2$ are the roots. By the distinct roots theorem, for some constants C and D

$$d_n = C \cdot (2^n) + D \cdot (-2)^n \quad \text{for all integers } n \geq 0.$$

Since $d_0 = 1$ and $d_1 = -1$, then

$$\left.\begin{aligned} d_0 &= C \cdot 2^0 + D \cdot (-2)^0 = C + D = 1 \\ d_1 &= C \cdot 2^1 + D \cdot (-2)^1 = 2C - 2D = -1 \end{aligned}\right\}$$

$$\Leftrightarrow \begin{Bmatrix} D = 1 - C \\ 2C - 2(1-C) = -1 \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} D = 1 - C \\ 4C - 2 = -1 \end{Bmatrix}$$

$$\Leftrightarrow \begin{cases} C = \frac{1}{4} \\ D = \frac{3}{4} \end{cases}$$

Thus $d_n = \frac{1}{4}(2^n) + \frac{3}{4}(-2)^n$ for all integers $n \geq 0$.

13. *Characteristic equation:* $t^2 - 2t + 1 = 0$. By the quadratic formula,

$$t = \frac{2 \pm \sqrt{4 - 4 \cdot 1}}{2} = \frac{2}{2} = 1.$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

By the single root theorem, for some constants C and D

$$\begin{aligned} r_n &= C\cdot(1^n) + Dn\cdot(1^n) \\ &= C + nD \quad \text{for all integers } n \geq 0. \end{aligned}$$

Since $r_0 = 1$ and $r_1 = 4$, then

$$\left.\begin{array}{l} r_0 = C + 0\cdot D = C = 1 \\ r_1 = C + 1\cdot D = C + D = 4 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C = 1 \\ 1 + D = 4 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} C = 1 \\ D = 3 \end{array}\right.$$

Thus $r_n = 1 + 3n$ for all integers $n \geq 0$.

16. *Hint:* For all integers $n \geq 0$,

$$s_n = \frac{\sqrt{3}+2}{2\sqrt{3}}\left(1+\sqrt{3}\right)^n + \frac{\sqrt{3}-2}{2\sqrt{3}}\left(1-\sqrt{3}\right)^n.$$

19. Proof: Suppose r, s, a_0, and a_1 are numbers with $r \neq s$. Consider the system of equations

$$\begin{aligned} C + D &= a_0 \\ Cr + Ds &= a_1. \end{aligned}$$

By solving for D and substituting, we find that

$$\begin{aligned} D &= a_0 - C \\ Cr + (a_0 - C)s &= a_1. \end{aligned}$$

Hence

$$C(r - s) = a_1 - a_0 s.$$

Since $r \neq s$, both sides may be divided by $r - s$. Thus the given system of equations has the unique solution

$$C = \frac{a_1 - a_0 s}{r - s}$$

and

$$\begin{aligned} D = a_0 - C &= a_0 - \frac{a_1 - a_0 s}{r - s} \\ &= \frac{a_0 r - a_0 s - a_1 + a_0 s}{r - s} = \frac{a_0 r - a_1}{r - s}. \end{aligned}$$

Alternative solution: Since the determinant of the system is $1\cdot s - r\cdot 1 = s - r$ and since $r \neq s$, the given system has a nonzero determinant and therefore has a unique solution.

21. *Hint:* Use strong mathematical induction. First note that the formula holds for $n = 0$ and $n = 1$. To prove the inductive step, suppose that for some $k \geq 2$, the formula holds for all i with $0 \leq i \leq k$. Then show that the formula holds for $k + 1$. Use the proof of Theorem 5.8.3 (the distinct roots theorem) as a model.

22. The characteristic equation is $t^2 - 2t + 2 = 0$. By the quadratic formula, its roots are

$$t = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = \begin{cases} 1 + i \\ 1 - i \end{cases}.$$

By the distinct roots theorem, for some constants C and D

$$a_n = C(1+i)^n + D(1-i)^n$$

for all integers $n \geq 0$.

Since $a_0 = 1$ and $a_1 = 2$, then

$$\begin{aligned} a_0 &= C(1+i)^0 + D(1-i)^0 = C + D = 1 \\ a_1 &= C(1+i)^1 + D(1-i)^1 \\ &= C(1+i) + D(1-i) = 2 \end{aligned}$$

$$\Leftrightarrow \left\{\begin{array}{l} D = 1 - C \\ C(1+i) + (1-C)(1-i) = 2 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} D = 1 - C \\ C(1 + i - 1 + i) + 1 - i = 2 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} D = 1 - C \\ C(2i) = 1 + i \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} D = 1 - C \\ C = \dfrac{1+i}{2i} = \dfrac{1+i}{2i}\cdot\dfrac{i}{i} = \dfrac{i-1}{-2} = \dfrac{1-i}{2} \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} D = 1 - \dfrac{1-i}{2} = \dfrac{2-1+i}{2} = \dfrac{1+i}{2} \\ C = \dfrac{1-i}{2} \end{array}\right\}$$

Thus for all integers $n \geq 0$,

$$a_n = \left(\frac{1-i}{2}\right)(1+i)^n + \left(\frac{1+i}{2}\right)(1-i)^n.$$

Section 5.9

1. a. (1) p, q, r, and s are Boolean expressions by I.
(2) $\sim s$ is a Boolean expression by (1) and II(c).
(3) $(r \vee \sim s)$ is a Boolean expression by (1), (2), and II(b).
(4) $(q \wedge (r \vee \sim s))$ is a Boolean expression by (1), (3), and II(a).
(5) $\sim p$ is a Boolean expression by (1) and II(c).
(6) $(\sim p \vee (q \wedge (r \vee \sim s)))$ is a Boolean expression by (4), (5), and II(b).

2. a. (1) $\epsilon \in S$ by I.
(2) $a = \epsilon a \in S$ by (1) and II(a).
(3) $aa \in S$ by (2) and II(a).
(4) $aab \in S$ by (3) and II(b).

3. a. (1) MI is in the MIU system by I.
(2) MII is in the MIU system by (1) and II(b).
(3) $MIIII$ is in the MIU system by (3) and II(b).
(4) $MIIIIIIII$ is in the MIU system by (3) and II(b).
(5) $MIUIIII$ is in the MIU system by (4) and II(c).
(6) $MIUUI$ is in the MIU system by (5) and II(c).
(7) $MIUI$ is in the MIU system by (6) and II(d).

4. a. (1) 2, 0.3, 4.2, and 7 are arithmetic expressions by I.
(2) $(0.3 - 4.2)$ is an arithmetic expression by (1) and II(d).
(3) $(2\cdot(0.3 - 4.2))$ is an arithemetic expression by (1), (2), and II(e).

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

(4) (-7) is an arithmetic expression by (1) and II(b).
(5) $((2 \cdot (0.3 - 4.2)) + (-7))$ is an arithmetic expression by (3), (4), and II(c).

5. Proof by structural induction: Let the property be the following sentence: The string ends in a 1.

Show that each object in the BASE for S satisfies the property:

The only object in the base is 1, and the string 1 ends in a 1.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the objects defined by the rule also satisfy the property:

The recursion for S consists of two rules denoted II(a) and II(b). Suppose s is a string in S that ends in a 1. In the case where rule II(a) is applied to s, the result is the string $1s$, which also ends in a 1. In the case where rule II(b) is applied to s, the result is the string $1s$, which also ends in a 1. Thus when each rule in the RECURSION is applied to a string in S that ends in a 1, the result is also a string that ends in a 1.

7. Proof by structural induction: Let the property be the following sentence: The string contains an even number of a's.

Show that each object in the BASE for S satisfies the property:

The only object in the base is ϵ, which contains 0 a's. Because 0 is an even number, ϵ contains an even number of a's.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the objects defined by the rule also satisfy the property:

The recursion for S consists of four rules denoted II(a), II(b), II(c), and II(d). Suppose s is a string in S that contains an even number of a's. In the case where either rule II(a) or rule II(b) is applied to s, the result is the string bs or the string sb, each of which contain the same number of a's as s and hence an even number of a's. In the case where either rule II(c) or rule II(d) is applied to s, the result is the string aas or the string saa, each of which contain two more a's than the number of a's in s. Because two more than any even integer is an even integer, both aas and saa contain an even number of a's. Thus when each rule in ‘the RECURSION is applied to a string in S that contains an even number of a's, the result is also a string that contains even number of a's.

9. *Hint:* Let the property be the following sentence: The string represents an odd integer. In the decimal notation, a string represents an odd integer if, and only if, it ends in 1, 3, 5, 7 or 9.

10. *Hint:* By divisibility results from Chapter 3 (exercises 15 and 16 of Section 3.3), if both s and t are divisible by 5, then so are $s + t$ and $s - t$.

12. *Hint:* Can the number of I's in a string in the MIU system be a multiple of 3? How do rules II(a)–(d) affect the number of I's in a string?

13. a. (1) ( ) is in P by I.
(2) (( )) is in P by (1) and II(a).
(3) ( )(( )) is in P by (1), (2), and II(b).

14. a. This structure is not in P. Define a function $f\colon P \to Z$ as follows: For each parenthesis structure S in P, let

$$f(S) = \begin{bmatrix}\text{the number of left}\\ \text{parentheses in } S\end{bmatrix} - \begin{bmatrix}\text{the number of right}\\ \text{parentheses in } S\end{bmatrix}.$$

Observe that for all S in P, $f(S) = 0$. To see why, use the reasoning of structural induction:

1. The base element of P is sent by f to 0: $f[()] = 0$ *[because there is one left and one right parenthesis in ( )]*.
2. For all $S \in P$, if $f[S] = 0$ then $f[(S)] = 0$ *[because if $k - m = 0$ then $(k+1) - (m+1) = 0$]*.
3. For all S and T in P, if $f[S] = 0$ and $f[T] = 0$, then $f[ST] = 0$ *[because if $k - m = 0$ and $n - p = 0$, then $(k+n) - (m+p) = 0$]*.

Items (1), (2), and (3) show that all parenthesis structures obtainable from the base structure ( ) by repeated application of II(a) and II(b) are sent to 0 by f. But by III (the restriction condition), there are no other elements of P besides those obtainable from the base element by applying II(a) and II(b). Hence $f(S) = 0$ for all $S \in P$.
Now if ( )(( ) were in P, then it would be sent to 0 by f. But $f[()(()] = 3 - 2 = 1 \neq 0$. Thus $()(() \notin P$.

15. Let S be the set of all strings of 0's and 1's with the same number of 0's and 1's. The following is a recursive definition of S.

I. BASE: The null string $\epsilon \in S$.
II. RECURSION: If $s \in S$, then
a. $01s \in S$ **b.** $s01 \in S$ **c.** $10s \in S$
d. $s10 \in S$ **e.** $0s1 \in S$ **f.** $1s0 \in S$
III. RESTRICTION: There are no elements of S other that those obtained from I and II.

17. Let T be the set of all strings of a's and b's that contain an odd number of a's. The following is a recursive definition of T.

I. BASE: The $a \in T$.
II. RECURSION: If $t \in T$, then
a. $bt \in T$ **b.** $tb \in T$ **c.** $aat \in T$
d. $ata \in T$ **e.** $taa \in T$
III. RESTRICTION: There are no elements of T other than those obtained from I and II.

19. a.
$$\begin{aligned}
M(86) &= M(M(97)) && \text{since } 86 \le 100\\
&= M(M(M(108))) && \text{since } 97 \le 100\\
&= M(M(98)) && \text{since } 108 > 100\\
&= M(M(M(109))) && \text{since } 98 < 100\\
&= M(M(99)) && \text{since } 109 > 100\\
&= M(91) && \text{by Example 5.9.6}
\end{aligned}$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

21. a. $A(1,1) = A(0, A(1,0))$ by (5.9.3) with $m = 1$ and $n = 1$

$= A(1,0) + 1$ by (5.9.1) with $n = A(1,0)$

$= A(0,1) + 1$ by (5.9.2) with $m = 1$

$= (1+1) + 1$ by (5.9.1) with $n = 1$

$= 3$

Alternative solution:

$A(1,1) = A(0, A(1,0))$ by (5.9.3) with $m = 1$ and $n = 1$

$= A(0, A(0,1))$ by (5.9.2) with $m = 1$

$= A(0,2)$ by (5.9.1) with $n = 1$

$= 3$ by (5.9.1) with $n = 2$

22. a. Proof by mathematical induction: Let the property, $P(n)$, be the equation $A(1,n) = n + 2$.

Show that P(0) is true:

When $n = 0$,

$A(1,n) = A(1,0)$ by substitution

$= A(0,1)$ by (5.9.2)

$= 1 + 1$ by (5.9.1)

$= 2.$

On the other hand, $n + 2 = 0 + 2$ also. Thus $A(1,n) = n + 2$ for $n = 0$.

Show that for all integers $k \geq 0$, if P(k) is true, then P(k + 1) is true:

Let k be an integer with $k \geq 1$ and suppose $P(k)$ is true. In other words, suppose $A(1,k) = k + 2$. *[This is the inductive hypothesis.]* We must show that $P(k+1)$ is true. In other words, we must show that $A(1, k+1) = (k+1) + 2 = k + 3$. But

$A(1, k+1) = A(0, A(1,k))$ by (5.9.3)

$= A(1,k) + 1$ by (5.9.1)

$= (k+2) + 1$ by inductive hypothesis

$= k + 3.$

[This is what was to be shown.]

[Since both the basis and the inductive steps have been proved, we conclude that the equation holds for all nonnegative integers n.]

24. Suppose F is a function. Then $F(1) = 1$, $F(2) = F(1) = 1$, $F(3) = 1 + F(5 \cdot 3 - 9) = 1 + F(6) = 1 + F(3)$. Subtracting $F(3)$ from the extreme left and extreme right of this sequence of equations gives $1 = 0$, which is false. Hence F is not a function.

Section 6.1

1. a. $A = \{2, \{2\}, (\sqrt{2})^2\} = \{2, \{2\}, 2\} = \{2, \{2\}\}$ and $B = \{2, \{2\}, \{\{2\}\}\}$. So $A \subseteq B$ because every element in A is in B, but $B \nsubseteq A$ because $\{\{2\}\} \in B$ and $\{\{2\}\} \notin A$. Also A is a proper subset of B because $\{\{2\}\}$ is in B but not A.

c. $A = \{\{1,2\}, \{2,3\}\}$ and $B = \{1,2,3\}$. So $A \nsubseteq B$ because $\{1,2\} \in A$ and $\{1,2\} \notin B$. Also $B \nsubseteq A$ because $1 \in B$ and $1 \notin A$.

e. $A = \left\{\sqrt{16}, \{4\}\right\} = \{4, \{4\}\}$ and $B = \{4\}$. Then $B \subseteq A$ because the only element in B is 4 and 4 is in A, but $A \nsubseteq B$ because $\{4\} \in A$ and $\{4\} \notin B$. Also B is a proper subset of A because $\{4\}$ is in A but not B.

2. ***Proof That*** $B \subseteq A$:

Suppose x is a particular but arbitrarily chosen element of B.

[We must show that $x \in A$. By definition of A, this means we must show that $x = 2 \cdot$ (some integer).]

By definition of B, there is an integer b such that $x = 2b - 2$.

[Given that $x = 2b - 2$, can x also be expressed as $2 \cdot$ (some integer)? I.e., is there an integer, say a, such that $2b - 2 = 2a$? Solve for a to obtain $a = b - 1$. Check to see if this works.]

Let $a = b - 1$.

[First check that a is an integer.]

Then a is an integer because it is a difference of integers.

[Then check that $x = 2a$.]

Also $2a = 2(b-1) = 2b - 2 = x$,

Thus, by definition of A, x is an element of A,

[which is what was to be shown].

3. a. No. $R \nsubseteq T$ because there are elements in R that are not in T. For example, the number 2 is in R but 2 is not in T since 2 is not divisible by 6.

b. Yes. $T \subseteq R$ because every number divisible by 6 is divisible by 2. To see why this is so, suppose n is any number that is divisible by 6. Then $n = 6m$ for some integer m. Since $6m = 2(3m)$ and since $3m$ is an integer (being a product of integers), it follows that $n = 2 \cdot$ (*some integer*), and, hence, that n is divisible by 2.

5. a. $C \subseteq D$ Proof: *[We will show that every element of C is in D.]* Suppose n is any element of C. Then $n = 6r - 5$ for some integer r. Let $s = 2r - 2$. Then s is an integer (because products and differences of integers are integers), and

$$3s + 1 = 3(2r - 2) + 1 = 6r - 6 + 1 = 6r - 5,$$

which equals n. Thus n satisfies the condition for being in D. Hence, every element in C is in D.

b. $D \nsubseteq C$ because there are elements of D that are not in C. For example, 4 is in D because $4 = 3 \cdot 1 + 1$. But 4 is not in C because if it were, then $4 = 6r - 5$ for some integer r, which would imply that $9 = 6r$, or, equivalently, that $r = 3/2$, and this contradicts the fact that r is an integer.

6. c. *Sketch of proof that* $B \subseteq C$: If r is any element of B then there is an integer b such that $r = 10b - 3$. To show that r is in C, you must show that there is an integer c such that $r = 10c + 7$. In scratch work, assume that c exists and use the information that $10b - 3$ would have

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

to equal $10c + 7$ to deduce the only possible value for c. Then show that this value is (1) an integer and (2) satisfies the equation $r = 10c + 7$, which will allow you to conclude that r is an element of C.

Sketch of proof that $C \subseteq B$: If s is any element of C then there is an integer c such that $s = 10c + 7$. To show that s is in B, you must show that there is an integer b such that $s = 10c - 3$. In scratch work, assume that b exists and use the information that $10c + 7$ would have to equal $10b - 3$ to deduce the only possible value for b. Then show that this value is (1) an integer and (2) satisfies the equation $s = 10b - 3$, which will allow you to conclude that s is an element of B.

8. a. The set of all x in U such that x is in A and x is in B. The shorthand notation is $A \cap B$.

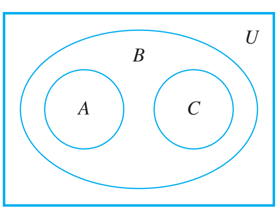
9. a. $x \notin A$ and $x \notin B$

10. a. $\{1, 3, 5, 6, 7, 9\}$ **b.** $\{3, 9\}$
c. $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **d.** $\emptyset$ **e.** $\{1, 5, 7\}$

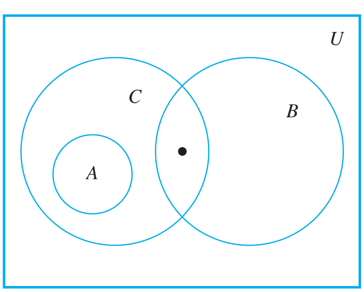
11. a. $A \cup B = \{x \in \mathbf{R} \mid 0 < x < 4\}$
b. $A \cap B = \{x \in \mathbf{R} \mid 1 \le x \le 2\}$
c. $A^c = \{x \in \mathbf{R} \mid x \le 0 \text{ or } x > 2\}$
d. $A \cup C = \{x \in \mathbf{R} \mid 0 < x \le 2 \text{ or } 3 \le x < 9\}$
e. $A \cap C = \emptyset$
f. $B^c = \{x \in \mathbf{R} \mid x < 1 \text{ or } x \ge 4\}$
g. $A^c \cap B^c = \{x \in \mathbf{R} \mid x \le 0 \text{ or } x \ge 4\}$
h. $A^c \cup B^c = \{x \in \mathbf{R} \mid x < 1 \text{ or } x > 2\}$
i. $(A \cap B)^c = \{x \in \mathbf{R} \mid x < 1 \text{ or } x > 2\}$
j. $(A \cup B)^c = \{x \in \mathbf{R} \mid x \le 0 \text{ or } x \ge 4\}$

13. b. False. Many negative real numbers are not rational. For example, $-\sqrt{2} \in \mathbf{R}$ but $-\sqrt{2} \notin \mathbf{Q}$.
d. False. $0 \in \mathbf{Z}$ but $0 \notin \mathbf{Z}^- \cup \mathbf{Z}^+$.

14. a.

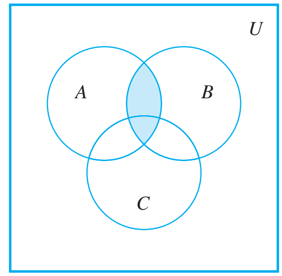


15. a.



16. a. $A \cup (B \cap C) = \{a, b, c\}$, $(A \cup B) \cap C = \{b, c\}$, and $(A \cup B) \cap (A \cup C) = \{a, b, c, d\} \cap \{a, b, c, e\} = \{a, b, c\}$.
Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

17. a.



18. a. The number 0 is not in $\emptyset$ because $\emptyset$ has no elements.
b. No. The left-hand set is the empty set; it does not have any elements. The right-hand set is a set with one element, namely $\emptyset$.

19. $A_1 = \{1, 1^2\} = \{1\}$, $A_2 = \{2, 2^2\} = \{2, 4\}$, $A_3 = \{3, 3^2\} = \{3, 9\}$, $A_4 = \{4, 4^2\} = \{4, 16\}$
a. $A_1 \cup A_2 \cup A_3 \cup A_4 = \{1\} \cup \{2, 4\} \cup \{3, 9\} \cup \{4, 16\} = \{1, 2, 3, 4, 9, 16\}$
b. $A_1 \cap A_2 \cap A_3 \cap A_4 = \{1\} \cap \{2, 4\} \cap \{3, 9\} \cap \{4, 16\} = \emptyset$
c. A_1, A_2, A_3, and A_4 are not mutually disjoint because $A_2 \cap A_4 = \{4\} = \emptyset$.

21. $C_0 = \{0, -0\} = \{0\}$, $C_1 = \{1, -1\}$, $C_1 = \{2, -2\}$, $C_1 = \{3, -3\}$, $C_1 = \{4, -4\}$
a. $\bigcup_{i=0}^{4} C_i = \{0\} \cup \{1, -1\} \cup \{2, -2\} \cup \{3, -3\} \cup \{4, -4\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
b. $\bigcap_{i=0}^{4} C_i = \{0\} \cap \{1, -1\} \cap \{2, -2\} \cap \{3, -3\} \cap \{4, -4\} = \emptyset$
c. $C_0, C_1, C_2, \ldots$ are mutually disjoint because no two of the sets have any elements in common.
d. $\bigcup_{i=0}^{n} C_i = \{-n, -(n-1), \ldots, -2, -1, 0, 1, 2, \ldots, (n-1), n\}$
e. $\bigcap_{i=0}^{n} C_i = \emptyset$
f. $\bigcup_{i=0}^{\infty} C_i = \mathbf{Z}$, the set of all integers
g. $\bigcap_{i=0}^{\infty} C_i = \emptyset$

22. $D_0 = [-0, 0] = \{0\}$, $D_1 = [-1, 1]$, $D_2 = [-2, 2]$, $D_3 = [-3, 3]$, $D_4 = [-4, 4]$
a. $\bigcup_{i=0}^{4} D_i = \{0\} \cup [-1, 1] \cup [-2, 2] \cup [-3, 3] \cup [-4, 4] = [-4, 4]$
b. $\bigcap_{i=0}^{4} D_i = \{0\} \cup [-1, 1] \cup [-2, 2] \cup [-3, 3] \cup [-4, 4] = \{0\}$
c. $D_0, D_1, D_2, \ldots$ are not mutually disjoint. In fact, each $D_k \subseteq D_{k+1}$.
d. $\bigcup_{i=0}^{n} D_i = [-n, n]$
e. $\bigcap_{i=0}^{n} D_i = \{0\}$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

f. $\bigcup_{i=0}^{\infty} D_i = \mathbf{R}$, the set of all real numbers

g. $\bigcap_{i=0}^{\infty} D_i = \{0\}$

24. $W_0 = (0, \infty)$, $W_1 = (1, \infty)$, $W_2 = (2, \infty)$, $W_3 = (3, \infty)$, $W_4 = (4, \infty)$

a. $\bigcup_{i=0}^{4} W_i = (0, \infty) \cup (1, \infty) \cup (2, \infty) \cup (3, \infty) \cup (4, \infty) = (0, \infty)$

b. $\bigcap_{i=0}^{4} W_i = (0, \infty) \cap (1, \infty) \cap (2, \infty) \cap (3, \infty) \cap (4, \infty) = (4, \infty)$

c. $W_0, W_1, W_2, \ldots$ are not mutually disjoint. In fact, $W_{k+1} \subseteq W_k$ for all integers $k \geq 0$.

d. $\bigcup_{i=0}^{n} W_i = (0, \infty)$

e. $\bigcap_{i=0}^{n} W_i = (n, \infty)$

f. $\bigcup_{i=0}^{\infty} W_i = (0, \infty)$

g. $\bigcap_{i=0}^{\infty} W_i = \emptyset$

27. a. No. The element d is in two of the sets.

b. No. None of the sets contains 6.

28. Yes. Every integer is either even or odd, and no integer is both even and odd.

31. a. $A \cap B = \{2\}$, so $\mathscr{P}(A \cap B) = \{\emptyset, \{2\}\}$.

b. $A = \{1, 2\}$, so $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

c. $A \cup B = \{1, 2, 3\}$, so $\mathscr{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

d. $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, so
$\mathscr{P}(A \times B) = \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(2, 2)\}, \{(2, 3)\},$
$\{(1, 2), (1, 3)\}, \{(1, 2), (2, 2)\},$
$\{(1, 2), (2, 3)\}, \{(1, 3), (2, 2)\}, \{(1, 3), (2, 3)\},$
$\{(2, 2), (2, 3)\}, \{(1, 2), (1, 3), (2, 2)\},$
$\{(1, 2), (1, 3), (2, 3)\},$
$\{(1, 2), (2, 2), (2, 3)\}, \{(1, 3), (2, 2), (2, 3)\},$
$\{(1, 2), (1, 3), (2, 2), (2, 3)\}\}$.

32. a. $\mathscr{P}(A \times B) = \{\emptyset, \{(1, u)\}, \{(1, v)\}, \{(1, u), (1, v)\}\}$

33. b. $\mathscr{P}(\mathscr{P}(\emptyset)) = \mathscr{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

34. a. $A_1 \times (A_2 \times A_3) = \{(1, (u, m)), (2, (u, m)), (3, (u, m)), (1, (u, n)), (2, (u, n)), (3, (u, n)), (1, (v, m)), (2, (v, m)), (3, (v, m)), (1, (v, n)), (2, (v, n)), (3, (v, n))\}$

35. a. $A \times (B \cup C) = \{a, b\} \times \{1, 2, 3\}$
$= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

b. $(A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (b, 1), (b, 2), (a, 2), (a, 3), (b, 2), (b, 3)\}$
$= \{(a, 1), (a, 2), (b, 1), (b, 2), (a, 3), (b, 3)\}$

36.

i	1 →	→	→	→	2 →	→	→	→	3 →	→	→	4
j		1	2	3	1	2	3	4	1 →	→	2	
found		no	yes		no		yes		no	yes		
answer	$A \subseteq B$											

Section 6.2

1. a. (1) A (2) $B \cup C$

b. (1) $A \cap B$ (2) C

2. a. (1) $A - B$ (2) A (3) A (4) B

b. (1) $x \in A$ (2) A (3) B (4) A

3. (a.) A **(b)** C **(c)** B **(d)** C **(e)** $B \subseteq C$

5. Proof: Suppose A and B are sets.

***$B - A \subseteq B \cap A^c$*:** Suppose $x \in B - A$. By definition of set difference, $x \in B$ and $x \notin A$. But then by definition of complement, $x \in B$ and $x \in A^c$, and so by definition of intersection, $x \in B \cap A^c$. *[Thus $B - A \subseteq B \cap A^c$ by definition of subset].*

***$B \cap A^c \subseteq B - A$*:** Suppose $x \in B \cap A^c$. By definition of intersection, $x \in B$ and $x \in A^c$. But then by definition of complement, $x \in B$ and $x \notin A$, and so by definition of set difference, $x \in B - A$. *[Thus $B \cap A^c \subseteq B - A$ by definition of subset.]*

[Since both set containments have been proved, $B - A = B \cap A^c$ by definition of set equality.]

6. *Partial answers*

a. $(A \cap B) \cup (A \cap C)$ **b.** A **c.** $B \cup C$
d. $x \in C$ **e.** $A \cap B$ **f.** by definition of intersection, $x \in A \cap C$, and so by definition of union, $x \in (A \cap B) \cup (A \cap C)$.

7. *Hint:* This is somewhat similar to the proof in Example 6.2.3.

8. Proof: Suppose A and B are any sets.

***Proof that $(A \cap B) \cup (A \cap B^c) \subseteq A$*:** Suppose $x \in (A \cap B) \cup (A \cap B^c)$. *[We must show that $x \in A$.]* By definition of union, $x \in A \cap B$ or $x \in (A \cap B^c)$.

***Case 1* ($\mathbf{x \in A \cap B}$):** In this case x is in A and x is in B, and so, in particular, $x \in A$.

***Case 2* ($\mathbf{x \in A \cap B^c}$):** In this case x is in A and x is not in B, and so, in particular, $x \in A$.

Thus, in either case, $x \in A$ *[as was to be shown]. [Thus $(A \cap B) \cup (A \cap B^c) \subseteq A$ by definition of subset.]*

***Proof that $A \subseteq (A \cap B) \cup (A \cap B^c)$*:** Suppose $x \in A$. *[We must show that $x \in (A \cap B) \cup (A \cap B^c)$.]* Either $x \in B$ or $x \notin B$.

***Case 1* ($\mathbf{x \in B}$):** In this case we know that x is in A and we are also assuming that x is in B. Hence, by definition of intersection, $x \in A \cap B$.

***Case 2* ($\mathbf{x \in A \cap B^c}$):** In this case we know that x is in A and we are also assuming that x is in B^c. Hence, by definition of intersection, $x \in A \cap B^c$.

Thus, $x \in A \cap B$ or $x \in A \cap B^c$, and so, by definition of union, $x \in (A \cap B) \cup (A \cap B^c)$ *[as was to be shown. Thus $A \subseteq (A \cap B) \cup (A \cap B^c)$ by definition of subset.]*

Conclusion: Since both set containments have been proved, it follows by definition of set equality that $(A \cap B) \cup (A \cap B^c) = A$.

9. Partial proof: Suppose A, B, and C are any sets. To show that $(A - B) \cup (C - B) = (A \cup C) - B$, we must show that $(A - B) \cup (C - B) \subseteq (A \cup C) - B$ and that $(A \cup C) - B \subseteq (A - B) \cup (C - B)$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$(A - B) \cup (C - B) \subseteq (A \cup C) - B$: Suppose that x is any element in $(A - B) \cup (C - B)$. *[We must show that $x \in (A \cup C) - B$.]* By definition of union, $x \in A - B$ or $x \in C - B$.

***Case 1* ($x \in A - B$):** Then, by definition of set difference, $x \in A$ and $x \notin B$. But because $x \in A$, we have that $x \in A \cup C$ by definition of union. Hence $x \in A \cup C$ and $x \notin B$, and so, by definition of set difference, $x \in (A \cup C) - B$.

***Case 2* ($x \in C - B$):** Then, by definition of set difference, $x \in C$ and $x \notin B$. But because $x \in C$, we have that $x \in A \cup C$ by definition of union. Hence $x \in A \cup C$ and $x \notin B$, and so, by definition of set difference, $x \in (A \cup C) - B$.

Thus, in both cases, $x \in (A \cup C) - B$ *[as was to be shown]*. So $(A - B) \cup (C - B) \subseteq (A \cup C) - B$.

11. Partial proof: Suppose A and B are any sets. We will show that $A \cup (A \cap B) \subseteq A$. Suppose x is any element in $A \cup (A \cap B)$. *[We must show that $x \in A$.]* By definition of union, $x \in A$ or $x \in A \cap B$. In the case where $x \in A$, clearly $x \in A$. In the case where $x \in A \cap B$, $x \in A$ and $x \in B$ (by definition of intersection). Thus, in particular, $x \in A$. Hence, in both cases $x \in A$ *[as was to be shown]*.
To complete the proof that $A \cup (A \cap B) = A$, you must show that $A \subseteq A \cup (B \cap A)$.

12. Proof: Let A be a set. *[We must show that $A \cup \emptyset = A$.]*

$A \cup \emptyset \subseteq A$: Suppose $x \in A \cup \emptyset$. Then $x \in A$ or $x \in \emptyset$ by definition of union. But $x \notin \emptyset$ since $\emptyset$ has no elements. Hence $x \in A$.

$A \subseteq A \cup \emptyset$: Suppose $x \in A$. Then the statement "$x \in A$ or $x \in \emptyset$" is true. Hence $x \in A \cup \emptyset$ by definition of union. *[Alternatively, $A \subseteq A \cup \emptyset$ by the inclusion in union property.]* Since $A \cup \emptyset \subseteq A$ and $A \subseteq A \cup \emptyset$, then $A \cup \emptyset = A$ by definition of set equality.

13. Proof: Suppose A, B, and C are sets and $A \subseteq B$. Let $x \in A \cap C$. By definition of intersection, $x \in A$ and $x \in C$. But since $A \subseteq B$ and $x \in A$, then $x \in B$. Hence $x \in B$ and $x \in C$, and so, by definition of intersection, $x \in B \cap C$. *[Thus $A \cap C \subseteq B \cap C$ by definition of subset.]*

16. *Hint:* The proof has the following outline:
Suppose A, B, and C are any sets such that $A \subseteq B$ and $A \subseteq C$.

$\vdots$

Therefore, $A \subseteq B \cap C$.

18. Proof: Suppose A, B, and C are arbitrarily chosen sets.

$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$: Suppose $(x, y) \in A \times (B \cup C)$. *[We must show that $(x, y) \in (A \times B) \cup (A \times C)$.]* Then $x \in A$ and $y \in B \cup C$. By definition of union, this means that $y \in B$ or $y \in C$.

***Case 1* ($y \in B$):** Then, since $x \in A$, $(x, y) \in A \times B$ by definition of Cartesian product. Hence $(x, y) \in (A \times B) \cup (A \times C)$ by the inclusion in union property.

***Case 2* ($y \in C$):** Then, since $x \in A$, $(x, y) \in A \times C$ by definition of Cartesian product. Hence $(x, y) \in (A \times B) \cup (A \times C)$ by the inclusion in union property.

Hence, in either case, $(x, y) \in (A \times B) \cup (A \times C)$ *[as was to be shown]*.

Thus $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ by definition of subset.

$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$: Suppose $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$.

***Case 1* ($(x, y) \in A \times B$):** In this case, $x \in A$ and $y \in B$. By definition of union, since $y \in B$, then $y \in B \cup C$. Hence $x \in A$ and $y \in B \cup C$, and so, by definition of Cartesian product, $(x, y) \in A \times (B \cup C)$.

***Case 2* ($(x, y) \in A \times C$):** In this case, $x \in A$ and $y \in C$. By definition of union, since $y \in C$, then $y \in B \cup C$. Hence $x \in A$ and $y \in B \cup C$, and so, by definition of Cartesian product, $(x, y) \in A \times (B \cup C)$.

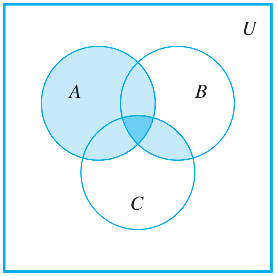
Thus, in either case, $(x, y) \in A \times (B \cup C)$. *[Hence, by definition of subset, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.]*

[Since both subset relations have been proved, we can conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ by definition of set equality.]

20. There is more than one error in this "proof." The most serious is the misuse of the definition of subset. To say that A is a subset of B means that for all x, **if** $x \in A$ **then** $x \in B$. It does not mean that there exists an element of A that is also an element of B. The second error in the proof occurs in the last sentence. Just because there is an element in A that is in B and an element in B that is in C, it does not follow that there is an element in A that is in C. For instance, suppose $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{3, 4\}$. Then there is an element in A that is in B (namely 2) and there is an element in B that is in C (namely 3), but there is no element in A that is in C.

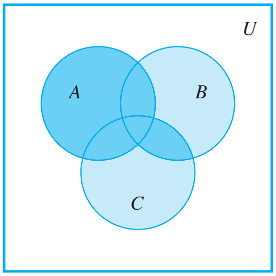
21. *Hint:* The statement "since $x \notin A$ or $x \notin B$, $x \notin A \cup B$" is fallacious. Try to think of an example of sets A and B and an element x such that the statement "$x \notin A$ or $x \notin B$" is true and the statement "$x \notin A \cup B$" is false.

23. a.



Entire shaded region is $A \cup (B \cap C)$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



Darkly shaded region is $(A \cup B) \cap (A \cup C)$.

24. **(a)** $(A - B) \cap (B - A)$ **(b)** intersection **(c)** $B - A$ **(d)** B **(e)** A **(f)** A **(g)** $(A - B) \cap (B - A) = \emptyset$

25. Proof by contradiction: Suppose not. That is, suppose there exist sets A and B such that $(A \cap B) \cap (A \cap B^c) \neq \emptyset$. Then there is an element x in $(A \cap B) \cap (A \cap B^c)$. By definition of intersection, $x \in (A \cap B)$ and $x \in (A \cap B^c)$. Applying the definition of intersection again, we have that since $x \in (A \cap B)$, $x \in A$ and $x \in B$, and since $x \in (A \cap B^c)$, $x \in A$ and $x \notin B$. Thus, in particular, $x \in B$ and $x \notin B$, which is a contradiction. It follows that the supposition is false, and so $(A \cap B) \cap (A \cap B^c) = \emptyset$.

27. Proof: Let A be a subset of a universal set U. Suppose $A \cap A^c \neq \emptyset$, that is, suppose there is an element x such that $x \in A \cap A^c$. Then by definition of intersection, $x \in A$ and $x \in A^c$, and so by definition of complement, $x \in A$ and $x \notin A$. This is a contradiction. *[Hence the supposition is false, and we conclude that $A \cap A^c = \emptyset$.]*

29. Proof: Let A be a set. Suppose $A \times \emptyset \neq \emptyset$. Then there would be an element (x, y) in $A \times \emptyset$. By definition of Cartesian product, $x \in A$ and $y \in \emptyset$. But there are no elements y such that $y \in \emptyset$. Hence there are no elements (x, y) such that $x \in A$ and $y \in \emptyset$. Consequently, $(x, y) \notin A \times \emptyset$. *[Thus the supposition is false, and so $A \times \emptyset = \emptyset$.]*

30. Proof: Let A and B be sets such that $A \subseteq B$. *[We must show that $A \cap B^c = \emptyset$.]* Suppose $A \cap B^c \neq \emptyset$; that is, suppose there were an element x such that $x \in A \cap B^c$. Then $x \in A$ and $x \in B^c$ by definition of intersection. So $x \in A$ and $x \notin B$ by definition of complement. But $A \subseteq B$ by hypothesis. So since $x \in A$, $x \in B$ by definition of subset. Thus $x \notin B$ and also $x \in B$, which is a contradiction. Hence the supposition that $A \cap B^c \neq \emptyset$ is false, and so $A \cap B^c = \emptyset$.

33. Proof: Let A, B, and C be any sets such that $C \subseteq B - A$. Suppose $A \cap C \neq \emptyset$. Then there is an element x such that $x \in A \cap C$. By definition of intersection, $x \in A$ and $x \in C$. Since $C \subseteq B - A$, then $x \in B$ and $x \notin A$. So $x \in A$ and $x \notin A$, which is a contradiction. Hence the supposition is false, and thus $A \cap C = \emptyset$.

36. a. *Start of proof that $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$*: Given any element x in $A \cup B$, by definition of union x is in at least one of A and B. Thus x satisfies exactly one of the following three conditions:
(1) $x \in A$ and $x \notin B$ (x is in A only)
(2) $x \in B$ and $x \notin A$ (x is in B only)
(3) $x \in A$ and $x \in B$ (x is in both A and B)

b. To show that $(A - B)$, $(B - A)$, and $(A \cap B)$ are mutually disjoint, we must show that the intersection of any two of them is the empty set. But, by definition of set difference and set intersection, saying that $x \in A - B$ means that (1) $x \in A$ and $x \notin B$, saying that $x \in B - A$ means that (2) $x \in B$ and $x \notin A$, and saying that $x \in A \cap B$ means that (3) $x \in A$ and $x \in B$. Conditions (1)–(3) are mutually exclusive, and so no two of them can be satisfied at the same time. Thus no element can be in the intersection of any two of the sets, and, therefore, the intersection of any two of the sets is the empty set. Hence, $(A - B)$, $(B - A)$, and $(A \cap B)$ are mutually disjoint.

37. Suppose A and $B_1, B_2, B_3, \ldots, B_n$ are any sets.

Proof that $A \cap \left(\bigcup_{i=1}^{n} B_i\right) \subseteq \bigcup_{i=1}^{n}(A \cap B_i)$:

Suppose x is any element in $A \cap \left(\bigcup_{i=1}^{n} B_i\right)$. *[We must show that $x \in \bigcup_{i=1}^{n}(A \cap B_i)$.]* By definition of intersection, $x \in A$ and $x \in \bigcup_{i=1}^{n} B_i$. Since $x \in \bigcup_{i=1}^{n} B_i$, the definition of general union implies that $x \in B_i$ for some $i = 1, 2, \ldots, n$, and so, since $x \in A$, the definition of intersection implies that $x \in A \cap B_i$. Thus, by definition of general union, $x \in \bigcup_{i=1}^{n}(A \cap B_i)$ *[as was to be shown].*

Proof that $\bigcup_{i=1}^{n}(A \cap B_i) \subseteq A \cap \left(\bigcup_{i=1}^{n} B_i\right)$:

Suppose x is any element in $\bigcup_{i=1}^{n}(A \cap B_i)$. *[We must show that $x \in A \cap \left(\bigcup_{i=1}^{n} B_i\right)$.]* By definition of general union, $x \in A \cap B_i$ for some $i = 1, 2, \ldots, n$. Thus, by definition of intersection, $x \in A$ and $x \in B_i$. Since $x \in B_i$ for some $i = 1, 2, \ldots, n$, by definition of general union, $x \in \bigcup_{i=1}^{n} B_i$. Thus we have that $x \in A$ and $x \in \bigcup_{i=1}^{n} B_i$, and so, by definition of intersection, $x \in A \cap \left(\bigcup_{i=1}^{n} B_i\right)$ *[as was to be shown].*

Conclusion: Since both set containments have been proved, it follows by definition of set equality that $A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n}(A \cap B_i)$.

38. *Proof sketch*: If $x \in \bigcup_{i=1}^{n}(A_i - B)$, then $x \in A_i - B$ for some $i = 1, 2, \ldots, n$, and so, (1) for some $i = 1, 2, \ldots, n$, $x \in A_i$ (which implies that $x \in \left(\bigcup_{i=1}^{n} A_i\right)$) and (2) $x \notin B$. Conversely, if $x \in \left(\bigcup_{i=1}^{n} A_i\right) - B$, then $x \in \bigcup_{i=1}^{n} A_i$ and $x \notin B$, and so, by definition of general union, $x \in A_i$ for some $i = 1, 2, \ldots, n$, $x \in A_i$ and $x \notin B$. This implies that

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

there is an integer i such that $x \in A_i - B$, and thus that $x \in \bigcup_{i=1}^{n}(A_i - B)$.

40. Suppose A and $B_1, B_2, B_3, \ldots, B_n$ are any sets.

Proof that $\bigcup_{i=1}^{n}(A \times B_i) \subseteq A \times \left(\bigcup_{i=1}^{n} B_i\right)$:

Suppose (x, y) is any element in $\bigcup_{i=1}^{n}(A \times B_i)$. *[We must show that* $(x, y) \in A \times \left(\bigcup_{i=1}^{n} B_i\right)$.*]* By definition of general union, $(x, y) \in A \times B_i$ for some $i = 1, 2, \ldots, n$. By definition of Cartesian product, this implies that (1) $x \in A$ and (2) $y \in B_i$ for some $i = 1, 2, \ldots, n$. By definition of general union, (2) implies that $y \in \bigcup_{i=1}^{n} B_i$. Thus $x \in A$ and $y \in \bigcup_{i=1}^{n} B_i$, and so by definition of Cartesian product, $(x, y) \in A \times \left(\bigcup_{i=1}^{n} B_i\right)$ *[as was to be shown]*.

Proof that $A \times \left(\bigcup_{i=1}^{n} B_i\right) \subseteq \bigcup_{i=1}^{n}(A \times B_i)$:

Suppose (x, y) is any element in $A \times \left(\bigcup_{i=1}^{n} B_i\right)$. *[We must show that* $(x, y) \in \bigcup_{i=1}^{n}(A \times B_i)$.*]* By definition of Cartesian product, (1) $x \in A$ and (2) $y \in \bigcup_{i=1}^{n} B_i$. By definition of general union, (2) implies that $y \in B_i$ for some $i = 1, 2, \ldots, n$. Thus $x \in A$ and $y \in B_i$ for some $i = 1, 2, \ldots, n$, and so, by definition of Cartesian product, $(x, y) \in A \times B_i$ for some $i = 1, 2, \ldots, n$. It follows from the definition of general union that $(x, y) \in \bigcup_{i=1}^{n}(A \times B_i)$ *[as was to be shown]*.

Conclusion: Since both set containments have been proved, it follows by definition of set equality that $\bigcup_{i=1}^{n}(A \times B_i) = A \times \left(\bigcup_{i=1}^{n} B_i\right)$.

Section 6.3

1. Counterexample: Any sets A, B, and C where C contains elements that are not in A will serve as a counterexample. For instance, let $A = \{1, 3\}, B = \{2, 3\}$, and $C = \{4\}$. Then $(A \cap B) \cup C = \{3\} \cup \{4\} = \{3, 4\}$, whereas $A \cap (B \cup C) = \{1, 3\} \cap \{2, 3, 4\} = \{3\}$. Since $\{3, 4\} \neq \{3\}$, $(A \cap B) \cup C \neq A \cap (B \cup C)$.

3. Counterexample: Any sets, A, B, and C where $A \subseteq C$ and B contains at least one element that is not in either A or C will serve as a counterexample. For instance, let $A = \{1\}, B = \{2\}$, and $C = \{1, 3\}$. Then $A \nsubseteq B$ and $B \nsubseteq C$ but $A \subseteq C$.

5. False. Counterexample: Any sets A, B, and C where A and C have elements in common that are not in B will serve as a counterexample. For instance, let $A = \{1, 2, 3\}$, $B = \{2, 3\}$, and $C = \{3\}$. Then $B - C = \{2\}$, and so $A - (B - C) = \{1, 2, 3\} - \{2\} = \{1, 3\}$. On the other hand $A - B = \{1, 2, 3\} - \{2, 3\} = \{1\}$, and so $(A - B) - C = \{1\} - \{3\} = \{1\}$. Since $\{1, 3\} \neq \{1\}$, $A - (B - C) \neq (A - B) - C$.

6. True. Proof: Let A and B be any sets.

$\mathbf{A \cap (A \cup B) \subseteq A}$**:** Suppose $x \in A \cap (A \cup B)$. By definition of intersection, $x \in A$ and $x \in A \cup B$. In particular $x \in A$. Thus, by definition of subset, $A \cap (A \cup B) \subseteq A$.

$\mathbf{A \subseteq A \cap (A \cup B)}$**:** Suppose $x \in A$. Then by definition of union, $x \in A \cup B$. Hence $x \in A$ and $x \in A \cup B$, and so, by definition of intersection $x \in A \cap (A \cup B)$. Thus, by definition of subset, $A \subseteq A \cap (A \cup B)$.

Because both $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$ have been proved, we conclude that $A \cap (A \cup B) = A$.

9. True. Proof: Suppose A, B, and C are sets and $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$. By definition of union, $x \in A$ or $x \in B$. But if $x \in A$ then $x \in C$ (because $A \subseteq C$), and if $x \in B$ then $x \in C$ (because $B \subseteq C$). Hence, in either case, $x \in C$. *[So, by definition of subset, $A \cup B \subseteq C$.]*

11. *Hint:* The statement is false. Consider sets *U, A, B*, and C as follows: $U = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{2\}$.

12. *Hint:* The statement is true. *Sketch of proof*: If $x \in A \cap (B - C)$, then $x \in A$ and $x \in B$ and $x \notin C$. So it is true that $x \in A$ and $x \in B$ and that $x \in A$ and $x \notin C$. Conversely, if $x \in (A \cap B) - (A \cap C)$, then $x \in A$ and $x \in B$, but $x \notin A \cap C$, and so $x \notin C$.

14. *Hint:* The statement is false. Show that the following is a counterexample: $A = \{1, 3\}$, $B = \{1, 2, 3\}$, and $C = \{2, 3\}$.

15. *Hint:* The statement is true. *Sketch of proof*: Suppose $x \in A$. *[We must show that $x \in B$.]* Either $x \in C$ or $x \notin C$. In case $x \in C$, make use of the fact that $A \cap C \subseteq B \cap C$ to show that $x \in B$. In case $x \notin C$, make use of the fact that $A \cup C \subseteq B \cup C$ to show that $x \in B$.

17. True. Proof: Suppose A and B are any sets with $A \subseteq B$. *[We must show that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.]* So suppose $X \in \mathscr{P}(A)$. Then $X \subseteq A$ by definition of power set. But because $A \subseteq B$, we also have that $X \subseteq B$ by the transitive property for subsets, and thus, by definition of power set, $X \in \mathscr{P}(B)$. This proves that for all X, if $X \in \mathscr{P}(A)$ then $X \in \mathscr{P}(B)$, and so $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ *[as was to be shown]*.

18. False. Counterexample: For any sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B)$ contains only sets that are subsets of either A or B, whereas the sets in $\mathscr{P}(A \cup B)$ can contain elements of both A and B. Thus, if at least one of A or B contains elements that are not in the other set, $\mathscr{P}(A) \cup \mathscr{P}(B)$ and $\mathscr{P}(A \cup B)$ will not be equal. For instance, let $A = \{1\}$ and $B = \{2\}$. Then $\{1, 2\} \in \mathscr{P}(A \cup B)$ but $\{1, 2\} \notin \mathscr{P}(A) \cup \mathscr{P}(B)$.

19. *Hint:* The statement is true. To prove it, suppose A and B are any sets, and suppose $X \in \mathscr{P}(A) \cup \mathscr{P}(B)$. Show that $X \subseteq A \cup B$, and deduce the conclusion from this result.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

22. **a.** *Statement:* $\forall$ sets S, $\exists$ a set T such that $S \cap T = \emptyset$.
Negation: $\exists$ a set S such that $\forall$ sets T, $S \cap T \neq \emptyset$.
The statement is true. Given any set S, take $T = S^c$. Then $S \cap T = S \cap S^c = \emptyset$ by the complement law for $\cap$. Alternatively, T could be taken to be $\emptyset$.

23. *Hint:* $S_0 = \{\emptyset\}$, $S_1 = \{\{a\}, \{b\}, \{c\}\}$

25. **a.** $S_1 = \{\emptyset, \{t\}, \{u\}, \{v\}, \{t, u\}, \{t, v\}, \{u, v\}, \{t, u, v\}\}$
b. $S_2 = \{\{w\}, \{t, w\}, \{u, w\}, \{v, w\}, \{t, u, w\}, \{t, v, w\}, \{u, v, w\}, \{t, u, v, w\}\}$
c. Yes

26. *Hint:* Use mathematical induction. In the inductive step, you will consider the set of all nonempty subsets of $\{2, \ldots, k\}$ and the set of all nonempty subsets of $\{2, \ldots, k+1\}$. Any subset of $\{2, \ldots, k+1\}$ either contains $k+1$ or does not contain $k+1$. Thus

$$\left[\begin{array}{l}\text{the sum of all products}\\ \text{of elements of nonempty}\\ \text{subsets of } \{2, \ldots, k+1\}\end{array}\right]$$

$$= \left[\begin{array}{l}\text{the sum of all products}\\ \text{of elements of nonempty}\\ \text{subsets of } \{2, \ldots, k+1\}\\ \text{that do not contain } k+1\end{array}\right] + \left[\begin{array}{l}\text{the sum of all products}\\ \text{of elements of nonempty}\\ \text{subsets of } \{2, \ldots, k+1\}\\ \text{that contain } k+1\end{array}\right]$$

But any subset of $\{2, \ldots, k+1\}$ that does not contain $k+1$ is a subset of $\{2, \ldots, k\}$. And any subset of $\{2, \ldots, k+1\}$ that contains $k+1$ is the union of a subset of $\{2, \ldots, k\}$ and $\{k+1\}$.

27. **a.** commutative law for $\cap$
b. distributive law
c. commutative law for $\cap$

28. Partial answer:
a. set difference law
b. set difference law
c. commutative law for $\cap$
d. De Morgan's law

29. *Hint:* Remember to use the properties in Theorem 6.2.2 exactly as they are written. For example, the distributive law does not state that for all sets A, B, and C, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

30. Proof: Let sets A, B, and C be given. Then

$$\begin{aligned}
&(A \cap B) \cup C\\
&\quad = C \cup (A \cap B) && \text{by the commutative law for } \cup\\
&\quad = (C \cup A) \cap (C \cup B) && \text{by the distributive law}\\
&\quad = (A \cup C) \cap (B \cup C) && \text{by the commutative law for } \cup.
\end{aligned}$$

31. Proof: Suppose A and B are sets. Then

$$\begin{aligned}
&A \cup (B - A)\\
&\quad = A \cup (B \cap A^c) && \text{by the set difference law}\\
&\quad = (A \cup B) \cap (A \cup A^c) && \text{by the distributive law}\\
&\quad = (A \cup B) \cap U && \text{by the complement law for } \cup\\
&\quad = A \cup B && \text{by the identity law for } \cap.
\end{aligned}$$

36. Proof: Let A, B, and C be any sets. Then

$$\begin{aligned}
&((A^c \cup B^c) - A)^c\\
&\quad = ((A^c \cup B^c) \cap A^c)^c && \text{by the set difference law}\\
&\quad = (A^c \cup B^c)^c \cup (A^c)^c && \text{by De Morgan's law}\\
&\quad = ((A^c)^c \cap (B^c)^c) \cup (A^c)^c && \text{by De Morgan's law}\\
&\quad = (A \cap B) \cup A && \text{by the double complement law}\\
&\quad = A \cup (A \cap B) && \text{by the commutative law for } \cup\\
&\quad = A && \text{by the absorption law}
\end{aligned}$$

39. Partial proof: Let A and B be any sets. Then

$$\begin{aligned}
&(A - B) \cup (B - A)\\
&= (A \cap B^c) \cup (B \cap A^c) && \text{by the set difference law}\\
&= [(A \cap B^c) \cup B] \cap [(A \cap B^c) \cup A^c)] && \text{by the distributive law}\\
&= [(B \cup (A \cap B^c)] \cap [A^c \cup (A \cap B^c)] && \text{by the commutative law for } \cup\\
&= [(B \cup A) \cap (B \cup B^c)] \cap [(A^c \cup A) \cap (A^c \cup B^c)] && \text{by the distributive law}\\
&= [(A \cup B) \cap (B \cup B^c)] \cap [(A \cup A^c) \cap (A^c \cup B^c)] && \text{by the commutative law for } \cup
\end{aligned}$$

41. *Hint:* The answer is $\emptyset$.

44. **a.** Proof: Suppose not. That is, suppose there exist sets A and B such that $A - B$ and B are not disjoint. *[We must derive a contradiction.]* Then $(A - B) \cap B \neq \emptyset$, and so there is an element x in $(A - B) \cap B$. By definition of intersection, $x \in A - B$ and $x \in B$, and by definition of difference, $x \in A$ and $x \notin B$. Hence $x \in B$ and also $x \notin B$, which is a contradiction. Thus the supposition is false, and we conclude that $A - B$ and B are disjoint.
b. Let A and B be any sets. Then

$$\begin{aligned}
&(A - B) \cap B\\
&\quad = (A \cap B^c) \cap B && \text{by the set difference law}\\
&\quad = A \cap (B^c \cap B) && \text{by the associative law for } \cap\\
&\quad = A \cap (B \cap B^c) && \text{by the commutative law for } \cap\\
&\quad = A \cap \emptyset && \text{by the complement law for } \cap\\
&\quad = \emptyset && \text{by the universal bound law for } \cap.
\end{aligned}$$

46. **a.** $A \triangle B = (A - B) \cup (B - A) = \{1, 2\} \cup \{5, 6\} = \{1, 2, 5, 6\}$

47. Proof: Let A and B be any subsets of a universal set. By definition of $\triangle$, showing that $A \triangle B = B \triangle A$ is equivalent to showing that $(A - B) \cup (B - A) = (B - A) \cup (A - B)$. But this follows immediately from the commutative law for $\cup$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

48. Proof: Let A be any subset of a universal set. Then

$$\begin{aligned} &A \triangle \emptyset \\ &= (A - \emptyset) \cup (\emptyset - A) && \text{by definition of } \triangle \\ &= (A \cap \emptyset^c) \cup (\emptyset \cap A^c) && \text{by the set difference law} \\ &= (A \cap U) \cup (A^c \cap \emptyset) && \text{by the complement of } U \text{ law and the commutative law for } \cap \\ &= A \cup \emptyset && \text{by the identity law for } \cap \text{ and the universal bound law for } \cap \\ &= A. && \text{by the identity law for } \cup \end{aligned}$$

51. *Hint:* First show that for any sets A and B and for any element x,

$$x \in A \triangle B \Leftrightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A),$$

and

$$x \notin A \triangle B \Leftrightarrow (x \notin A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \in A).$$

52. Same hint as for exercise 51.

53. *Start of proof*: Suppose A and B are any subsets of a universal set U. By the universal bound law for union, $B \cup U = U$, and so, by the commutative law for union, $U \cup B = U$. Take the intersection of both sides of the equation with A.

Section 6.4

1. a. because 1 is an identity for $\cdot$
b. by the complement law for $+$
c. by the distributive law for $+$ over $\cdot$
d. by the complement law for $\cdot$
e. because 0 is an identity for $+$

4. Proof: For all elements a in B,

$$\begin{aligned} a \cdot 0 &= a \cdot (a \cdot \overline{a}) && \text{by the complement law for } \cdot \\ &= (a \cdot a) \cdot \overline{a} && \text{by the associative law for } \cdot \\ &= a \cdot \overline{a} && \text{by exercise 48} \\ &= 0. && \text{by the complement law for } \cdot \end{aligned}$$

6. a. Proof: $0 \cdot 1 = 0$ because 1 is an identity for $\cdot$, and $0 + 1 = 1 + 0 = 1$ because $+$ is commutative and 0 is an identity for $+$. Thus, by the uniqueness of the complement law, $\overline{0} = 1$.

7. a. Proof: Suppose 0 and $0'$ are elements of B both of which are identities for $+$. Then both 0 and $0'$ satisfy the identity, complement, and universal bound laws. *[We will show that* $0 = 0'$.*]* By the identity law for $+$, for all $a \in B$,

$$a + 0 = a \qquad \text{and} \qquad a + 0' = a.$$

It follows that

$$\begin{aligned} \Rightarrow\quad a + 0 &= a + 0' && \text{because both quantities equals } a \\ \Rightarrow\quad \bar{a} \cdot (a + 0) &= \bar{a} \cdot (a + 0') && \text{by “multiplying” both sides by } \bar{a} \\ \Rightarrow\quad (\bar{a} \cdot a) + (\bar{a} \cdot 0) &= (\bar{a} \cdot a) + (\bar{a} \cdot 0') && \text{by the distributive law} \\ \Rightarrow\quad (a \cdot \bar{a}) + 0 &= (a \cdot \bar{a}) + 0' && \text{by the universal bound law for} \cdot \\ \Rightarrow\quad 0 \cdot 0 &= 0' \cdot 0' && \text{by the complement law for} \cdot \\ \Rightarrow\quad 0 &= 0' && \text{by the universal bound law for} \cdot \end{aligned}$$

[This is what was to be shown.]

b. *Hint:* Suppose 1 and $1'$ are elements of B both of which are identities for $\cdot$. Then for all $a \in B$, by the identity law for $\cdot$, $a \cdot 1 = a$ and $a \cdot 1' = a$. It follows that $a \cdot 1 = a \cdot 1'$, and $\bar{a} + a \cdot 1 = \bar{a} + a \cdot 1'$. Etc.

8. Proof: Suppose B is a Boolean algebra and a and b are any elements of B. We first prove that $(a \cdot b) + (\overline{a} + \overline{b}) = 1$.

$$\begin{aligned} &a \cdot b + (\overline{a} + \overline{b}) \\ &= (\overline{a} + \overline{b}) + (a \cdot b) && \text{by the commutative law for } + \\ &= ((\overline{a} + \overline{b}) + a) \cdot ((\overline{a} + \overline{b}) + b) && \text{by the distributive law of } + \text{ over } \cdot \\ &= ((\overline{b} + \overline{a}) + a) \cdot (\overline{a} + (\overline{b} + b)) && \text{by the commutative and associative laws for } + \\ &= (\overline{b} + (\overline{a} + a)) \cdot (\overline{a} + (b + \overline{b})) && \text{by the associative and commutative laws for } + \\ &= (\overline{b} + (a + \overline{a})) \cdot (\overline{a} + 1) && \text{by the commutative and complement laws for } + \\ &= (\overline{b} + 1) \cdot 1 && \text{by the complement and universal bound laws for } + \\ &= 1 \cdot 1 && \text{by the universal bound law for } + \\ &= 1 && \text{by the identity law for } \cdot. \end{aligned}$$

Next we prove that $(a \cdot b) \cdot (\overline{a} + \overline{b}) = 0$.

$$\begin{aligned} &(a \cdot b) \cdot (\overline{a} + \overline{b}) \\ &= ((a \cdot b) \cdot \overline{a}) + (((a \cdot b) \cdot \overline{b}) && \text{by the distributive law of } \cdot \text{ over } + \\ &= ((b \cdot a) \cdot \overline{a}) + ((a \cdot (b \cdot \overline{b})) && \text{by the commutative and associative laws for } \cdot \\ &= (b \cdot (a \cdot \overline{a})) + (a \cdot 0) && \text{by the associative and complement laws for } \cdot \\ &= (b \cdot 0) + 0 && \text{by the complement and universal bound laws for } \cdot \\ &= 0 + 0 && \text{by the universal bound law for } \cdot \\ &= 0 && \text{by the identity law for } +. \end{aligned}$$

Because both $(a \cdot b) + (\overline{a} + \overline{b}) = 1$ and $(a \cdot b) \cdot (\overline{a} + \overline{b}) = 0$, it follows, by the uniqueness of the complement law, that $\overline{a \cdot b} = \overline{a} + \overline{b}$.

10. *Hint:* One way to prove the statement is to use the result of exercise 3. Some stages in the proof are the following:

$$y = (y + x) \cdot y = (x \cdot y) + (z \cdot y) = z \cdot (x + y) = z.$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

11. a. (i) Because S has only two distinct elements, 0 and 1, we only need to check that $0+1=1+0$. But this is true because both sums equal 1.
(v) *Partial answer:*

$$0+(0\cdot 0)=0+0=0 \text{ and } (0+0)\cdot(0+0)=0\cdot 0=0 \text{ also}$$
$$0+(0\cdot 1)=0+0=0 \text{ and } (0+0)\cdot(0+1)=0\cdot 1=0 \text{ also}$$
$$0+(1\cdot 0)=0+0=0 \text{ and } (0+1)\cdot(0+0)=1\cdot 0=0 \text{ also}$$
$$0+(1\cdot 1)=0+1=1 \text{ and } (0+1)\cdot(0+1)=1\cdot 1=1 \text{ also}$$

b. *Hint:* Verify that $0+x=x$ and that $1\cdot x=x$ for all $x\in S$.

12. *Hints:* (1) Because the proofs of the absorption laws do not use the associative laws, the absorption laws may be used at any stage of the derivation.
(2) Show that for all x, y, and z in B, $x\,(x+(y+z))\cdot x=x$ and $((x+y)+z))\cdot x=x$.
(3) Show that for all a, b, and c in B, both $a+(b+c)$ and $(a+b)+c$ equal $((a+b)+c)\cdot(a+(b+c))$.
(4) Use De Morgan's laws and the double complement law to deduce the associative law for $\cdot$.

13. The sentence is not a statement because it is neither true nor false. If the sentence were true, then because it declares itself to be false, the sentence would be false. Therefore, the sentence is not true. On the other hand, if the sentence were false, then it would be false that "This sentence is false," and so the sentence would be true. Consequently, the sentence is not false.

14. This sentence is a statement because it is true. Recall that the only way for an if-then statement to be false is for the hypothesis to be true and the conclusion false. In this case the hypothesis is not true. So regardless of what the conclusion states, the sentence is true. (This is an example of a statement that is vacuously true, or true by default.)

17. This sentence is not a statement because it is neither true nor false. If the sentence were true, then either the sentence is false or $1+1=3$. But $1+1\neq 3$, and so the sentence is false. Therefore, the sentence is not true. On the other hand, if the sentence were false, then it would be true that "This sentence is false or $1+1=3$," and so the sentence would be true. Consequently, the sentence is not false.

20. *Hint:* Suppose that apart from statement (ii), all of Nixon's other assertions about Watergate are evenly split between true and false.

21. No. Suppose there were a computer program P that had as output a list of all computer programs that do not list themselves in their output. If P lists itself as output, then it would be on the output list of P, which consists of all computer programs that do not list themselves in their output. Hence P would not list itself as output. But if P does not list itself as output, then P would be a member of the list of all computer programs that do not list themselves in their output, and this list is exactly the output of P. Hence P would list itself as output. This analysis shows that the assumption of the existence of such a program P is contradictory, and so no such program exists.

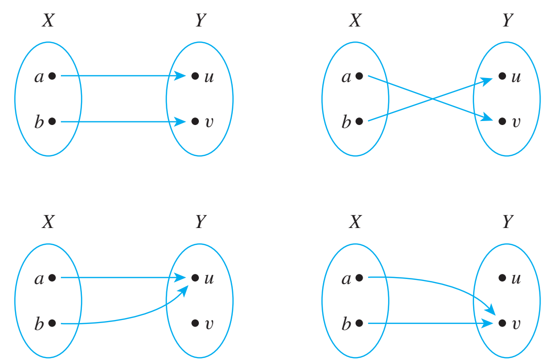
25. *Hint:* Show that any algorithm that solves the printing problem can be adapted to produce an algorithm that solves the halting problem.

Section 7.1

1. a. domain of $f=\{1,3,5\}$, co-domain of $f=\{s,t,u,v\}$
b. $f(1)=v$, $f(3)=s$, $f(5)=v$
c. range of $f=\{s,v\}$
d. yes, no
e. inverse image of $s=\{3\}$, inverse image of $u=\emptyset$, inverse image of $v=\{1,5\}$
f. $\{(1,v),(3,s),(5,v)\}$

3. a. True. The definition of function says that for any input there is one and only one output, so if two inputs are equal, their outputs must also be equal.
c. True. The definition of function does not prohibit this occurrence.

4. a. There are four functions from X to Y as shown below.



5. a. $I_{\mathbf{Z}}(e)=e$
b. $I_{\mathbf{Z}}\left(b_i^{jk}\right)=b_i^{jk}$

6. a. The sequence is given by the function $f\colon \mathbf{Z}^{nonneg}\to\mathbf{R}$ defined by the rule

$$f(n)=\frac{(-1)^n}{2n+1} \quad \text{for all nonnegative integers } n.$$

7. a. 1 *[because there is an odd number of elements in* $\{1,3,4\}$*]*
c. 0 *[because there is an even number of elements in* $\{2,3\}$*]*

8. a. $F(0)=(0^3+2\cdot 0+4)\ mod\ 5=4\ mod\ 5=4$
b. $F(1)=(1^3+2\cdot 1+4)\ mod\ 5=7\ mod\ 5=2$

9. a. $S(1)=1$ **b.** $S(15)=1+3+5+15=24$
c. $S(17)=1+17=18$

10. a. $T(1)=\{1\}$ **b.** $T(15)=\{1,3,5,15\}$
c. $T(17)=\{1,17\}$

11. a. $F(4,4)=(2\cdot 4+1,3\cdot 4-2)=(9,10)$
b. $F(2,1)=(2\cdot 2+1,3\cdot 1-2)=(5,1)$

12. a. $G(4,4)=((2\cdot 4+1)\ mod\ 5,(3\cdot 4-2)\ mod\ 5)=(9\ mod\ 5,10\ mod\ 5)=(4,0)$
b. $G(2,1)=((2\cdot 2+1)\ mod\ 5,(3\cdot 1-2)\ mod\ 5)=(5\ mod\ 5,1\ mod\ 5)=(0,1)$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

13.

x	$f(x)$	$g(x)$
0	$4^2 \, mod \, 5 = 1$	$(0^2 + 3 \cdot 0 + 1) \, mod \, 5 = 1$
1	$5^2 \, mod \, 5 = 0$	$(1^2 + 3 \cdot 1 + 1) \, mod \, 5 = 0$
2	$6^2 \, mod \, 5 = 1$	$(2^2 + 3 \cdot 2 + 1) \, mod \, 5 = 1$
3	$7^2 \, mod \, 5 = 4$	$(3^2 + 3 \cdot 3 + 1) \, mod \, 5 = 4$
4	$8^2 \, mod \, 5 = 4$	$(4^2 + 3 \cdot 4 + 1) \, mod \, 5 = 4$

The table shows that $f(x) = g(x)$ for all x in J_5. Thus, by definition of equality of functions, $f = g$.

15. $F \cdot G$ and $G \cdot F$ are equal because for all real numbers x,

$$\begin{aligned}(F \cdot G)(x) &= F(x) \cdot G(x) && \text{by definition of } F \cdot G \\ &= G(x) \cdot F(x) && \text{by the commutative law for multiplication of real numbers} \\ &= (G \cdot F)(x) && \text{by definition of } G \cdot F.\end{aligned}$$

17. a. $2^3 = 8$ **c.** $4^1 = 4$

18. a. $\log_3 81 = 4$ because $3^4 = 81$

c. $\log_3 \left(\frac{1}{27}\right) = -3$ because $3^{-3} = \frac{1}{27}$

19. Let b be any positive real number with $b \neq 1$. Since $b^1 = b$, by definition of logarithm, $\log_b b = 1$.

21. Proof: Suppose b and u are any positive real numbers. *[We must show that* $\log_b \left(\frac{1}{u}\right) = -\log_b(u)$.*]* Let $v = \log_b \left(\frac{1}{u}\right)$. By definition of logarithm, $b^v = \frac{1}{u}$. Multiplying both sides by u and dividing by b^v gives $u = b^{-v}$, and thus, by definition of logarithm, $-v = \log_b(u)$. Now multiply both sides of this equation by -1 to obtain $v = -\log_b(u)$. Therefore, $\log_b \left(\frac{1}{u}\right) = -\log_b(u)$ because both expressions equal v. *[This is what was to be shown.]*

22. *Hint:* Use a proof by contradiction. Suppose $\log_3 7$ is rational. Then $\log_3 7 = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Apply the definition of logarithm to rewrite $\log_3 7 = \frac{a}{b}$ in exponential form.

23. Suppose b and y are positive real numbers with $\log_b y = 3$. By definition of logarithm, this implies that $b^3 = y$. Then

$$y = b^3 = \frac{1}{\frac{1}{b^3}} = \frac{1}{\left(\frac{1}{b}\right)^3} = \left(\frac{1}{b}\right)^{-3}.$$

Thus, by definition of logarithm (with base $1/b$), $\log_{1/b}(y) = -3$.

25. a. $p_1(2, y) = 2$, $p_1(5, x) = 5$, range of $p_1 = \{2, 3, 5\}$

26. a. $mod(67, 10) = 7$ and $div(67, 10) = 6$ since $67 = 10 \cdot 6 + 7$.

27. $f(aba) = 0$ *[because there are no b's to the left of the left-most a in aba]*

$f(bbab) = 2$ *[because there are two b's to the left of the left-most a in bbab]*

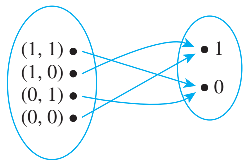
$f(b) = 0$ *[because the string b contains no a's]*

range of $f = Z^{nonneg}$

28. a. $E(0110) = 000111111000$ and $D(111111000111) = 1101$

29. a. $H(10101, 00011) = 3$

30. a. Domain of f Co-domain of f

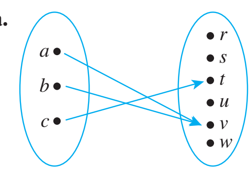


32. a. $f(1, 1, 1) = (4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1) \, mod \, 2 = 9 \, mod \, 2 = 1$
$f(0, 0, 1) = (4 \cdot 0 + 3 \cdot 0 + 2 \cdot 1) \, mod \, 2 = 2 \, mod \, 2 = 0$

33. If g were well defined, then $g(1/2) = g(2/4)$ because $1/2 = 2/4$. However, $g(1/2) = 1 - 2 = -1$ and $g(2/4) = 2 - 4 = -2$. Since $-1 \neq -2$, $g(1/2) \neq g(2/4)$. Thus g is not well defined.

35. Student B is correct. If R were well defined, then $R(3)$ would have a uniquely determined value. However, on the one hand, $R(3) = 2$ because $(3 \cdot 2) \, mod \, 5 = 1$, and, on the other hand, $R(3) = 7$ because $(3 \cdot 7) \, mod \, 5 = 1$. Hence $R(3)$ does not have a uniquely determined value, and so R is not well defined.

38. a.

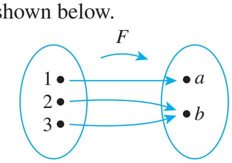


b. $f(A) = \{v\}$, $f(X) = \{t, v\}$, $f^{-1}(C) = \{c\}$, $f^{-1}(D) = \{a, b\}$, $f^{-1}(E) = \emptyset$ $f^{-1}(Y) = \{a, b, c\} = X$

40. *Partial answer:* **(a)** $y \in F(A)$ or $y \in F(B)$, **(b)** some, **(c)** $A \cup B$, **(d)** $F(A \cup B)$

41. The statement is true. Proof: Let F be a function from X to Y, and suppose $A \subseteq X$, $B \subseteq X$, and $A \subseteq B$. Let $y \in F(A)$. *[We must show that* $y \in F(B)$.*]* Then, by definition of image of a set, $y = F(x)$ for some $x \in A$. Since $A \subseteq B$, $x \in B$, and so $y = F(x)$ for some $x \in B$. Hence $y \in F(B)$ *[as was to be shown]*.

43. The statement is false. Counterexample: Let $X = \{1, 2, 3\}$, let $Y = \{a, b\}$, and define a function $F: X \to Y$ by the arrow diagram shown below.



Let $A = \{1, 2\}$ and $B = \{1, 3\}$. Then $F(A) = \{a, b\} = F(B)$, and so $F(A) \cap F(B) = \{a, b\}$. But $F(A \cap B) = F(\{1\}) = \{a\} \neq \{a, b\}$. And so $F(A) \cap F(B) \nsubseteq F(A \cap B)$.
(This is just one of many possible counterexamples.)

45. The statement is true. Proof: Let F be a function from a set X to a set Y, and suppose $C \subseteq Y$, $D \subseteq Y$, and $C \subseteq D$. *[We must show that* $F^{-1}(C) \subseteq F^{-1}(D)$.*]* Suppose $x \in F^{-1}(C)$. Then $F(x) \in C$. Since $C \subseteq D$, $F(x) \in D$ also. Hence by definition of inverse image, $x \in F^{-1}(D)$. *[So* $F^{-1}(C) \subseteq F^{-}1(D)$.*]*

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

46. *Hint:* $x \in F^{-1}(C \cup D) \Leftrightarrow F(x) \in C \cup D \Leftrightarrow F(x) \in C$ or $F(x) \in D$

51. **a.** $\phi(15) = 8$ *[because* $1, 2, 4, 7, 8, 11, 13,$ *and* 14 *have no common factors with* 15 *other than* ± 1*]*

b. $\phi(2) = 1$ *[because the only positive integer less than or equal to* 2 *having no common factors with* 2 *other than* ± 1 *is* 1*]*

c. $\phi(5) = 4$ *[because* $1, 2, 3,$ *and* 4 *have no common factors with* 5 *other than* ± 1*]*

52. Proof: Let p be any prime number and n any integer with $n \geq 1$. There are p^{n-1} positive integers less than or equal to p^n that have a common factor other than ± 1 with p^n, namely $p, 2p, 3p, \ldots, (p^{n-1})p$. Hence, by the difference rule, there are $p^n - p^{n-1}$ positive integers less than or equal to p^n that have no common factor with p^n except ± 1.

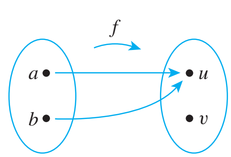
53. *Hint:* Use the result of exercise 52 with $p = 2$.

Section 7.2

1. The second statement is the contrapositive of the first.

2. **a.** most **b.** least

3. *Hint:* One counterexample is given and explained below. Give a different counterexample and accompany it with an explanation. Counterexample: Consider the function defined by the following arrow diagram:



Observe that a is sent to exactly one element of Y, namely, u, and b is also sent to exactly one element of Y, namely, u also. So it is true that every element of X is sent to exactly one element of Y. But f is not one-to-one because $f(a) = f(b)$ but $a \neq b$. *[Note that to say, "Every element of X is sent to exactly one element of Y" is just another way of saying that in the arrow diagram for the function there is only one arrow coming out of each element of X. But this statement is part of the definition of* any *function, not just of a one-to-one function.]*

4. *Hint:* The statement is true.

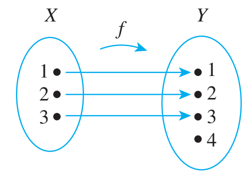
5. *Hint:* One of the incorrect ways is (b).

6. **a.** f is not one-to-one because $f(1) = 4 = f(9)$ and $1 \neq 9$. f is not onto because $f(x) \neq 3$ for any x in X.

b. g is one-to-one because $g(1) \neq g(5)$, $g(1) \neq g(9)$, and $g(5) \neq g(9)$. g is onto because each element of Y is the image of some element of X: $3 = g(5)$, $4 = g(9)$, and $7 = g(1)$.

7. **a.** F is not one-to-one because $F(c) = x = F(d)$ and $c \neq d$. F is onto because each element of Y is the image of some element of X: $x = F(c) = F(d)$, $y = F(a)$, and $z = F(b)$.

9. **a.** One example of many is the following:



10. **a.** (i) f *is one-to-one:* Suppose $f(n_1) = f(n_2)$ for some integers n_1 and n_2. *[We must show that* $n_1 = n_2$*.]* By definition of f, $2n_1 = 2n_2$, and dividing both sides by 2 gives $n_1 = n_2$, as was to be shown.

(ii) f *is not onto:* Consider $1 \in \mathbf{Z}$. We claim that $1 \neq f(n)$, for any integer n, because if there were an integer n such that $1 = f(n)$, then, by definition of f, $1 = 2n$. Dividing both sides by 2 would give $n = 1/2$. But $1/2$ is not an integer. Hence $1 \neq f(n)$ for any integer n, and so f is not onto.

b. h *is onto:* Suppose $m \in 2\mathbf{Z}$. *[We must show that there exists an integer n such that* $h(n) = m$*.] Since* $m \in 2\mathbf{Z}$, $m = 2k$ for some integer k. Let $n = k$. Then $h(n) = 2n = 2k = m$. Hence there exists an integer (namely, n) such that $h(n) = m$. This is what was to be shown.

11. *Hints:* **a.** (i) g is one-to-one (ii) g is not onto

b. G is onto. Proof: Suppose y is any element of $\mathbf{R}$. *[We must show that there is an element x in* $\mathbf{R}$ *such that* $G(x) = y$*. What would x be if it exists? Scratch work shows that x would have to equal* $(y + 5)/4$*. The proof must then show that x has the necessary properties.]* Let $x = (y + 5)/4$. Then (1) $x \in \mathbf{R}$, and (2) $G(x) = G((y + 5)/4) = 4[(y + 5)/4] - 5 = (y + 5) - 5 = y$ *[as was to be shown]*.

13. **a.** (i) H *is not one-to-one:* $H(1) = 1 = H(-1)$ but $1 \neq -1$.

(ii) H *is not onto:* $H(x) \neq -1$ for any real number x (since no real numbers have negative squares).

14. The "proof" claims that f is one-to-one because for each integer n there is only one possible value for $f(n)$. But to say that for each integer n there is only one possible value for $f(n)$ is just another way of saying that f satisfies one of the conditions necessary for it to be a function. To show that f is one-to-one, one must show that any integer n has a *different* function value from that of the integer m whenever $n \neq m$.

15. f is one-to-one. Proof: Suppose $f(x_1) = f(x_2)$ where x_1 and x_2 are nonzero real numbers. *[We must show that* $x_1 = x_2$*.]* By definition of f,

$$\frac{x_1 + 1}{x_1} = \frac{x_2 + 1}{x_2}$$

cross-multiplying gives

$$x_1 x_2 + x_2 = x_1 x_2 + x_1,$$

and so

$$x_1 = x_2 \qquad \text{by subtracting } x_1x_2 \text{ from both sides}$$

[This is what was to be shown.]

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

16. f is not one-to-one. Note that

$$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1} \Rightarrow x_1x_2^2 + x_1 = x_2x_1^2 + x_2$$
$$\Rightarrow x_1x_2^2 - x_2x_1^2 = x_2 - x_1$$
$$\Rightarrow x_1x_2(x_2 - x_1) = x_2 - x_1$$
$$\Rightarrow x_1 = x_2 \text{ or } x_1x_2 = 1.$$

Thus for a counterexample take any x_1 and x_2 with $x_1 \neq x_2$ but $x_1x_2 = 1$. For instance, take $x_1 = 2$ and $x_2 = 1/2$. Then $f(x_1) = f(2) = 2/5$ and $f(x_2) = f(1/2) = 2/5$, but $2 \neq 1/2$.

19. a. Note that because $\frac{417302072}{7} \cong 59614581.7$ and $417302072 - 7 \cdot 59614581 = 5$, $h(417\text{-}30\text{-}2072) = 5$. But position 5 is already occupied, so the next position is checked. It is free, and thus the record is placed in position 6.

20. Recall that $\lfloor x \rfloor =$ that unique integer n such that $n \leq x < n+1$.

a. *Floor is not one-to-one:*
Floor(0) $= 0 =$ Floor $(1/2)$ but $0 \neq 1/2$.

b. *Floor is onto:* Suppose $m \in \mathbf{Z}$. *[We must show that there exists a real number y such that Floor(y) = m.]* Let $y = m$. Then Floor(y) $=$ Floor(m) $= m$ since m is an integer. (Actually, Floor takes the value m for *all* real numbers in the interval $m \leq x < m+1$.) Hence there exists a real number y such that Floor(y) $= m$. This is what was to be shown.

21. a. *l is not one-to-one:* $l(0) = l(1) = 1$ but $1 \neq 0$.

b. *l is onto:* Suppose n is a nonnegative integer. *[We must show that there exists a string s in S such that $l(s) = n$.]* Let

$$s = \begin{cases} \epsilon \text{ (the null string)} & \text{if } n = 0 \\ \underbrace{00\ldots0}_{n\ 0\text{'s}} & \text{if } n > 0 \end{cases}.$$

Then $l(s) =$ the length of $s = n$. This is what was to be shown.

23. a. *F is not one-to-one:* Let $A = \{a\}$ and $B = \{b\}$. Then $F(A) = F(B) = 1$ but $A \neq B$.

24. b. *N is not onto:* The number -1 is in $\mathbf{Z}$ but $N(s) \neq -1$ for any string s in S because no string has a negative number of a's.

26. S is not one-to-one. Counterexample: $S(6) = 1+2+3+6 = 12$ and $S(11) = 1+11 = 12$. So $S(6) = S(11)$ but $6 \neq 11$.
S is not onto. Counterexample: In order for there to be a positive integer n such that $S(n) = 5$, n would have to be less than 5. But $S(1) = 1$, $S(2) = 3$, $S(3) = 4$, and $S(4) = 7$. Hence there is no positive integer n such that $S(n) = 5$.

27. *Hint:* **a**. T is not one-to-one. **b**. T is not onto.

28. a. G is one-to-one. Proof: Suppose (x_1, y_1) and (x_2, y_2) are any elements of $\mathbf{R} \times \mathbf{R}$ such that $G(x_1, y_1) = G(x_2, y_2)$. *[We must show that $(x_1, y_1) = (x_2, y_2)$.]* Then, by definition of G, $(2y_1, -x_1) = (2y_2, -x_2)$, and, by definition of ordered pair,

$$2y_1 = 2y_2 \quad \text{and} \quad -x_1 = -x_1.$$

Dividing both sides of the left equation by 2 and both sides of the right equation by -1 gives that

$$y_1 = y_2 \quad \text{and} \quad x_1 = x_2,$$

and so, by definition of ordered pair, $(x_1, y_1) = (x_2, y_2)$ *[as was to be shown]*.

b. G is onto. Proof: Suppose (u, v) is any element of $\mathbf{R} \times \mathbf{R}$. *[We must show that there is an element (x, y) in $\mathbf{R} \times \mathbf{R}$ such that $G(x, y) = (u, v)$.]* Let $(x, y) = (-v, u/2)$. Then (1) $(x, y) \in \mathbf{R} \times \mathbf{R}$ and (2) $G(x, y) = (2y, -x) = (2(u/2), -(-v)) = (u, v)$ *[as was to be shown.]*

31. a. *Hint:* F is one-to-one. Use the unique factorization of integers theorem in the proof.

32. a. Let $x = \log_8 27$ and $y = \log_2 3$. *[The question is: Is $x = y$?]* By definition of logarithm, both of these equations can be written in exponential form as

$$8^x = 27 \quad \text{and} \quad 2^y = 3.$$

Now $8 = 2^3$. So

$$8^x = (2^3)^x = 2^{3x}.$$

Also $27 = 3^3$ and $3 = 2^y$. So

$$27 = 3^3 = (2^y)^3 = 2^{3y}.$$

Hence, since $8^x = 27$,

$$2^{3x} = 2^{3y}.$$

By (7.2.5), then,

$$3x = 3y,$$

and so

$$x = y.$$

But $x = \log_8 27$ and $y = \log_2 3$, and so $\log_8 27 = y = \log_2 3$ and the answer to the question is yes.

33. Proof: Suppose that b, x, and y are positive real numbers and $b \neq 1$. Let $u = \log_b(x)$ and $v = \log_b(y)$. By definition of logarithm, $b^u = x$ and $b^v = y$. By substitution, $\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v}$ *[by (7.2.3) and the fact that $b^{-v} = \frac{1}{b^v}$]*. Translating $\frac{x}{y} = b^{u-v}$ into logarithmic form gives $\log_b\left(\frac{x}{y}\right) = u - v$, and so, by substitution, $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ *[as was to be shown]*.

35. *Start of Proof:* Suppose a, b, and x are *[particular but arbitrarily chosen]* real numbers such that b and x are positive and $b \neq 1$. *[We must show that $\log_b(x^a) = a \log_b x$.]* Let

$$r = \log_b(x^a) \text{ and } s = \log_b x.$$

36. No. Counterexample: Define $f\colon \mathbf{R} \to \mathbf{R}$ and $g\colon \mathbf{R} \to \mathbf{R}$ as follows: $f(x) = x$ and $g(x) = -x$ for all real numbers x. Then f and g are both one-to-one *[because for all real number x_1 and x_2, if $f(x_1) = f(x_2)$ then $x_1 = x_2$, and if $g(x_1) = g(x_2)$ then $-x_1 = -x_2$ and so $x_1 = x_2$ also]*, but

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

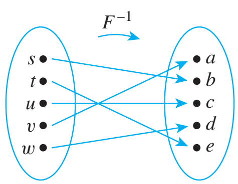
$f+g$ is not one-to-one *[because $f+g$ satisfies the equation $(f+g)(x) = x + (-x) = 0$ for all real numbers x, and so, for instance, $(f+g)(1) = (f+g)(2)$ but $1 \neq 2$].*

38. Yes. Proof: Let b be a one-to-one function from $\mathbf{R}$ to $\mathbf{R}$, and let c be any nonzero real number. Suppose $(cf)(x_1) = (cf)(x_2)$. *[We must show that $x_1 = x_2$.]* It follows by definition of cf that $cf(x_1) = cf(x_2)$. Since $c \neq 0$, we may divide both sides of the equation by c to obtain $f(x_1) = f(x_2)$. But since f is one-to-one, this implies that $x_1 = x_2$ *[as was to be shown].*

40. **a.** *Hint:* The assumption that F is one-to-one is needed in the proof that $F^{-1}(F(A)) \subseteq A$. If $F(r) \in F(A)$, the definition of image of a set implies that there is an element x in A such that $F(r) = F(x)$.

b. *Hint:* The assumption that F is one-to-one is needed in the proof that $F(A_1) \cap F(A_2) \subseteq F(A_1 \cap A_2)$. If $u \in F(A_1)$ and $u \in F(A_2)$, then the definition of image of a set implies that there are elements x_1 in A_1 and x_2 in A_2 such that $F(x_1) = u$ and $F(x_2) = u$ and, thus, that $F(x_1) = F(x_2)$.

42.



44. The function is not onto. Hence it is not a one-to-one correspondence.

45. The answer to exercise 10(b) shows that h is onto. To show that h is one-to-one, suppose $h(n_1) = h(n_2)$. By definition of h, this implies that $2n_1 = 2n_2$. Dividing both sides by 2 gives $n_1 = n_2$. Hence h is one-to-one.

Given any even integer m, if $m = h(n)$, then by definition of h, $m = 2n$, and so $n = m/2$. Thus

$$h^{-1}(m) = \frac{m}{2} \text{ for all } m \in 2\mathbf{Z}.$$

46. The function g is not a one-to-one correspondence because it is not onto. For instance, if $m = 2$, it is impossible to find an integer n such that $g(n) = m$. (This is because if $g(n) = m$, then $4n - 5 = 2$, which implies that $n = 7/4$. Thus the only number n with the property that $g(n) = m$ is $7/4$. But $7/4$ is not an integer.)

47. The answer to exercise 11b shows that G is onto. In addition, G is one-to-one. To prove this, suppose $G(x_1) = G(x_2)$ for some x_1 and x_2 in $\mathbf{R}$. *[We must show that $x_1 = x_2$.]* By definition of G, $4x_1 - 5 = 4x_2 - 5$. Add 5 to both sides of this equation and divide both sides by 4 to obtain $x_1 = x_2$ *[as was to be shown].* We claim that $G^{-1}(y) = (y+5)/4$. By definition of inverse function, this is true if, and only if, $G((y+5)/4) = y$. But $G((y+5)/4) = 4((y+5)/4) - 5 = (y+5) - 5 = y$, so it is the case that $G^{-1}(y) = (y+5)/4$.

50. The function is not one-to-one. Hence it is not a one-to-one correspondence.

52. The answer to exercise 15 shows that f is one-to-one, and if the co-domain is taken to be the set of all real numbers not equal to 1, then f is also onto. *[The reason is that given any real number $y \neq 1$, if we take $x = \frac{1}{y-1}$, then*

$$f(x) = f\left(\frac{1}{y-1}\right) = \frac{\frac{1}{y-1} + 1}{\frac{1}{y-1}} = \frac{1 + (y-1)}{1} = y.]$$

$$f^{-1}(y) = \frac{1}{y-1} \text{ for each real number } y \neq 1.$$

53. *Hint:* Is there a real number x such that $f(x) = 1$?

57. *Hint:* Let a function F be given and suppose the domain of F is represented as a one-dimensional array $a[1], a[2], \ldots, a[n]$. Introduce a variable *answer* whose initial value is "one-to-one." The main part of the body of the algorithm could be written as follows:

while ($i \leq n-1$ and *answer* = "one-to-one")
  $j := i + 1$
  while ($j \leq n$ and *answer* = "one-to-one")
    if ($F(a[i]) = F(a[j])$ and $a[i] \neq a[j]$)
    then *answer* := "not one-to-one"
    $j := j + 1$
  end while
  $i := i + 1$
end while

What can you say if execution reaches this point?

58. *Hint:* Let a function F be given and suppose the domain and co-domain of F are represented by the one-dimensional arrays $a[1], a[2], \ldots, a[n]$ and $b[1], b[2], \ldots, b[m]$, respectively. Introduce a variable *answer* whose initial value is "onto." For each $b[i]$ from $i = 1$ to m, make a search through $a[1], a[2], \ldots, a[n]$ to check whether $b[i] = F(a[j])$ for some $a[j]$. Introduce a Boolean variable to indicate whether a search has been successful. (Set the variable equal to 0 before the start of each search, and let it have the value 1 if the search is successful.) At the end of each search, check the value of the Boolean variable. If it is 0, then F is not onto. If all searches are successful, then F is onto.

Section 7.3

1. $g \circ f$ is defined as follows:

$$(g \circ f)(1) = g(f(1)) = g(5) = 1,$$
$$(g \circ f)(3) = g(f(3)) = g(3) = 5,$$
$$(g \circ f)(5) = g(f(5)) = g(1) = 3.$$

$f \circ g$ is defined as follows:

$$(f \circ g)(1) = f(g(1)) = f(3) = 3,$$
$$(f \circ g)(3) = f(g(3)) = f(5) = 1,$$
$$(f \circ g)(5) = f(g(5)) = f(1) = 5.$$

Then $g \circ f \neq f \circ g$ because, for example, $(g \circ f)(1) \neq (f \circ g)(1)$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

3. $(G \circ F)(x) = G(F(x)) = G(x^3) = x^3 - 1$ for all real numbers x.
$(F \circ G)(x) = F(G(x)) = F(x-1) = (x-1)^3$ for all real numbers x.
$G \circ F \neq F \circ G$ because, for instance, $(G \circ F)(2) = 2^3 - 1 = 7$, whereas $(F \circ G)(2) = (2-1)^3 = 1$.

6. $(G \circ F)(0) = G(F(0)) = G(7.0) = G(0) = 0 \ mod\ 5 = 0$
$(G \circ F)(1) = G(F(1)) = G(7.1) = G(7) = 7 \ mod\ 5 = 2$
$(G \circ F)(2) = G(F(2)) = G(7.2) = G(14) = 14 \ mod\ 5 = 4$
$(G \circ F)(3) = G(F(3)) = G(7.3) = G(21) = 21 \ mod\ 5 = 1$
$(G \circ F)(4) = G(F(4)) = G(7.4) = G(28) = 28 \ mod\ 5 = 3$

8. a.
$$\begin{aligned}(L \circ M)(12) &= L(M(12)) = L(12 \ mod\ 5) = L(2) \\ &= 2^2 = 4 \\ (M \circ L)(12) &= M(L(12)) = M(12^2) = M(144) \\ &= 144 \ mod\ 5 = 4 \\ (L \circ M)(9) &= L(M(9)) = L(9 \ mod\ 5) = L(4) \\ &= 4^2 = 16 \\ (M \circ L)(9) &= M(L(9)) = M(9^2) = M(81) \\ &= 81 \ mod\ 5 = 1\end{aligned}$$

9.
$$\begin{aligned}(F^{-1} \circ F)(x) &= F^{-1}(F(x)) = F^{-1}(3x+2) \\ &= \frac{(3x+2)-2}{3} = \frac{3x}{3} = x = I_{\mathbf{R}}(x)\end{aligned}$$

for all x in $\mathbf{R}$. Hence $F^{-1} \circ F = I_{\mathbf{R}}$ by definition of equality of functions.

$$\begin{aligned}(F \circ F^{-1})(y) &= F(F^{-1}(y)) = F\left(\frac{y-2}{3}\right) \\ &= 3\left(\frac{y-2}{3}\right) + 2 = (y-2) + 2 \\ &= y = I_{\mathbf{R}}(y)\end{aligned}$$

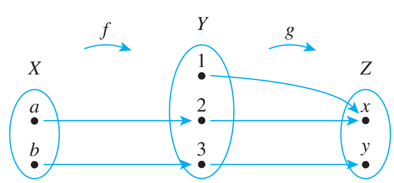
for all y in $\mathbf{R}$. Hence $F \circ F^{-1} = I_{\mathbf{R}}$ by definition of equality of functions.

12. a. By definition of logarithm with base b, for each real number x, $\log_b(b^x)$ is the exponent to which b must be raised to obtain b^x. But this exponent is just x. So $\log_b(b^x) = x$.

13. *Hint:* Suppose f is any function from a set X to a set Y, and show that for all x in X, $(I_Y \circ f)(x) = f(x)$.

15. a. $s_k = s_m$

16. No. Counterexample: Define f and g by the arrow diagrams below.



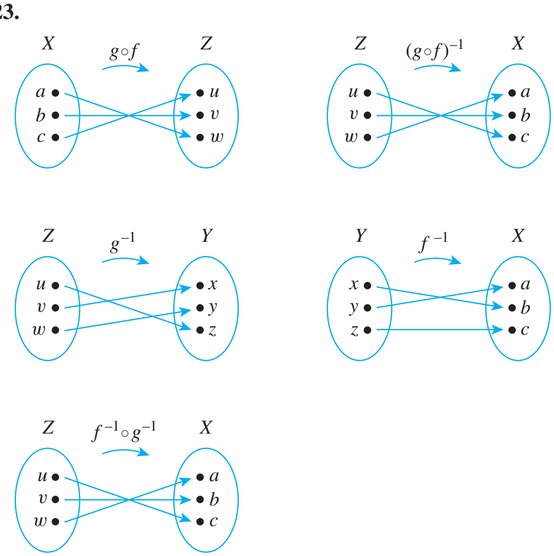
Then $g \circ f$ is one-to-one but g is not one-to-one. (So it is false that *both* f and g are one-to-one by De Morgan's law!) (This is one counterexample among many. Can you construct a different one?)

18. *Hint:* Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and $g \circ f$ is one-to-one. Given x_1 and x_2 in X, if $f(x_1) = f(x_2)$ then $(g \circ f)(x_1) = (g \circ f)(x_2)$. (Why?) Then use the fact that $g \circ f$ is one-to-one.

19. *Hint:* Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and $g \circ f$ is onto. Given $z \in Z$, there is an element x in X such that $(g \circ f)(x) = z$. (Why?) Let $y = f(x)$. Then $g(y) = z$.

21. True. Proof: Suppose X is any set and f, g, and h are functions from X to X such that h is one-to-one and $h \circ f = h \circ g$. *[We must show that for all x in X, $f(x) = g(x)$.]* Suppose x is any element in X. Because $h \circ f = h \circ g$, we have that $(h \circ f)(x) = (h \circ g)(x)$ by definition of equality of functions. Then, by definition of composition of functions, $h(f(x) = h(g(x))$. But since h is one-to-one, this implies that $f(x) = g(x)$ *[as was to be shown]*.

23.



The functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are equal.

26. *Hints:* (1) Theorems 7.3.3 and 7.3.4 taken together insure that $g \circ f$ is one-to-one and onto. (2) Use the inverse function property: $F^{-1}(b) = a \Leftrightarrow F(a) = b$, for all a in the domain of F and b in the domain of F^{-1}.

Section 7.4

1. The student should have replied that for A to have the same cardinality as B means that there is a function from A to B that is one-to-one and onto. A set cannot have the property of being one-to-one or onto another set; only a function can have these properties.

2. Define a function $f\colon \mathbf{Z}^+ \to S$ as follows: For all positive integers k, $f(k) = k^2$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

f is *one-to-one: [We must show that for all* $k_1, k_2 \in \mathbf{Z}^+$, *if* $f(k_1) = f(k_2)$ *then* $k_1 = k_2$.*]* Suppose k_1 and k_2 are positive integers and $f(k_1) = f(k_2)$. By definition of f, $(k_1)^2 = (k_2)^2$, so $k_1 = \pm k_2$. But k_1 and k_2 are *positive*. Hence $k_1 = k_2$.

f *is onto: [We must show that for all* $n \in S$, *there exists* $k \in \mathbf{Z}^+$ *such that* $n = f(k)$.*]* Suppose $n \in S$. By definition of S, $n = k^2$ for some positive integer k. But then by definition of f, $n = f(k)$.

Since there is a one-to-one, onto function (namely, f) from $\mathbf{Z}^+$ to S, the two sets have the same cardinality.

3. Define $f: \mathbf{Z} \to 3\mathbf{Z}$ by the rule $f(n) = 3n$ for all integers n. The function f is one-to-one because for any integers n_1 and n_2, if $f(n_1) = f(n_2)$ then $3n_1 = 3n_2$ and so $n_1 = n_2$. Also f is onto because if m is any element in $3\mathbf{Z}$, then $m = 3k$ for some integer k. But then $f(k) = 3k = m$ by definition of f. Thus, since there is a function $f: \mathbf{Z} \to 3\mathbf{Z}$ that is one-to-one and onto, $\mathbf{Z}$ has the same cardinality as $3\mathbf{Z}$.

6. *Hint:* If $m \in 2\mathbf{Z}$, show that $J(m) = J(m+1) = m$.

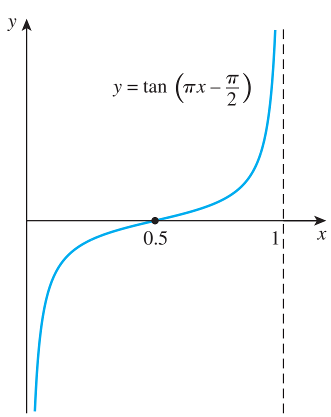
7. b. For each positive integer n, $F(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor$.

8. It was shown in Example 7.4.2 that $\mathbf{Z}$ is countably infinite, which means that $\mathbf{Z}^+$ has the same cardinality as $\mathbf{Z}$. By exercise 3, $\mathbf{Z}$ has the same cardinality as $3\mathbf{Z}$. It follows by the transitive property of cardinality (Theorem 7.4.1 (c)) that $\mathbf{Z}^+$ has the same cardinality as $3\mathbf{Z}$. Thus $3\mathbf{Z}$ is countably infinite *[by definition of countably infinite]*, and hence $3\mathbf{Z}$ is countable *[by definition of countable]*.

10. Proof: Define $f: S \to U$ by the rule $f(x) = 2x$ for all real numbers x in S. Then f is one-to-one by the same argument as in exercise 10a of Section 7.2 with $\mathbf{R}$ in place of $\mathbf{Z}$. Furthermore, f is onto because if y is any element in U, then $0 < y < 2$ and so $0 < y/2 < 1$. Consequently, $y/2 \in S$ and $f(y/2) = 2(y/2) = y$. Hence f is a one-to-one correspondence, and so S and U have the same cardinality.

11. *Hint:* Define $h: S \to V$ as follows: $h(x) = 3x + 2$, for all $x \in S$.

13.



It is clear from the graph that f is one-to-one (since it is increasing) and that the image of f is all of $\mathbf{R}$ (since the lines $x = 0$ and $x = 1$ are vertical asymptotes). Thus S and $\mathbf{R}$ have the same cardinality.

16. In Example 7.4.4 it was shown that there is a one-to-one correspondence from $\mathbf{Z}^+$ to $\mathbf{Q}^+$. This implies that the positive rational numbers can be written as an infinite sequence: $r_1, r_2, r_3, r_4, \ldots$. Now the set $\mathbf{Q}$ of all rational numbers consists of the numbers in this sequence together with 0 and the negative rational numbers: $-r_1, -r_2, -r_3, -r_4, \ldots$. Let $r_0 = 0$. Then the elements of the set of all rational numbers can be "counted" as follows:

$$r_0, r_1, -r_1, r_2, -r_2, r_3, -r_3, r_4, -r_4, \ldots.$$

In other words, we can define a one-to-one correspondence

$$G(n) = \begin{cases} r_{n/2} & \text{if } n \text{ is even} \\ -r_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases} \quad \text{for all integers } n \geq 1.$$

Therefore, $\mathbf{Q}$ is countably infinite and hence countable.

18. *Hint:* No.

19. *Hint:* Suppose r and s are real numbers with $s > r > 0$. Let n be an integer such that $n > \frac{\sqrt{2}}{s-r}$. Then $s - r > \frac{\sqrt{2}}{n}$. Let $m = \left\lfloor \frac{nr}{\sqrt{2}} \right\rfloor + 1$. Then $m > \frac{nr}{\sqrt{2}} \geq m - 1$. Use the fact that $s = r + (s - r)$ to show that $r < \frac{\sqrt{2}m}{n} < s$.

22. *Hint:* Use the unique factorization of integers theorem (Theorem 4.3.5) and Theorem 7.4.3.

23. a. Define a function $G: \mathbf{Z}^{nonneg} \to \mathbf{Z}^{nonneg} \times \mathbf{Z}^{nonneg}$ as follows: Let $G(0) = (0, 0)$, and then follow the arrows in the diagram, letting each successive ordered pair of integers be the value of G for the next successive integer. Thus, for instance, $G(1) = (1, 0)$, $G(2) = (0, 1)$, $G(3) = (2, 0)$, $G(4) = (1, 1)$, $G(5) = (0, 2)$, $G(6) = (3, 0)$, $G(7) = (2, 1)$, $G(8) = (1, 2)$, and so forth.

b. *Hint:* Observe that if the top ordered pair of any given diagonal is $(k, 0)$, the entire diagonal (moving from top to bottom) consists of $(k, 0), (k-1, 1), (k-2, 2), \ldots, (2, k-2), (1, k-1), (0, k)$. Thus for all the ordered pairs (m, n) within any given diagonal, the value of $m + n$ is constant, and as you move down the ordered pairs in the diagonal, starting at the top, the value of the second element of the pair keeps increasing by 1.

25. *Hint:* There are at least two different approaches to this problem. One is to use the method discussed in Section 4.2. Another is to suppose that $1.999999\ldots < 2$ and derive a contradiction. (Show that the difference between 2 and $1.999999\ldots$ can be made smaller than any given positive number.)

26. Proof: Let A be an infinite set. Construct a countably infinite subset $a_1, a_2, a_3, \ldots$ of A, by letting a_1 be any element of A, letting a_2 be any element of A other than a_1, letting a_3 be any element of A other than a_1 or a_2, and so forth. This process never stops (and hence $a_1, a_2, a_3, \ldots$ is an infinite sequence) because A is an infinite set. More formally,

1. Let a_1 be any element of A.
2. For each integer $n \geq 2$, let a_n be any element of $A - \{a_1, a_2, a_3, \ldots, a_{n-1}\}$. Such an element exists, for otherwise $A - \{a_1, a_2, a_3, \ldots, a_{n-1}\}$ would be empty and A would be finite.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

27. Proof: Suppose A is any countably infinite set, B is any set, and $g: A \to B$ is onto. Since A is countably infinite, there is a one-to-one correspondence $f: \mathbf{Z}^+ \to A$. Then, in particular, f is onto, and so by Theorem 7.3.4, $g \circ f$ is an onto function from $\mathbf{Z}^+$ to B. Define a function $h: B \to \mathbf{Z}^+$ as follows: Suppose x is any element of B. Since $g \circ f$ is onto, $\{m \in \mathbf{Z}^+ \mid (g \circ f)(m) = x\} \neq \emptyset$. Thus, by the well-ordering principle for the integers, this set has a least element. In other words, there is a least positive integer n with $(g \circ f)(n) = x$. Let $h(x)$ be this integer.

We claim that h is a one-to-one. For suppose $h(x_1) = h(x_2) = n$. By definition of h, n is the least positive integer with $(g \circ f)(n) = x_1$. But also by definition of h, n is the least positive integer with $(g \circ f)(n) = x_2$. Hence $x_1 = (g \circ f)(n) = x_2$.

Thus h is a one-to-one correspondence between B and a subset S of positive integers (the range of h). Since any subset of a countable set is countable (Theorem 7.4.3), S is countable, and so there is a one-to-one correspondence between B and a countable set. Hence, by the transitive property of cardinality, B is countable.

29. *Hint:* Suppose A and B are any two countably infinite sets. Then there are one-to-one correspondences $f: \mathbf{Z}^+ \to A$ and $g: \mathbf{Z}^+ \to B$.

Case 1 $(A \cap B = \emptyset)$: In this case define $h: \mathbf{Z}^+ \to A \cup B$ as follows: For all integers $n \geq 1$,

$$h(n) \begin{cases} f(n/2) & \text{if } n \text{ is even} \\ g((n+1)/2) & \text{if } n \text{ is odd.} \end{cases}$$

Show that h is one-to-one and onto.

Case 2 $(A \cap B \notin \emptyset)$: In this case let $C = B - A$. Then $A \cup B = A \cup C$ and $A \cap C = \emptyset$. If C is countably infinite, use the result of case 1 to complete the proof. If C is finite, use the result of exercise 28 to complete the proof.

30. *Hint:* Use proof by contradiction and the fact that the set of all real numbers is uncountable.

31. *Hint:* Consider the following cases: (1) A and B are both finite, (2) at least one of A or B is infinite and $A \cap B = \emptyset$, (3) at least one of A or B is infinite and $A \cap B \neq \emptyset$. In case 3 use the fact that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ and that the sets $(A - B)$, $(B - A)$, and $(A \cap B)$ are mutually disjoint.

32. *Hint:* Use the one-to-one correspondence $F: \mathbf{Z}^+ \to \mathbf{Z}$ of Example 7.4.2 to define a function $G: \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z} \times \mathbf{Z}$ by the formula $G(m, n) = (F(m), F(n))$. Show that G is a one-to-one correspondence, and use the result of exercise 22 and the transitive property of cardinality.

34. *Hint for Solution 1:* Define a function $f: \mathscr{P}(S) \to T$ as follows: For each subset A of S, let $f(A) = \chi_A$, the *characteristic function* of A, where $\chi_A: S \to \{0, 1\}$ is defined by the rule

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \text{ for all } x \in S \end{cases}.$$

Show that f is one-to-one (for all $A_1, A_2 \subseteq S$, if $\chi_{A_1} = \chi_{A_2}$ then $A_1 = A_2$) and that f is onto (given any function $g: S \to \{0, 1\}$, there is a subset A of S such that $g = \chi_A$).

Hint for Solution 2: Define $H: T \to \mathscr{P}(S)$ by letting $H(f) = \{x \in S \mid f(x) = 1\}$. Show that H is a one-to-one correspondence?

35. Partial proof (by contradiction): Suppose not. Suppose there is a one-to-one, onto function $f: S \to \mathscr{P}(S)$. Let

$$A = \{x \in S \mid x \notin f(x)\}.$$

Then $A \in \mathscr{P}(S)$ and since f is onto, there is a $z \in S$ such that $A = f(z)$. *[Now derive a contradiction!]*

37. *Hint:* Since A and B are countable, their elements can be listed as

$$A: a_1, a_2, a_3, \ldots \quad \text{and} \quad B: b_1, b_2, b_3, \ldots$$

Represent the elements of $A \times B$ in a grid:

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \ldots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \ldots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Now use a counting method similar to that of Example 7.4.4.

Section 8.1

1. a. $0\ E\ 0$ because $0 - 0 = 0 = 2 \cdot 0$, so $2 \mid (0 - 0)$.
$5\ \not{E}\ 2$ because $5 - 2 = 3$ and $3 \neq 2k$ for any integer k so $2 \nmid (5 - 2)$.
$(6, 6) \in E$ because $6 - 6 = 0 = 2 \cdot 0$, so $2 \mid (6 - 6)$.
$(-1, 7) \in E$ because $-1 - 7 = -8 = 2 \cdot (-4)$, so $2 \mid (-1 - 7)$.

2. *Hint:* To show a statement of the form $p \leftrightarrow (q \vee r)$, you need to show $p \to (q \vee r)$ and $(q \vee r) \to p$. To show a statement of the form $p \to (q \vee r)$, you can show $(p \wedge \sim q) \to r$ (since these two statement forms are logically equivalent). To show a statement of the form $(q \vee r) \to p$, you can show $(q \to p) \wedge (r \to p)$ (since these two statement forms are logically equivalent). In this case, suppose m and n are any integers, and let p be "$m - n$ is even," let q be "m and n are both even," and let r be "$m - n$ is even," let q be "m and n are both even," and let r be "m and n are both odd."

3. a. $10\ T\ 1$ because $10 - 1 = 9 = 3 \cdot 3$, so $3 \mid (10 - 1)$.
$1\ T\ 10$ because $1 - 10 = -9 = 3 \cdot (-3)$, so $3 \mid (1 - 10)$.
$2\ T\ 2$ because $2 - 2 = 0 = 3 \cdot 0$, so $3 \mid (2 - 2)$.
$8\ \not{T}\ 1$ because $8 - 1 = 7 \neq 3k$, for any integer k. So $3 \nmid (8 - 1)$.

b. *One possible answer:* 3, 6, 9, −3, −6

e. *Hint:* All integers of the form $3k + 1$, for some integer k, are related by T to 1.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

4. a. Yes, because 15 and 25 are both divisible by 5, which is prime.

b. No, because 22 and 27 have no common prime factor.

5. a. Yes, because both $\{a, b\}$ and $\{b, c\}$ have two elements.

6. a. No, because $\{a\} \cap \{c\} = \emptyset$.

7. a. Yes. $1\ R(-9) \Leftrightarrow 5 | (1^2 - (-9)^2)$. But $1^2 - (-9)^2 = 1 - 81 = -80$, and $5 | (-80)$ because $-80 = 5 \cdot (-16)$.

8. a. Yes, because both *abaa* and *abba* have the same first two characters *ab*.

b. No, because the first two characters of *aabb* are different from the first two characters of *bbaa*.

9. a. Yes, because the sum of the characters in 0121 is 4 and the sum of the characters in 2200 is also 4.

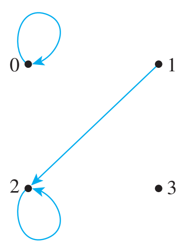
b. No, because the sum of the characters in 1011 is 3 whereas the sum of the characters in 2101 is 4.

10. $R = \{(3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}$

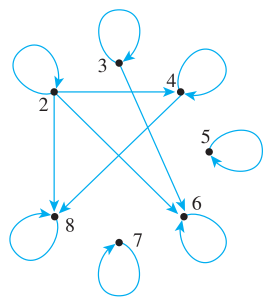
$R^{-1} = \{(4, 3), (5, 3), (6, 3), (5, 4), (6, 4), (6, 5)\}$

12. a. No. If $F: X \to Y$ is not onto, then F^{-1} is not defined on all of Y. In other words, there is an element y in Y such that $(y, x) \notin F^{-1}$ for any $x \in X$. Consequently, F^{-1} does not satisfy property (1) of the definition of function.

13.



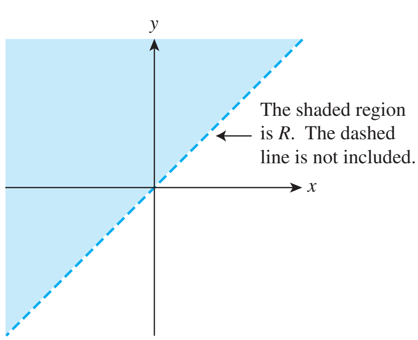
15.

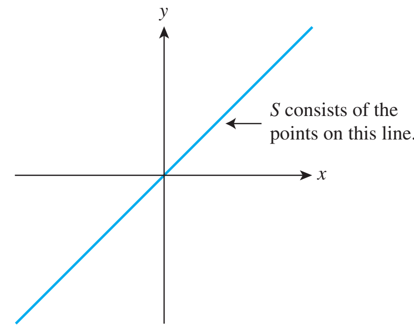


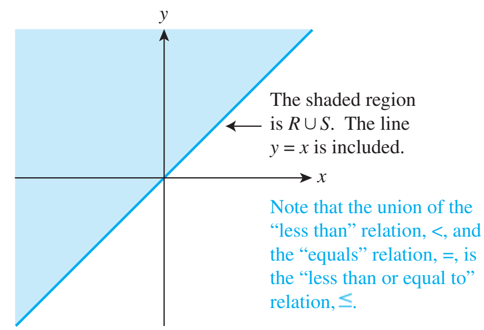
16. *Hint:* See Example 8.1.6.

19. $A \times B = \{(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)\}$
$R = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$
$S = \{(2, 6), (4, 8)\}$
$R \cup S = R,\ R \cap S = S$

21.







The graph of the intersection of R and S is obtained by finding the set of all points common to both graphs. But there are no points for which both $x < y$ and $x = y$. Hence $R \cap S = \emptyset$ and the graph consists of no points at all.

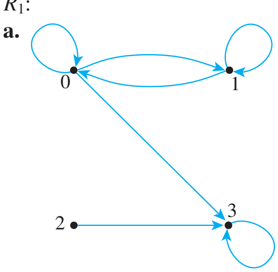
24. a.

574329	Tak Kurosawa
011985	John Schmidt

Section 8.2

1. R_1:

a.



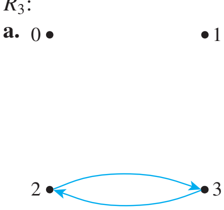
b. R_1 is not reflexive: $2 \not R_1\ 2$.

c. R_1 is not symmetric: $2\ R_1\ 3$ *but* $3 \not R_1\ 2$.

d. R_1 is not transitive: $1\ R_1\ 0$ and $0\ R_1\ 3$ but $1 \not R_1\ 3$.

3. R_3:

a.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

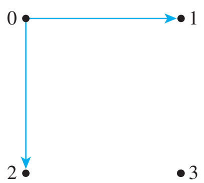
b. R_3 is not reflexive: $(0, 0) \notin R_3$

c. R_3 is symmetric. (If R_3 were not symmetric, there would be elements x and y in $A = \{0, 1, 2, 3\}$ such that $(x, y) \in R_3$ but $(y, x) \notin R_3$. It is clear by inspection that no such elements exist.)

d. R_3 is not transitive: $(2, 3) \in R_3$ and $(3, 2) \in R_3$ but $(2, 2) \notin R_3$

6. R_6:

a.



b. R_6 is not reflexive: $(0, 0) \notin R_6$

c. R_6 is not symmetric: $(0, 1) \in R_6$ but $(1, 0) \notin R_6$.

d. R_6 is transitive. (If R_6 were not transitive, there would be elements x, y, and z in $\{0, 1, 2, 3\}$ such that $(x, y) \in R_6$ and $(y, z) \in R_6$ and $(x, z) \notin R_6$. It is clear by inspection that no such elements exist.)

9. ***R is reflexive:*** R is reflexive $\Leftrightarrow$ for all real numbers $x, x\ R\ x$. By definition of R, this means that for all real numbers $x, x \geq x$. In other words, for all real numbers $x, x > x$ or $x = x$. But this is true.

R is not symmetric: R is symmetric $\Leftrightarrow$ for all real numbers x and y, if $x\ R\ y$ then $y\ R\ x$. By definition of R, this means that for all real numbers x and y, if $x \geq y$ then $y \geq x$. But this is false. As a counterexample, take $x = 1$ and $y = 0$. Then $x \geq y$ but $y \not\geq x$ because $1 \geq 0$ but $0 \not\geq 1$.

R is transitive: R is transitive $\Leftrightarrow$ for all real numbers x, y, and z, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$. By definition of R, this means that for all real numbers x, y and z, if $x \geq y$ and $y \geq z$ then $x \geq z$. But this is true by definition of $\geq$ and the transitive property of order for the real numbers. (See Appendix A, T18.)

11. ***D is reflexive:*** For D to be reflexive means that for all real numbers $x, x\ D\ x$. But by definition of D, this means that for all real numbers $x, xx = x^2 \geq 0$, which is true.

D is symmetric: For D to be symmetric means that for all real numbers x and y, if $x\ D\ y$ then $y\ D\ x$. But by definition of D, this means that for all real numbers x and y, if $xy \geq 0$ then $yx \geq 0$, which is true by the commutative law of multiplication.

D is not transitive: For D to be transitive means that for all real numbers x, y, and z, if $x\ D\ y$ and $y\ D\ z$ then $x\ D\ z$. By definition of D, this means that for all real numbers x, y, and z, if $xy \geq 0$ and $yz \geq 0$ then $xz \geq 0$. But this is false: there exist real numbers x, y, and z such that $xy \geq 0$ and $yz \geq 0$ but $xz \not\geq 0$. As a counterexample, let $x = 1$, $y = 0$, and $z = -1$. Then $x\ D\ y$ and $y\ D\ z$ because $1 \cdot 0 \geq 0$ and $0 \cdot (-1) \geq 0$. But $x \not D\ z$ because $1 \cdot (-1) \not\geq 0$.

12. ***E is reflexive:*** *[We must show that for all integers m, m E m.]* Suppose m is any integer. Since $m - m = 0$ and $2 \mid 0$, we have that $2 \mid (m - m)$. Consequently, $m\ E\ m$ by definition of E.

E is symmetric: *[We must show that for all integers m and n, if m E n then n E m.]* Suppose m and n are any integers such that $m\ E\ n$. By definition of E, this means that $2 \mid (m - n)$, and so, by definition of divisibility, $m - n = 2k$ for some integer k. Now $n - m = -(m - n)$. Hence, by substitution, $n - m = -(2k) = 2(-k)$. It follows that $2 \mid (n - m)$ by definition of divisibility (since $-k$ is an integer), and thus $n\ E\ m$ by definition of E.

E is transitive: *[We must show that for all integers m, n and p if m E n and n E p then m E p.]* Suppose m, n, and p are any integers such that $m\ E\ n$ and $n\ E\ p$. By definition of E this means that $2 \mid (m - n)$ and $2 \mid (n - p)$, and so, by definition of divisibility, $m - n = 2k$ for some integer k and $n - p = 2l$ for some integer l. Now $m - p = (m - n) + (n - p)$. Hence, by substitution, $m - p = 2k + 2l = 2(k + l)$. It follows that $2 \mid (m - p)$ by definition of divisibility (since $k + l$ is an integer), and thus mEp by definition of E.

15. ***D is reflexive:*** *[We must show that for all positive integers m, m D m.]* Suppose m is any positive integer. Since $m = m \cdot 1$, by definition of divisibility $m \mid m$. Hence $m\ D\ m$ by definition of D.

D is not symmetric: For D to be symmetric would mean that for all positive integers m and n, if $m\ D\ n$ then $n\ D\ m$. By definition of divisibility, this would mean that for all positive integers m and n, if $m \mid n$ then $n \mid m$. But this is false. As a counterexample, take $m = 2$ and $n = 4$. Then $m \mid n$ because $2 \mid 4$ but $n \nmid m$ because $4 \nmid 2$.

D is transitive: To prove transitivity of D, we must show that for all positive integers m, n, and p, if $m\ D\ n$ and $n\ D\ p$ then $m\ D\ p$. By definition of D, this means that for all positive integers m, n, and p, if $m \mid n$ and $n \mid p$ then $m \mid p$. But this is true by Theorem 4.3.3 (the transitivity of divisibility).

18. *Hint:* Q is reflexive, symmetric, and transitive.

20. ***E is reflexive:*** $\mathbf{E}$ is reflexive $\Leftrightarrow$ for all subsets A of X, A $\mathbf{E}$ A. By definition of $\mathbf{E}$, this means that for all subsets A of X, A has the same number of elements as A. But this is true.

E is symmetric: $\mathbf{E}$ is symmetric $\Leftrightarrow$ for all subsets A and B of X, if A $\mathbf{E}$ B then B $\mathbf{E}$ A. By definition of $\mathbf{E}$, this means that if A has the same number of elements as B, then B has the same number of elements as A. But this is true.

E is transitive: $\mathbf{E}$ is transitive $\Leftrightarrow$ for all subsets A, B, and C of X, if A $\mathbf{E}$ B and B $\mathbf{E}$ C, then A $\mathbf{E}$ C. By definition of $\mathbf{E}$, this means that for all subsets, A, B, and C of X, if A has the same number of elements as B and B has the number of elements as C, then A has the same number of elements as C. But this is true.

23. ***S is reflexive:*** $\mathbf{S}$ is reflexive $\Leftrightarrow$ for all subsets A of X, A$\mathbf{S}$A. By definition of $\mathbf{S}$, this means that for all subsets A of X, A $\subseteq$ A. But this is true because every set is a subset of itself.

S is not symmetric: $\mathbf{S}$ is symmetric $\Leftrightarrow$ for all subsets A and B of X, if A$\mathbf{S}$B then B$\mathbf{S}$A. By definition of $\mathbf{S}$, this means that for all subsets A and B of X, if A $\subseteq$ B then B $\subseteq$ A. But

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

this is false because $X \neq \emptyset$ and so there is an element a in X. As a counterexample, take $A = \emptyset$, and $B = \{a\}$. Then $A \subseteq B$ but $B \nsubseteq A$.

S is transitive: **S** is transitive $\Leftrightarrow$ for all subsets A, B, and C of X, if A**S**B and B**S**C, then A **S** C. By definition of **S**, this means that for all subsets A, B, and C of X, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. But this is true by the transitive property of subsets (Theorem 6.2.1 (3)).

25. ***R is reflexive:*** Suppose s is any string in A. Then $s\ R\ s$ because s has the same first two characters as s.

R is symmetric: Suppose s and t are any strings in A such that $s\ R\ t$. By definition of R, s has the same first two characters as t. It follows that t has the same first two characters as s, and so $t\ R\ s$.

R is transitive: Suppose s, t, and u, are any strings in A such that $s\ R\ t$ and $t\ R\ u$. By definition of R, s has the same first two characters as t and t has the same first two characters as u. It follows that s has the same two characters as u, and so $s\ R\ u$.

27. ***I is reflexive:*** *[We must show that for all statements p, p **I** p.]* Suppose p is a statement. The only way a conditional statement can be false is for its hypothesis to be true and its conclusion false. Consider the statement $p \to p$. Both the hypothesis and the conclusion have the same truth value. Thus it is impossible for $p \to p$ to be false, and so $p \to p$ must be true.

I is not symmetric: **I** is symmetric $\Leftrightarrow$ for all statements p and q, if p **I** q then q **I** p. By definition of **I**, this means that for all statements p and q, if $p \to q$ then $q \to p$. But this false. As a counterexample, let p be the statement "10 is divisible by 4" and let q be "10 is divisible by 2." Then $p \to q$ is the statement "If 10 is divisible by 4, then 10 is divisible by 2." This is true because its hypothesis, p, is false. On the other hand, $q \to p$ is the statement "If 10 is divisible by 2, then 10 is divisible by 4." This is false because its hypothesis, q, is true and its conclusion, p, is false.

I is transitive: *[We must show that for all statements p, q, and r, if p **I** q and q **I** r then p **I** r.]* Suppose p, q, and r are statements such that p **I** q and q **I** r. By definition of **I**, this means that $p \to q$ and $q \to r$ are both true. By transitivity of if-then (Example 2.3.6 and exercise 20 of Section 2.3), we can conclude that $p \to r$ is true. Hence, by definition of I, p, **I** r.

28. ***F is reflexive:*** **F** is reflexive $\Leftrightarrow$ for all elements (x, y) in $\mathbf{R} \times \mathbf{R}$, (x, y) **F** (x, y). By definition of **F**, this means that for all elements (x, y) in $\mathbf{R} \times \mathbf{R}$, $x = x$. But this is true.

F is symmetric: *[We must show that for all elements (x_1, y_1) and (x_2, y_2) in $\mathbf{R} \times \mathbf{R}$, if $(x_1, y_1)\mathbf{F}(x_2, y_2)$ then $(x_2, y_2)\mathbf{F}(x_1, y_1)$.]* Suppose (x_1, y_1) and (x_2, y_2) are elements of $\mathbf{R} \times \mathbf{R}$ such that (x_1, y_1), $\mathbf{F}(x_2, y_2)$. By definition of **F**, this means that $x_1 = x_2$. By symmetry of equality, $x_2 = x_1$. Thus, by definition of **F**, $(x_2, y_2)\mathbf{F}(x_1, y_1)$.

F is transitive: *[We must show that for all elements $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in $\mathbf{R} \times \mathbf{R}$, if $(x_1, y_1)\mathbf{F}(x_2, y_2)$ and $(x_2, y_2)\mathbf{F}(x_3, y_3)$ then $(x_1, y_1)\mathbf{F}(x_3, y_3)$.]* Suppose $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) are elements of $\mathbf{R} \times \mathbf{R}$ such that $(x_1, y_1)\mathbf{F}(x_2, y_2)$ and $(x_2, y_2)\mathbf{F}(x_3, y_3)$. By definition of **F**, this means that $x_1 = x_2$ and $x_2 = x_3$. By transitivity of equality, $x_1 = x_3$. Hence, by definition of **F**, $(x_1, y_1)\mathbf{F}(x_3, y_3)$.

31. ***R is reflexive:*** R is reflexive $\Leftrightarrow$ for all people p in A, $p\ R\ p$. By definition of R, this means that for all people p living in the world today, p lives within 100 miles of p. But this is true.

R is symmetric: *[We must show that for all people p and q in A, if $p\ R\ q$ then $q\ R\ p$.]* Suppose p and q are people in A such that $p\ R\ q$. By definition of R, this means that p lives within 100 miles of q. But this implies that q lives within 100 miles of p. So, by definition of $R, q\ R\ p$.

R is not transitive: R is transitive $\Leftrightarrow$ for all people p, q and r, if $p\ R\ q$ and $q\ R\ r$ then $p\ R\ r$. But this is false. As a counterexample, take p to be an inhabitant of Chicago, Illinois, q an inhabitant of Kankakee, Illinois, and r an inhabitant of Champaign, Illinois. Then $p\ R\ q$ because Chicago is less then 100 miles from Kankakee, and $q\ R\ r$ because Kankakee is less than 100 miles from Champaign, but $p \not R\ r$ because Chicago is not less than 100 miles from Champaign.

34. Proof: Suppose R is any reflexive relation on a set A. *[We must show that R^{-1} is reflexive. To show this, we must show that for all x in A, $x\ R^{-1}\ x$.]* Given any element x in A, since R is reflexive, $x\,R\,x$, and by definition of relation, this means that $(x, x) \in R$. It follows, by definition of the inverse of a relation, that $(x, x) \in R^{-1}$, and so, by definition of relation, $x\,R^{-1}\ x$ *[as was to be shown]*.

37. a. ***R ∩ S is reflexive:*** Suppose R and S are reflexive. *[To show that $R \cap S$ is reflexive, we must show that $\forall x \in A$, $(x, x) \in R \cap S$.]* So suppose $x \in A$. Since R is reflexive, $(x, x) \in R$, and since S is reflexive, $(x, x) \in S$. Thus, by definition of intersection, $(x, x) \in R \cap S$ *[as was to be shown]*.

38. *Hint:* The answer is yes.

41. Yes. To prove this we must show that for all x and y in A, if $(x, y) \in R \cup S$ then $(y, x) \in R \cup S$. So suppose (x, y) is a particular but arbitrarily chosen element in $R \cup S$. *[We must show that $(y, x) \in R \cup S$.]* By definition of union, $(x, y) \in R$ or $(x, y) \in S$. If $(x, y) \in R$, then $(y, x) \in R$ because R is symmetric. Hence $(y, x) \in R \cup S$ by definition of union. But also, if $(x, y) \in S$ then $(y, x) \in S$ because S is symmetric. Hence $(y, x) \in R \cup S$ by definition of union. Thus, in either case, $(y, x) \in R \cup S$ *[as was to be shown]*.

43. R_1 is not irreflexive because $(0, 0) \in R_1$. R_1 is not asymmetric because $(0, 1) \in R_1$ and $(1, 0) \in R_1$. R_1 is not intransitive because $(0, 1) \in R_1$ and $(1, 0) \in R_1$ and $(0, 0) \in R_1$.

45. R_3 is irreflexive. R_3 is not asymmetric because $(2, 3) \in R_3$ and $(3, 2) \in R_3$. R_3 is intransitive.

48. R_6 is irreflexive. R_6 is asymmetric. R_6 is intransitive (by default).

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

51. $R^t = R \cup \{(0,0), (0,3), (1,0), (3,1), (3,2), (3,3), (0,2), (1,2)\}$
$= \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (2,2), (3,0), (3,1), (3,2)(3,3)\}$

54. Algorithm—Test for Reflexivity
[The input for this algorithm is a binary relation R defined on a set A, that is represented as the one-dimensional array $a[1], a[2], \ldots, a[n]$*. To test whether R is reflexive, the variable answer is initially set equal to "yes," and each element* $a[i]$ *of A is examined in turn to see whether it is related by R to itself. If any element is not related to itself by R, then answer is set equal to "no," the while loop is not repeated, and processing terminates.]*

Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of* $A \times A$*]*

Algorithm Body:

```
i := 1, answer := "yes"
while (answer = "yes" and i ≤ n)
    if (a[i], a[i]) ∉ R then answer := "no"
    i := i + 1
end while
```

Output: *answer [a string]*

Section 8.3

1. a. cRc **b.** bRa, cRb, eRd **c.** aRc
d. $cRc, bRa, cRb, eRd, aRc, cRa$

2. a. $R = \{(0,0), (0,2), (1,1), (2,0), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

3. $\{0,4\}, \{1,3\}, \{2\}$

5. $\{1,5,9,13,17\}, \{2,6,10,14,18\}, \{3,7,11,15,19\}, \{4,8,12,16,20\}$

7. $\{(1,3),(3,9)\}, \{(2,4),(-4,-8),(3,6)\}, \{(1,5)\}$

8. $\{\emptyset\}, \{\{a\},\{b\},\{c\}\}, \{\{a,b\},\{a,c\},\{b,c\}\}, \{\{a,b,c\}\}$

11. $[0] = \{x \in A \mid 4 \mid (x^2 - 0)\} = \{x \in A \mid 4 \mid x^2\} = \{-4,-2,0,2,4\}$ $[1] = \{x \in A \mid 4 \mid (x^2 - 1^2)\} = \{x \in A \mid 4 \mid (x^2 - 1)\} = \{-3,-1,1,3\}$

13. $\{aaaa, aaab, aaba, aabb\}, \{abaa, abab, abba, abbb\}, \{baaa, baab, baba, babb\}, \{bbaa, bbab, bbba, bbbb\}$

15. a. True. $17 - 2 = 15$ and $5 \mid 15$.

16. a. $[7] = [4] = [19], [-4] = [17], [-6] = [27]$

17. a. Proof: Suppose that m and n are integers such that $m \equiv n\ (mod\ 3)$. *[We must show that* $m\ mod\ 3 = n\ mod\ 3$.*]* By definition of congruence, $3 \mid (m-n)$, and so by definition of divisibility, $m - n = 3k$ for some integer k. Let $m\ mod\ 3r =$. Then $m = 3l + r$ for some integer l. Since $m - n = 3k$, then by substitution, $(3l + r) - n = 3k$, or, equivalently, $n = 3(l-k) + r$. Since $l - k$ is an integer and $0 \le r < 3$, it follows, by definition of *mod*, that $n\ mod\ 3 = r$ also. So $m\ mod\ 3 = n\ mod\ 3$.

Suppose that m and n are integers such that $m\ mod\ 3 = n\ mod\ 3$. *[We must show that* $m \equiv n\ (mod\ 3)$.*]* Let $r = m\ mod\ 3 = n \bmod 3$. Then, by definition of *mod*, $m = 3p + r$ and $n = 3q + r$ for some integers p and q. By substitution, $m - n = (3p + r) - (3q + r) = 3(p - q)$. Since $p - q$ is an integer, it follows that $3 \mid (m - n)$, and so, by definition of congruence, $m \equiv n \pmod 3$.

18. a. For example, let $A = \{1,2\}$ and $B = \{2,3\}$. Then $A \neq B$, so A and B are distinct. But A and B are not disjoint since $2 \in A \cap B$.

19. a. (1) Proof: R is reflexive because it is true that for each student x at a college, x has the same major (or double major) as x.
R is symmetric because it is true that for all students x and y at a college, if x has the same major (or double major) as y, then y has the same major (or double major) as x.
R is transitive because it is true that for all students x, y, and z at a college, if x has the same major (or double major) as y and y has the same major (or double major) as z, then x has the same major (or double major) as z.
R is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one equivalence class for each major and double major at the college. Each class consists of all students with that major (or double major).

20. (1) *Hint:* See the solution to exercise 15 in Section 10.2.
(2) Two distinct classes: $\{x \in \mathbf{Z} \mid x = 2k$, for some integer $k\}$ and $\{x \in \mathbf{Z} \mid x = 2k + 1$, for some integer $k\}$.

25. (1) Proof: A is reflexive because each real number has the same absolute value as itself.
A is symmetric because for all real numbers x and y, if $|x| = |y|$ then $|y| = |x|$.
A is transitive because for all real numbers x, y, and z, if $|x| = |y|$ and $|y| = |z|$ then $|x| = |z|$.
A is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) The distinct classes are all sets of the form $\{x, -x\}$, where x is a real number.

26. *Hints*: (1) D is reflexive, symmetric, and transitive. The proofs are very similar to the proofs in exercise 17.

(2) There are two distinct equivalence classes. Note that $m^2 - n^2 = (m-n)(m+n)$ for all integers m and n. In addition, $3 \mid (m-n)$ or $3 \mid (m+n) \Leftrightarrow$ either $m - n = 3r$ or $m + n = 3r$, for some integer r

28. (1) Proof: I is reflexive because the difference between each real number and itself is 0, which is an integer.

I is symmetric because for all real numbers x and y, if $x - y$ is in integer, then $y - x = (-1)(x - y)$, which is also an integer.

I is transitive because for all real numbers x, y, and z, if $x - y$ is an integer and $y - z$ is an integer, then $x - z = (x - y) + (y - z)$ is the sum of two integers and thus an integer.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

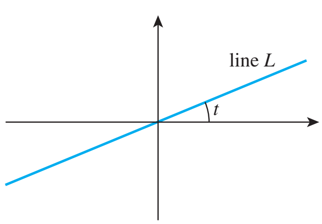
I is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one class for each real number x with $0 \le x < 1$. The distinct classes are all sets of the form $\{y \in \mathbf{R} \mid y = n + x$, for some integer $n\}$, where x is a real number such that $0 \le x < 1$.

29. (1) Proof: P is reflexive because each ordered pair of real numbers has the same first element as itself.
P is symmetric for the following reason: Suppose (w, x) and (y, z) are ordered pairs of real numbers such that $(w, x) P (y, z)$. Then, by definition of P, $w = y$. But by the symmetric property of equality, this implies that $y = w$, and so, by definition of P, $(y, z) P (w, x)$.
P is transitive for the following reason: Suppose (u, v), (w, x), and (y, z) are ordered pairs of real numbers such that $(u, v) P (w, x)$ and $(w, x) P (y, z)$. Then, by definition of P, $u = w$ and $w = y$. But by the transitive property of equality, this implies that $u = w$, and so, by definition of P, $(u, v) P (w, x)$.
P is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one equivalence class for each real number. The distinct equivalence classes are all sets of ordered pairs $\{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x = a\}$, for each real number a. Equivalently, the equivalence classes consist of all vertical lines in the Cartesian plane.

32. *Solution for (2):* There is one equivalence class for each real number t such that $0 \le t < \pi$. One line in each class goes through the origin, and that line makes an angle of t with the positive horizontal axis.



Alternatively, there is one equivalence class for every possible slope: all real numbers plus "undefined."

34. No. If points p, q, and r all lie on a straight line with q in the middle, and if p is c units from q and q is c units from r, than p is more then c units from r.

36. Proof: Suppose R is an equivalence relation on a set A and $a \in A$. Because R is an equivalence relation, R is reflexive, and because R is reflexive, each element of A is related to itself by R. In particular, $a \; R \; a$. Hence by definition of equivalence class, $a \in [a]$.

38. Proof: Suppose R is an equivalence relation on a set A and a, b, and c are elements of A with $b \; R \; c$ and $c \in [a]$. Since $c \in [a]$, then $c \; R \; a$ by definition of equivalence class. But R is transitive since R is an equivalence relation. Thus since $b \; R \; c$ and $c \; R \; a$, then $b \; R \; a$. It follows that $b \in [a]$ by definition of class.

40. Proof: Suppose a, b and x are in A, $a \; R \; b$, and $x \in [a]$. By definition of equivalence class, $x \; R \; a$. So $x \; R \; a$ and $a \; R \; b$, and thus, by transitivity, $x \; R \; b$. Hence $x \in [b]$.

41. *Hint:* To show that $[a] = [b]$, show that $[a] \subseteq [b]$ and $[b] \subseteq [a]$. To show that $[a] \subseteq [b]$, show that for all x in A, if $x \in [a]$ then $x \in [b]$.

42. c. For example $(2, 6)$, $(-2, -6)$, $(3, 9)$, $(-3, -9)$.

43. a. Suppose that (a, b), (a', b'), (c, d) and (c', d') are any elements of A such that $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. By definition of the relation, $ab' = ba'$ (*) and $cd' = dc'$ (**). We must show that $[(a, b)] + [(c, d)] = [(a', b')] + [(c', d')]$. By definition of the addition, this equation is true if, and only if,

$$[(ad + bc, bd)] = [(a'd' + b'c', b'd')].$$

And, by definition of the relation, this equation is true if, and only if,

$$(ad + bc)b'd' = bd(a'd' + b'c'),$$

which is equivalent to

$$adb'd' + bcb'd' = bda'd' + bdb'c', \quad \text{by multiplying out.}$$

But this equation is equivalent to

$$\begin{aligned}(ab')(dd') + (cd')(bb') \\ = (ba')(dd') + (dc')(bb') \quad \text{by regrouping}\end{aligned}$$

and, by substitution from (*) and (**), this last equation is true.

c. Suppose that (a, b) is any element of A. We must show that $[(a, b)] + [(0, 1)] = [(a, b)]$. By definition of the addition, this equation is true if, and only if,

$$[(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)].$$

But this last equation is true because $a \cdot 1 + b \cdot 0 = a$ and $b \cdot 1 = b$.

e. Suppose that (a, b) is any element of A. We must show that $[(a, b)] + [(-a, b)] = [(-a, b)] + [(a, b)] = [(0, 1)]$. By definition of the addition, this equation is true if, and only if,

$$[(ab + b(-a), bb)] = [(0, 1)],$$

or, equivalently,

$$[(0, bb)] = [(0, 1)]$$

By definition of the relation, this last equation is true if, and only if, $0 \cdot 1 = bb \cdot 0$, which is true.

44. a. Let (a, b) be any element of $\mathbf{Z}^+ \times \mathbf{Z}^+$. We must show that $(a, b) R (a, b)$. By definition of R, this relationship holds if, and only if, $a + b = b + a$. But this equation is true by the commutative law of addition for real numbers. Hence R is reflexive.

c. *Hint:* You will need to show that for any positive integers a, b, c, and d, if $a + d = c + b$ and $c + f = d + e$, then $a + f = b + e$.

d. *One possible answer:* $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(5, 5)$

g. Observe that for any positive integers a and b, the equivalence class of (a, b) consists of all ordered pairs

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

in $\mathbf{Z}^+ \times \mathbf{Z}^+$ for which the difference between the first and second coordinates equals $a - b$. Thus there is one equivalence class for each integer: positive, negative, and zero. Each positive integer n corresponds to the class of $(n+1, 1)$; each negative integer $-n$ corresponds to the class of $(1, n+1)$; and zero corresponds to the class $(1, 1)$.

47. c. "Ways and Means"

Section 8.4

1. a. ZKUHUH VKDOO ZH PHHW
b. IN THE CAFETERIA

3. a. The relation $3 \mid (25 - 19)$ is true because $25 - 19 = 6$ and $3 \mid 6$ (since $6 = 3 \cdot 2$).

b. By definition of congruence modulo n, to show that $25 \equiv 19 \pmod 3$, one must show that $3 \mid (25 - 19)$. This was verified in part (a).

c. To show that $25 = 19 + 3k$ for some integer k, one solves the equation for k and checks that the result is an integer. In this case, $k = (25 - 19)/3 = 2$, which is an integer. Thus $25 = 19 + 2 \cdot 3$.

d. When 25 is divided by 3, the remainder is 1 because $25 = 3 \cdot 8 + 1$. When 19 is divided by 3, the remainder is also 1 because $19 = 3 \cdot 6 + 1$. Thus 25 and 19 have the same remainder when divided by 3.

e. By definition, 25 *mod* 3 is the remainder obtained when 25 is divided by 3, and 19 *mod* 3 is the remainder obtained when 19 is divided by 3. In part (d) these two numbers were shown to be equal.

6. *Hints:* (1) Use the quotient-remainder theorem and Theorem 8.4.1 to show that given any integer a, a is in one of the classes $[0], [1], [2], \ldots [n-1]$. (2) Use Theorem 4.3.1 to prove that if $0 \le a < n$, $0 \le b < n$, and $a \equiv b \pmod n$, then $a = b$.

7. a. $128 \equiv 2 \pmod 7$ because $128 - 2 = 126 = 7 \cdot 18$, and $61 \equiv 5 \pmod 7$ because $61 - 5 = 56 = 7 \cdot 8$

b. $128 + 61 \equiv (2 + 5) \pmod 7$ because $128 + 61 = 189$, $2 + 5 = 7$, and $189 - 7 = 182 = 7 \cdot 26$

c. $128 - 61 \equiv (2 - 5) \pmod 7$ because $128 - 61 = 67$, $2 - 5 = -3$, and $67 - (-3) = 70 = 7 \cdot 10$

d. $128 \cdot 61 \equiv (2 \cdot 5) \pmod 7$ because $128 \cdot 61 = 7808$, $2 \cdot 5 = 10$, and $7808 - (10) = 7798 = 7 \cdot 1114$

e. $128^2 \equiv 2^2 \pmod 7$ because $128^2 = 16384$, $2^2 = 4$, and $16384 - 4 = 16380 = 7 \cdot 2340$.

9. a. Proof: Suppose a, b, c, d, and n are integers with $n > 1$, $a \equiv c \pmod n$, and $b \equiv d \pmod n$. By Theorem 8.4.1, $a - c = nr$ and $b - d = ns$ for some integers r and s. Then

$$\begin{aligned}(a+b) - (c+d) &= (a-c) + (b-d) = nr + ns \\ &= n(r+s).\end{aligned}$$

But $r + s$ is an integer, and so, by Theorem 8.4.1, $a + b \equiv (c + d) \pmod n$.

12. a. Proof (by mathematical induction): Let the property $P(n)$ be the congruence $10^n \equiv 1 \pmod 9$.

***Show that P(0) is true*:**

When $n = 0$, the left-hand side of the congruence is $10^0 = 1$ and the right-hand side is also 1.

***Show that for all integers $k \ge 0$, if $P(k)$ is true, then $P(k+1)$ is true*.**

Let k be any integer with $k \ge 0$, and suppose $P(k)$ is true. That is, suppose $10^k \equiv 1 \pmod 9$. (*) *[This is the inductive hypothesis.]* By Theorem 8.4.1, $10 \equiv 1 \pmod 9$(**) because $10 - 1 = 9 = 9 \cdot 1$. And by Theorem 8.4.3, we can multiply the left- and right-hand sides of (*) and (**) to obtain $10^k \cdot 10 \equiv 1 \cdot 1 \pmod 9$, or, equivalently, $10^{k+1} \equiv 1 \pmod 9$. Hence $P(k+1)$ is true.

Alternative Proof: Note that $10 \equiv 1 \pmod 9$ because $10 - 1 = 9$ and $9|9$. Thus by Theorem 8.4.3(4), $10^n \equiv 1^n \equiv 1 \pmod 9$.

14. $14^1 \; mod \; 55 = 14$
$14^2 \; mod \; 55 = 196 \; mod \; 55 = 31$
$14^4 \; mod \; 55 = (14^2 \; mod \; 55)^2 \; mod \; 55 = 31^2 \; mod \; 55 = 26$
$14^8 \; mod \; 55 = (14^4 \; mod \; 55)^2 \; mod \; 55 = 26^2 \; mod \; 55 = 16$
$14^{16} \; mod \; 55 = (14^8 \; mod \; 55)^2 \; mod \; 55 = 16^2 \; mod \; 55 = 36$

15. $$\begin{aligned}4^{27} \; mod \; 55 &= 14^{16+8+2+1} \; mod \; 55 \\ &= \left\{(14^{16} \; mod \; 55)(14^8 \; mod \; 55)(14^2 \; mod \; 55)(14^1 \; mod \; 55)\right\} mod \; 55 \\ &= (36 \cdot 16 \cdot 31 \cdot 14) \; mod \; 55 = 249984 \; mod \; 55 = 9\end{aligned}$$

16. Note that $307 = 256 + 32 + 16 + 2 + 1$.
$675^1 \; mod \; 713 = 675$
$675^2 \; mod \; 713 = 18$
$675^4 \; mod \; 713 = 18^2 \; mod \; 713 = 324$
$675^8 \; mod \; 713 = 324^2 \; mod \; 713 = 165$
$675^{16} \; mod \; 713 = 165^2 \; mod \; 713 = 131$
$675^{32} \; mod \; 713 = 131^2 \; mod \; 713 = 49$
$675^{64} \; mod \; 713 = 49^2 \; mod \; 713 = 262$
$675^{128} \; mod \; 713 = 262^2 \; mod \; 713 = 196$
$675^{256} \; mod \; 713 = 196^2 \; mod \; 713 = 627$

Thus

$$\begin{aligned}675^{307} \; mod \; 713 &= 675^{256+32+16+2+1} \; mod \; 713 \\ &= (675^{256} \cdot 675^{32} \cdot 675^{16} \cdot 675^2 \cdot 675^1) \; mod \; 713 \\ &= (627 \cdot 49 \cdot 131 \cdot 18 \cdot 675) \; mod \; 713 = 3.\end{aligned}$$

19. The letters in HELLO translate numercially into 08, 05, 12, 12, and 15. By Example 8.4.9, the H is encrypted as 17. To encrypt E, we compute $5^3 \; mod \; 55 = 15$. To encrypt L, we compute $12^3 \; mod \; 55 = 23$. And to encrypt 0, we compute $15^3 \; mod \; 55 = 20$. Thus the ciphertext is 17 15 23 23 20. (In practice, individual letters of the alphabet are grouped together in blocks during encryption so that deciphering cannot be accomplished through knowledge of frequency patterns of letters or words.)

22. By Example 8.4.10, the decryption key is 27. Thus the residues modulo 55 for 13^{27}, 20^{27}, and 9^{27} must be found and then translated into letters of the alphabet.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Because $27 = 16 + 8 + 2 + 1$, we first perform the following computations:

$13^1 \equiv 13 \pmod{55}$ $\quad$ $20^1 \equiv 20 \pmod{55}$
$13^2 \equiv 4 \pmod{55}$ $\quad$ $20^2 \equiv 15 \pmod{55}$
$13^4 \equiv 4^2 \equiv 16 \pmod{55}$ $\quad$ $20^4 \equiv 15^2 \equiv 5 \pmod{55}$
$13^8 \equiv 16^2 \equiv 36 \pmod{55}$ $\quad$ $20^8 \equiv 25^2 \equiv 5 \pmod{55}$
$13^{16} \equiv 36^2 \equiv 31 \pmod{55}$ $\quad$ $20^{16} \equiv 25^2 \equiv 20 \pmod{55}$
$9^1 \equiv 9 \pmod{55}$
$9^2 \equiv 26 \pmod{55}$
$9^4 \equiv 26^2 \equiv 16 \pmod{55}$
$9^8 \equiv 16^2 \equiv 36 \pmod{55}$
$9^{16} \equiv 36^2 \equiv 31 \pmod{55}$

Then we compute

$13^{27}\ mod\ 55 = (31 \cdot 36 \cdot 4 \cdot 13)\ mod\ 55 = 7,$
$20^{27}\ mod\ 55 = (20 \cdot 25 \cdot 15 \cdot 20)\ mod\ 55 = 15,$
$9^{27}\ mod\ 55 = (31 \cdot 36 \cdot 26 \cdot 9)\ mod\ 55 = 4.$

Finally, because 7, 15, and 4 translate into letters as G, O, and D, we see that the message is GOOD.

25. *Hint:* By Theorem 5.2.3, using a in place of r and $n-1$ in place of n, we have $1 + a + a^2 + \cdots + a^{n-1} = \frac{a^n - 1}{a-1}$. Multiplying both sides by $a-1$ gives

$$a^n - 1 = (a-1)(1 + a + a^2 + \cdots + a^{n-1}).$$

26. Step 1: $6664 = 765 \cdot 8 + 544$, and so $544 = 6664 - 765 \cdot 8$
Step 2: $765 = 544 \cdot 1 + 221$, and so $221 = 765 - 544$
Step 3: $544 = 221 \cdot 2 + 102$, and so $102 = 544 - 221 \cdot 2$
Step 4: $221 = 102 \cdot 2 + 17$, and so $17 = 221 - 102 \cdot 2$
Step 5: $102 = 17 \cdot 6 + 0$

Thus $\gcd(6664, 765) = 17$ (which is the remainder obtained just before the final division). Substitute back through steps 4–1 to express 17 as a linear combination of 6664 and 765:

$$\begin{aligned} 17 &= 221 - 102 \cdot 2 \\ &= 221 - (544 - 221 \cdot 2) = 221 \cdot 5 - 544 \cdot 2 \\ &= (765 - 544) \cdot 5 - 544 \cdot 2 = 765 \cdot 5 - 544 \cdot 7 \\ &= 765 \cdot 5 - (6664 - 765 \cdot 8) \cdot 7 = (-7) \cdot 6664 + 61 \cdot 765. \end{aligned}$$

(When you have finished this final step, it is wise to verify that you have not made a mistake by checking that the final expression really does equal the greatest common divisor.)

28.

a	330	156	18	12	6
b	156	18	12	6	0
r		18	12	6	0
q		2	8	1	2
s	1	0	1	−8	9
t	0	1	−2	17	−19
u	0	1	−8	9	−26
v	1	−2	17	−19	55
newu		1	−8	9	−26
newv		−2	17	−19	55
sa + tb	330	18	−6	6	6

31. a. Step 1: $210 = 13 \cdot 16 + 2$, and so $2 = 210 - 16 \cdot 13$
Step 2: $13 = 2 \cdot 6 + 1$, and so $1 = 13 - 2 \cdot 6$
Step 3: $6 = 1 \cdot 6 + 0$, and so $\gcd(210, 13) = 1$

Substitute back through steps 2–1:

$$\begin{aligned} 1 &= 13 - 2 \cdot 6 \\ &= 13 - (210 - 16 \cdot 13) \cdot 6 = (-6) \cdot 210 + 97 \cdot 13 \end{aligned}$$

Thus $210 \cdot (-6) \equiv 1 \pmod{13}$, and so -6 is an inverse for 210 modulo 13.

b. Compute $13 - 6 = 7$, and note that $7 \equiv -6 \pmod{13}$ because $7 - (-6) = 13 = 13 \cdot 1$. Thus, by Theorem 8.4.3(3), $210 \cdot 7 \equiv 210 \cdot (-6) \pmod{13}$. It follows, by the transitive property of congruence, that $210 \cdot 7 \equiv 1 \pmod{13}$, and so 7 is a positive inverse for 210 modulo 13.

c. This problem can be solved using either the result of part (a) or that of part (b). By part (b) $210 \cdot 7 \equiv 1 \pmod{13}$. Multiply both sides by 8 and apply Theorem 8.4.3(3) to obtain $210 \cdot 56 \equiv 8 \pmod{13}$. Thus a positive solution for $210x \equiv 8 \pmod{13}$ is $x = 56$. Note that the least positive residue corresponding to this solution is also a solution. By Theorem 8.4.1, $56 \equiv 4 \pmod{13}$ because $56 = 13 \cdot 4 + 4$, and so, by Theorem 8.4.3(3), $210 \cdot 56 \equiv 210 \cdot 4 \equiv 9 \pmod{13}$. This shows that 4 is also a solution for the congruence, and because $0 \le 4 < 13$, 4 is the least positive solution for the congruence.

33. *Hint:* If $as + bt = 1$ and $c = au = bv$, then $c = asc + btc = as(bv) + bt(au)$.

35. Proof: Suppose a, n, s and s' are integers such that $as \equiv as' \equiv 1 \pmod{n}$. Consider the quantity $as's$, and note that $as's = (as') \cdot s = (as) \cdot s'$. By Theorem 8.4.3(3), $(as') \cdot s \equiv 1 \cdot s = s \pmod{n}$ and $(as') \cdot s' \equiv 1 \cdot s' = s' \pmod{n}$. Thus by transitivity of congruence modulo n, $s \equiv s' \pmod{n}$. This shows that any two inverses for a are congruent modulo n.

36. The numeric equivalents of H, E, L, and P are 08, 05, 12 and 16. To encrypt these letters, the following quantities must be computed: $8^{43}\ mod\ 713$, $5^{43}\ mod\ 713$, $12^{43}\ mod\ 713$, and $16^{43}\ mod\ 713$. We use the fact that $43 = 32 + 8 + 2 + 1$.

H: $8 \equiv 8 \pmod{713}$
$8^2 \equiv 64 \pmod{713}$
$8^4 \equiv 64^2 \equiv 531 \pmod{713}$
$8^8 \equiv 531^2 \equiv 326 \pmod{713}$
$8^{16} \equiv 326^2 \equiv 39 \pmod{713}$
$8^{32} \equiv 39^2 \equiv 95 \pmod{713}$
Thus the ciphertext is
$8^{43}\ mod\ 713$
$= (95 \cdot 326 \cdot 64 \cdot 8)\ mod\ 713 = 233.$

E: $5 \equiv 5 \pmod{713}$
$5^2 \equiv 25 \pmod{713}$
$5^4 \equiv 625 \pmod{713}$
$5^8 \equiv 625^2 \equiv 614 \pmod{713}$
$5^{16} \equiv 614^2 \equiv 532 \pmod{713}$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$5^{32} \equiv 532^2 \equiv 676 \pmod{713}$
Thus the ciphertext is
5^{43} *mod* 713
$= (676 \cdot 614 \cdot 25 \cdot 5)$ *mod* $713 = 129$.

L: $12 \equiv 12 \pmod{713}$
$12^2 \equiv 144 \pmod{713}$
$12^4 \equiv 144^2 \equiv 59 \pmod{713}$
$12^8 \equiv 59^2 \equiv 629 \pmod{713}$
$12^{16} \equiv 629^2 \equiv 639 \pmod{713}$
$12^{32} \equiv 639^2 \equiv 485 \pmod{713}$
Thus the ciphertext is
12^{43} *mod* 713
$= (485 \cdot 629 \cdot 144 \cdot 12)$ *mod* $713 = 48$.

P: $16 \equiv 16 \pmod{713}$
$16^2 \equiv 256 \pmod{713}$
$16^4 \equiv 256^2 \equiv 653 \pmod{713}$
$16^8 \equiv 653^2 \equiv 35 \pmod{713}$
$16^{16} \equiv 35^2 \equiv 512 \pmod{713}$
$16^{32} \equiv 512^2 \equiv 473 \pmod{713}$
Thus the ciphertext is
16^{43} *mod* 713
$= (473 \cdot 35 \cdot 256 \cdot 16) \bmod 713 = 128$.

Therefore, the encrypted message is 233 129 048 128. (Again, note that in practice, individual letters of the alphabet are grouped together in blocks during encryption so that deciphering cannot be accomplished through knowledge of frequency patterns of letters or words. We kept them separate so that the numbers in the computations would be smaller and easier to work with.)

39. By exercise 38, the decryption key, d, is 307. Hence, to decrypt the message, the following quantities must be computed: 675^{307} *mod* 713, 89^{307} *mod* 713, and 48^{307} *mod* 713. We use the fact that $307 = 256 + 32 + 16 + 2 + 1$.

$675 \equiv 675 \pmod{713}$
$675^2 \equiv 18 \pmod{713}$
$675^4 \equiv 18^2 \equiv 324 \pmod{713}$
$675^8 \equiv 324^2 \equiv 165 \pmod{713}$
$675^{16} \equiv 165^2 \equiv 131 \pmod{713}$
$675^{32} \equiv 131^2 \equiv 49 \pmod{713}$
$675^{64} \equiv 49^2 \equiv 262 \pmod{713}$
$675^{128} \equiv 262^2 \equiv 196 \pmod{713}$
$675^{256} \equiv 196^2 \equiv 627 \pmod{713}$

$89 \equiv 89 \pmod{713}$
$89^2 \equiv 78 \pmod{713}$
$89^4 \equiv 78^2 \equiv 380 \pmod{713}$
$89^8 \equiv 380^2 \equiv 374 \pmod{713}$
$89^{16} \equiv 374^2 \equiv 128 \pmod{713}$
$89^{32} \equiv 128^2 \equiv 698 \pmod{713}$
$89^{64} \equiv 698^2 \equiv 225 \pmod{713}$
$89^{128} \equiv 225^2 \equiv 2 \pmod{713}$
$89^{256} \equiv 2^2 \equiv 4 \pmod{713}$

$48 \equiv 48 \pmod{713}$
$48^2 \equiv 165 \pmod{713}$
$48^4 \equiv 131 \pmod{713}$
$48^8 \equiv 49 \pmod{713}$
$48^{16} \equiv 262 \pmod{713}$
$48^{32} \equiv 196 \pmod{713}$
$48^{64} \equiv 627 \pmod{713}$
$48^{128} \equiv 627^2 \equiv 266 \pmod{713}$
$48^{256} \equiv 266^2 \equiv 169 \pmod{713}$

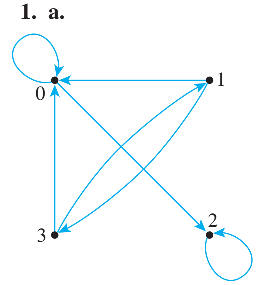
Thus the decryption for 675 is
675^{307} *mod* $713 = (675^{256+32+16+2+1})$ *mod* 713
$= (627 \cdot 49 \cdot 131 \cdot 18 \cdot 675)$ *mod* $713 = 3$, which corresponds to the letter C.
The decryption for 89 is
89^{307} *mod* $713 = (89^{256+32+16+2+1})$ *mod* 713
$= (4 \cdot 698 \cdot 128 \cdot 78 \cdot 89)$ *mod* $713 = 15$, which corresponds to the letter O.
The decryption for 48 is
48^{307} *mod* $713 = (48^{256+32+16+2+1})$ *mod* 713
$= (169 \cdot 196 \cdot 262 \cdot 165 \cdot 48)$ *mod* $713 = 12$, which corresponds to the letter L.
Thus the decrypted message is COOL.

41. a. *Hint:* For the inductive step, assume $p \mid q_1 q_2 \ldots q_{s+1}$ and let $a = q_1 q_2 \ldots q_s$. Then $p \mid a q_{s+1}$, and either $p = q_{s+1}$ or Euclid's lemma and the inductive hypothesis can be applied.

42. a. When $a = 15$ and $p = 7$, $a^{p-1} = 15^6 = 11390625 \equiv 1 \pmod 7$ because $11390625 - 1 = 7 \cdot 1627232$.

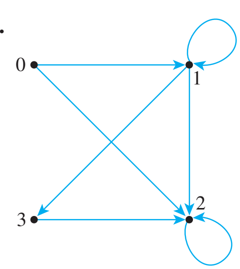
Section 8.5

1. a.



R_1 is not antisymmetric: $1\ R_1\ 3$ and $3\ R_1\ 1$ and $1 \neq 3$.

b.



R_2 is antisymmetric: There are no cases where $a\ R\ b$ and $b\ R\ a$ and $a \neq b$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

2. R is not antisymmetric. Let x and y be any two distinct people of the same age. Then $x\ R\ y$ and $y\ R\ x$ but $x \neq y$.

5. R is a partial order relation.

Proof:
***R* is reflexive:** Suppose $(a, b) \in \mathbf{R} \times \mathbf{R}$. Then $(a, b)\ R\ (a, b)$ because $a = a$ and $b \leq b$.
***R* is antisymmetric:** Suppose (a, b) and (c, d) are ordered pairs of real numbers such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (a, b)$. Then

$$\text{either } a < c \quad \text{or} \quad \text{both } a = c \text{ and } b \leq d$$

and

$$\text{either } c < a \quad \text{or} \quad \text{both } c = a \text{ and } d \leq b.$$

Thus

$$a \leq c \text{ and } c \leq a$$

and so

$$a = c.$$

Consequently,

$$b \leq d \quad \text{and} \quad d \leq b$$

and so

$$b = d.$$

Hence $(a, b) = (c, d)$.
***R* is transitive:** Suppose $(a, b), (c, d)$, and (e, f) are ordered pairs of real numbers such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then

$$\text{either } a < c \quad \text{or} \quad \text{both } a = c \text{ and } b \leq d$$

and

$$\text{either } c < e \quad \text{or} \quad \text{both } c = e \text{ and } d \leq f.$$

It follows that one of the following cases must occur.
***Case 1* ($a < c$ and $c < e$):** Then by transitivity of $<$, $a < e$, and so $(a, b)\ R\ (e, f)$ by definition of R.
***Case 2* ($a < c$ and $c = e$):** Then by substitution, $a < e$, and so $(a, b)\ R\ (e, f)$ by definition of R.
***Case 3* ($a = c$ and $c < e$):** Then by substitution, $a < e$, and so $(a, b)\ R\ (e, f)$ by definition of R.
***Case 4* ($a = c$ and $c = e$):** Then by definition of R, $b \leq d$ and $d \leq f$, and so by transitivity of $\leq$, $b \leq f$. Hence $a = e$ and $b \leq f$, and so $(a, b)\ R\ (e, f)$ by definition of R.
In each case, $(a, b)\ R\ (e, f)$. Therefore, R is transitive. Since R is reflexive, antisymmetric, and transitive, R is a partial order relation.

8. R is not a partial order relation because R is not antisymmetric.

Counterexample: $1\ R\ 3$ (because $1 + 3$ is even) and $3\ R\ 1$ (because $3 + 1$ is even) but $1 \neq 3$.

10. No. Counterexample: Define relations R and S on the set $\{1, 2\}$ as follows: $R = \{(1, 2)\}$ and $S = \{(2, 1)\}$. Then both R and S are antisymmetric, but $R \cup S = \{(1, 2), (2, 1)\}$ is not antisymmetric because $(1, 2) \in R \cup S$ and $(2, 1) \in R \cup S$ but $1 \neq 2$.

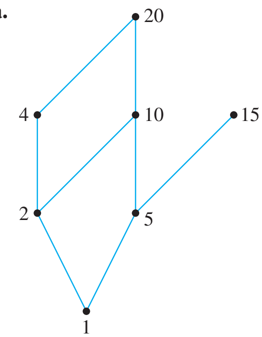
11. a. This follows from (1).
b. False. By (1), $bba \preceq bbab$.

13. $R_1 = \{(a, a), (b, b)\}$, $R_2 = \{(a, a), (b, b), (a, b)\}$, $R_3 = \{(a, a), (b, b), (b, a)\}$

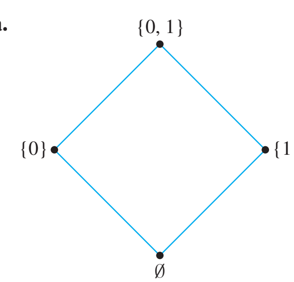
14. a.
$R_1 = \{(a, a), (b, b), (c, c)\}$,
$R_2 = \{(a, a), (b, b), (c, c), (b, a)\}$,
$R_3 = \{(a, a), (b, b), (c, c), (c, a)\}$,
$R_4 = \{(a, a), (b, b), (c, c), (b, a), (c, a)\}$,
$R_5 = \{(a, a), (b, b), (c, c), (c, b), (c, a)\}$,
$R_6 = \{(a, a), (b, b), (c, c), (b, c), (b, a)\}$,
$R_7 = \{(a, a), (b, b), (c, c), (c, b), (b, a), (c, a)\}$,
$R_8 = \{(a, a), (b, b), (c, c), (b, c), (b, a), (c, a)\}$,
$R_9 = \{(a, a), (b, b), (c, c), (b, c)\}$,
$R_{10} = \{(a, a), (b, b), (c, c), (c, b)\}$

15. *Hint:* R is the identity relation on A: $x\ R\ x$ for all $x \in A$ and $x \not R\ y$ if $x \neq y$.

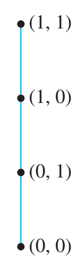
16. a.



17. a.



18.

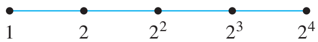


21. a. Proof: *[We must show that for all a and b in A, $a \mid b$ or $b \mid a$.]* Let a and b be particular but arbitrarily chosen elements of A. By definition of A, there are nonnegative integers r and s such that $a = 2^r$ and $b = 2^s$. Now either $r \leq s$ or $s < r$. If $r \leq s$, then

$$b = 2^s = 2^r \cdot 2^{s-r} = a \cdot 2^{s-r},$$

where $s - r \geq 0$. It follows, by definition of divisibility, that $a \mid b$. By a similar argument, if $s < r$, then $b \mid a$. Hence either $a \mid b$ or $b \mid a$ *[as was to be shown]*.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

b. 

22. Greatest element: none; least element: 1;
Maximal elements: 15, 20; minimal element: 1

24. Greatest element: {0, 1}; least element: ∅;
Maximal elements: {0, 1}; minimal elements: ∅

26. Greatest element: (1, 1); least element: (0, 0);
Maximal elements: (1, 1); minimal elements: (0, 0)

30. a. No greatest element, no least element
b. Least element is 0, greatest element is 1

31. R is a total order relation because it is reflexive, antisymmetric, and transitive (so it is a partial order) and because $[b, a, c, d]$ is a chain that contains every element of A: bRc, cRa, and aRd.

34. *Hint:* Let R' be the restriction of R to B and show that R' is reflexive, antisymmetric, and transitive. In each case, this follows almost immediately from the fact that R is reflexive, antisymmetric, and transitive.

35. $\emptyset \subseteq \{w\} \subseteq \{w, x\} \subseteq \{w, x, y\} \subseteq \{w, x, y, z\}$

36. Proof: Suppose A is a partially ordered set with respect to a relation $\preceq$. By definition of total order, A is totally ordered if, and only if, any two elements of A are comparable. By definition of chain, this is true if, and only if, A is a chain.

39. Proof (by mathematical induction): Let A be a set that is totally ordered with respect to a relation $\preccurlyeq$, and let the property $P(n)$ be the sentence "Every subset of A with n elements has both a least element and a greatest element."

Show that $P(1)$ is true:

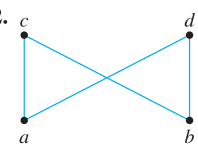
If $A = \emptyset$, then $P(1)$ is true by default. So assume that A has at least one element, and suppose $S = \{a_1\}$ is a subset of A with one element. Because $\preccurlyeq$ is reflexive, $a_1 \preccurlyeq a_1$. So, by definition of least element and greatest element, a_1 is both a least element and a greatest element of S, and thus the property is true for $n = 1$.

Show that for all integers $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose that any subset of A with k elements has both a least element and a greatest element. *[Inductive hypothesis]* We must show that any subset of A with $k + 1$ elements has both a least element and a greatest element. If A has fewer than $k + 1$ elements, then the statement is true by default. So assume that A has at least $k + 1$ elements and that $S = \{a_1, a_2, \ldots, a_{k+1}\}$ is a subset of A with $k + 1$ elements. By inductive hypothesis, $S - \{a_{k+1}\}$ has both a least element s and a greatest element b. Now because A is totally ordered, a_{k+1} and s are comparable. If $a_{k+1} \preccurlyeq s$, then, by transitivity of $\preccurlyeq$, a_{k+1} is the least element of S; otherwise, s remains the least element of S. And if $b \preccurlyeq a_{k+1}$, then, by transitivity of $\preccurlyeq$, a_{k+1} is the greatest element of S; otherwise, b remains the greatest element of S. Thus S has both a greatest element and a least element *[as was to be shown]*.

40. a. Proof by contradiction: Suppose not. Suppose A is a finite set that is partially ordered with respect to a relation $\preceq$ and A has no minimal element. Construct a sequence of elements $x_1, x_2, x_3, \ldots$ of A as follows:

1. Pick any element of A and call it x_1.
2. For each $i = 2, 3, 4, \ldots$, pick x_i to be an element of A for which $x_i \preceq x_{i-1}$ and $x_i \neq x_{i-1}$. *[Such an element must exist because otherwise x_{i-1} would be minimal, and we are supposing that no element of A is minimal.]* Now $x_i \neq x_j$ for any $i \neq j$. *[If $x_i = x_j$ where $i < j$, then on the one hand, $x_j \preceq x_{j-1} \preceq \ldots \preceq x_{i+1} \preceq x_i$ and so $x_i \preceq x_{i+1}$, and on the other hand, since $x_i = x_j$ then $x_j = x_i \succeq x_{i+1}$, and so $x_j \succeq x_{i+1}$. Hence by antisymmetry, $x_j = x_{i+1}$, and so $x_i = x_{i+1}$. But this contradicts the definition of the sequence $x_1, x_2, x_3, \ldots$.]* Thus $x_1, x_2, x_3, \ldots$ is an infinite sequence of distinct elements, and consequently $\{x_1, x_2, x_3, \ldots\}$ is an infinite subset of the finite set A. This is impossible. Hence the supposition is false and we conclude that any partially ordered subset of a finite set has a minimal element.

42. 

44. One such total order is 1, 5, 2, 15, 10, 4, 20.

46. One such total order is (0, 0), (1, 0), (0, 1), (1, 1).

50. a. *One possible answer:* 1, 6, 10, 9, 5, 7, 2, 4, 8, 3

51. b. Critical path: 1, 2, 5, 8, 9.

Section 9.1

2. 3/4, 1/2, 1/2

3. {1♦, 2♦, 3♦, 4♦, 5♦, 6♦, 7♦, 8♦, 9♦, 10♦, 1♥, 2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥}, probability $= 20/52 \cong 38.5\%$

5. {10♣, J♣, Q♣, K♣, A♣, 10♦, J♦, Q♦, K♦, A♦, 10♥, J♥, Q♥, K♥, A♥, 10♠, J♠, Q♠, K♠, K♠, A♠} probability $= 20/52 = 5/13 \cong 38.5\%$.

7. {26, 35, 44, 53, 62}, probability $= 5/36 \cong 13.9\%$

9. {11, 12, 13, 14, 15, 21, 22, 23, 24, 31, 32, 33, 41, 42, 51} probability $= 15/36 = 41\frac{2}{3}\%$

11. a. $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
b. (i) $\{HTT, THT, TTH\}$, probability $= 3/8 \cong 37.5\%$

12. a. $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$
b. (i) $\{GBB, BGB, BBG\}$ probability $= 3/8 = 37.5\%$

13. a. $\{CCC, CCW, CWC, CWW, WCC, WCW, WWC, WWW\}$
b. (i) $\{CWW, WCW, WWC\}$, probability $= 3/8 = 37.5\%$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

14. a. probability $= 3/8 = 37.5\%$

16. a. $\{RRR, RRB, RRY, RBR, RBB, RBY, RYR, RYB, RYY, BRR, BRB, BRY, BBR, BBB, BBY, BYR, BYB, BYY, YRR, YRB, YRY, YBR, YBB, YBY, YYR, YYB, YYY\}$

b. $\{RBY, RYB, YBR, BRY, BYR, YRB\}$, probability $= 6/27 = 2/9 \cong 22.2\%$

c. $\{RRB, RBR, BRR, RRY, RYR, YRR, BBR, BRB, RBB, BBY, BYB, YBB, YYR, YRY, RYY, YYB, YBY, BYY\}$ probability $= 18/27 = 2/3 = 66\frac{2}{3}\%$

18. a. $\{B_1B_1, B_1B_2, B_1W, B_2B_1, B_2B_2, B_2W, WB_1, WB_2, WW\}$

b. $\{B_1B_1, B_1B_2, B_2B_1, B_2B_2\}$ probability $= 4/9 \cong 44.4\%$

c. $\{B_1W, B_2W, WB_1, WB_2\}$ probability $= 4/9 \cong 44.4\%$

21. a. 10 11 12 13 14 15 16 17 18 . . . 96 97 98 99

$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$

$3\cdot 4$ $3\cdot 5$ $3\cdot 6$ $3\cdot 32$ $3\cdot 33$

The above diagram shows that there are as many positive two-digit integers that are multiples of 3 as there are integers from 4 to 33 inclusive. By Theorem 9.1.1, there are $33 - 4 + 1$, or 30, such integers.

b. There are $99 - 10 + 1 = 90$ positive two-digit integers in all, and by part (a), 30 of these are multiples of 3. So the probability that a randomly chosen positive two-digit integer is a multiple of 3 is $30/90 = 1/3 = 33\frac{1}{3}\%$.

c. Of the integers from 10 through 99 that are multiples of 4, the smallest is $12\,(= 4\cdot 3)$ and the largest is $96\,(= 4\cdot 24)$. Thus there are $24 - 3 + 1 = 22$ two-digit integers that are multiples of 4. Hence the probability that a randomly chosen two-digit integer is a multiple of 4 is $22/90 = 36\frac{2}{3}\%$.

23. c. Probability $= \frac{m-3+1}{n} = \frac{m-2}{n}$

d. Because $\left\lfloor \frac{39}{2} \right\rfloor = 19$, the probability is $\frac{39-19+1}{39} = \frac{21}{39}$.

24. a. (i) If n is even, there are $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$ elements in the sub-array.

(ii) If n is odd, there are $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$ elements in the sub-array.

b. There are n elements in the array, so

(i) The probability that an element is in the given sub-array when n is even is $\frac{\frac{n}{2}}{n} = \frac{1}{2}$,

(ii) The probability that an element is in the given sub-array when n is odd is $\frac{\frac{n-1}{2}}{n} = \frac{n-1}{2n}$.

26. Let k be the 27th element in the array. By Theorem 9.1.1, $k - 42 + 1 = 27$, and so $k = 42 + 27 - 1 = 68$. Thus the 27th element in the array is $A[68]$.

28. Let m be the smallest of the integers. By Theorem 9.1.1, $279 - m + 1 = 56$, and so $m = 279 - 56 + 1 = 224$. Thus the smallest of the integers is 224.

31. 1 2 3 4 5 6 7 8 9 . . . 999 1000 1001

$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$

$3\cdot 1$ $3\cdot 2$ $3\cdot 3$ $3\cdot 333$

Thus there are 333 multiples of 3 between 1 and 1001.

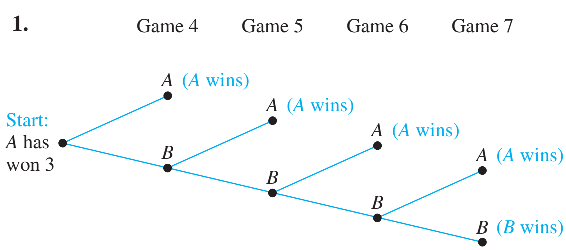
32. a.

M	Tu	W	Th	F	Sa	Su	M	Tu	W	Th	F	Sa	Su	⋯	F	Sa	Su	M
1	2	3	4	5	6	7	8	9	10	11	12	13	14		362	363	364	365
						$\updownarrow$							$\updownarrow$				$\updownarrow$	
						$7\cdot 1$							$7\cdot 2$				$7\cdot 52$	

Sundays occur on the 7th day of the year, the 14th day of the year, and in fact on all days that are multiples of 7. There are 52 multiples of 7 between 1 and 365, and so there are 52 Sundays in the year.

Section 9.2

1.

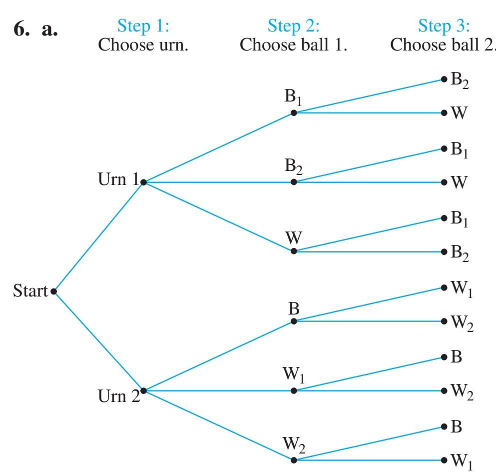


There are five ways to complete the series: *A, B–A, B–B–A, B–B–B–A*, and *B–B–B–B*.

3. Four ways: *A–A–A–A, B–A–A–A–A, B–B–A–A–A–A*, and *B–B–B–A–A–A–A*.

4. Two ways: *A–B–A–B–A–B–A* and *B–A–B–A–B–A–B*

6. a.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

b. There are 12 equally likely outcomes of the experiment.

c. $2/12 = 1/6 = 16\frac{2}{3}\%$ **d.** $8/12 = 2/3 = 66\frac{2}{3}\%$

8. By the multiplication rule, the answer is $3 \cdot 2 \cdot 2 = 12$.

9. a. In going from city A to city B, one may take any of the 3 roads. In going from city B to city C, one may take any of the 5 roads. So, by the multiplication rule, there are $3 \cdot 5 = 15$ ways to travel from city A to city C via city B.

b. A round-trip journey can be thought of as a four-step operation:

Step 1: Go from A to B.
Step 3: Go from B to C.
Step 2: Go from C to B.
Step 4: Go from B to A.

Since there are 3 ways to perform step 1, 5 ways to perform step 2, 5 ways to perform step 3, and 3 ways to perform step 4, by the multiplication rule, there are $3 \cdot 5 \cdot 5 \cdot 3 = 225$ round-trip routes.

c. In this case the steps for making a round-trip journey are the same as in part (b), but since no route segment may be repeated, there are only 4 ways to perform step 3 and only 2 ways to perform step 4. So, by the multiplication rule, there are $3 \cdot 5 \cdot 4 \cdot 2 = 120$ round-trip routes in which no road is traversed twice.

11. a. Imagine constructing a bit string of length 8 as an eight-step process:

Step 1: Choose either a 0 or a 1 for the left-most position,
Step 2: Choose either a 0 or a 1 for the next position to the right.
Step 3: Choose either a 0 or a 1 for the next position to the right.

Since there are 2 ways to perform each step, the total number of ways to accomplish the entire operation, which is the number of different bit strings of length 8, is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot\ 2 \cdot 2 \cdot 2 = 2^8 = 256$.

b. Imagine that there are three 0's in the three left-most positions, and imagine filling in the remaining 5 positions as a 5-step process, where step i is to fill in the $(i+3)$rd position. Since there are 2 ways to perform each of the 5 steps, there are 2^5 ways to perform the entire operation. So there are 2^5, or 32, 8-bit strings that begin with three 0's.

12. a. There are 9 hexadecimal digits from 3 through B and 11 hexadecimal digits from 5 through F. Thus the answer is $9 \cdot 16 \cdot 16 \cdot 16 \cdot 11 = 405{,}504$.

13. a. In each of the four tosses there are two possible results: Either a head (H) or a tail (T) is obtained. Thus, by the multiplication rule, the number of outcomes is $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.

b. There are six outcomes with two heads: $HHTT, HTHT, HTTH, THHT, THTH, TTHH$. Thus the probability of obtaining exactly two heads is $6/16 = 3/8$.

14. a. Let each of steps 1–4 be to choose a letter of the alphabet to put in positions 1–4, and let each of steps 5–7 be to choose a digit to put in positions 5–7. Since there are 26 letters and 10 digits (0–9), the number of license plates is

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 456{,}976{,}000.$$

b. In this case there is only one way to perform step 1 (because the first letter must be an A) and only one way to perform step 7 (because the last digit must be a 0). Therefore, the number of license plates is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 = 17{,}576{,}000$.

d. In this case there are 26 ways to perform step 1, 25 ways to perform step 2, 24 ways to perform step 3, 10 ways to perform step 4, 9 ways to perform step 5, and 8 ways to perform step 6, so the number of license plates is $26 \cdot 25 \cdot 24 \cdot 23 \cdot 10 \cdot 9 \cdot 8 = 258{,}336{,}000$.

16. a. Two solutions:

(i) number of integers

$$= \begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{first digit}\end{bmatrix}\begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{second digit}\end{bmatrix} = 9 \cdot 10 = 90$$

(ii) Using Theorem 9.1.1, number of integers $= 99 - 10 + 1 = 90$.

b. Odd integers end in 1, 3, 5, 7, or 9.
number of odd integers

$$= \begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{first digit}\end{bmatrix}\begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{second digit}\end{bmatrix} = 9 \cdot 5 = 45$$

Alternative solution: Use the listing method shown in the solution for Example 9.1.4.

c. $\begin{bmatrix}\text{number of integers}\\ \text{with distinct digits}\end{bmatrix}$

$$= \begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{first digit}\end{bmatrix}\begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{second digit}\end{bmatrix}$$
$$= 9 \cdot 9 = 81$$

d. $\begin{bmatrix}\text{number of odd integers}\\ \text{with distinct digits}\end{bmatrix}$

$$= \begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{second digit}\end{bmatrix}\begin{bmatrix}\text{number of}\\ \text{ways to pick}\\ \text{first digit}\end{bmatrix}$$
$$= 5 \cdot 8 = 40$$ because the first digit can't equal 0, nor can it equal the second digit

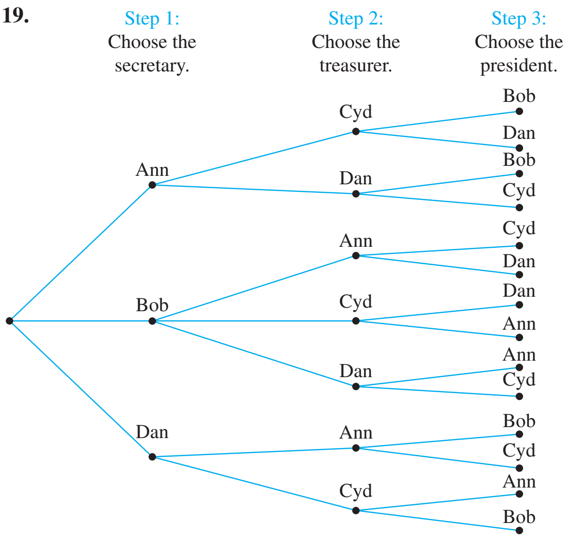
e. $81/90 = 9/10$, $40/90 = 4/9$

18. a. Let step 1 be to choose either the number 2 or one of the letters corresponding to the number 2 on the keypad, let step 2 be to choose either the number 1 or one of the letters corresponding to the number 1 on the keypad, and let steps 3 and 4 be to choose either the number 3 or one of the letters corresponding to the number 3 on the

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

keypad. There are 4 ways to perform step 1, 3 ways to perform step 2, and 4 ways to perform each of steps 3 and 4. So by the multiplication rule, there are $4 \cdot 3 \cdot 4 \cdot 4 = 192$ ways to perform the entire operation. Thus there are 192 different PINs that are keyed the same as 2133. Note that on a computer keyboard, these PINs would not be keyed the same way.

19.



There are 14 different paths from "root" to "leaf" of this possibility tree, and so there are 14 ways the officers can be chosen. Because $14 = 2 \cdot 7$, reordering the steps will not make it possible to use the multiplication rule alone to solve this problem.

20. a. The number of ways to perform step 4 is not constant; it depends on how the previous steps were performed. For instance, if 3 digits had been chosen in steps 1–3, then there would be $10 - 3 = 7$ ways to perform step 4, but if 3 letters had been chosen in steps 1–3, then there would be 10 ways to perform step 4.

21. *Hint*:

a. The answer is 2^{mn}. **b.** The answer is n^m.

22. a. The answer is $4 \cdot 4 \cdot 4 = 4^3 = 64$. Imagine creating a function from a 3-element set to a 4-element set as a three-step process: Step 1 is to send the first element of the 3-element set to an element of the 4-element set (there are four ways to perform this step); step 2 is to send the second element of the 3-element set to an element of the 4-element set (there are also four ways to perform this step); and step 3 is to send the third element of the 3-element set to an element of the 4-element set (there are four ways to perform this step). Thus the entire process can be performed in $4 \cdot 4 \cdot 4$ different ways.

24. The outer loop is iterated 30 times, and during each iteration of the outer loop there are 15 iterations of the inner loop. Hence, by the multiplication rule, the total number of iterations of the inner loop is $30 \cdot 15 = 450$.

27. The outer loop is iterated $50 - 5 + 1 = 46$ times, and during each iteration of the outer loop there are $20 - 10 + 1 = 11$ iterations of the inner loop. Hence, by the multiplication rule, the total number of iterations of the inner loop is $46 \cdot 11 = 506$.

29. *Hints:* One solution is to add leading zeros as needed to make each number five digits long. For instance, write 1 as 00001. Let some of the steps be to choose positions for the given digits. The answer is 720. Another solution is to consider separately the cases of four-digit and five-digit numbers.

31. a. There are $a + 1$ divisors: $1, p, p^2, \ldots, p^a$.

b. A divisor is a product of any one of the $a + 1$ numbers listed in part (a) times any one of the $b + 1$ numbers $1, q, q^2, \ldots, q^b$. So, by the multiplication rule, there are $(a + 1)(b + 1)$ divisors in all.

32. a. Since the nine letters of the word $ALGORITHM$ are all distinct, there are as many arrangements of these letters in a row as there are permutations of a set with nine elements: $9! = 362,880$.

b. In this case there are effectively eight symbols to be permuted (because $\boxed{AL}$ may be regarded as a single symbol). So the number of arrangements is $8! = 40,320$.

34. The same reasoning as in Example 9.2.9 gives an answer of $4! = 24$.

35. $WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY$

37. a. $P(6, 4) = \dfrac{6!}{(6-4)!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2 \cdot 1}}{\cancel{2 \cdot 1}} = 360$

38. a. $P(5, 3) = \dfrac{5 \cdot 4 \cdot 3 \cdot \cancel{2}!}{\cancel{2}!} = 60$

39. a. $P(9, 3) = \dfrac{9 \cdot 8 \cdot 7 \cdot \cancel{6}!}{\cancel{6}!} = 504$

c. $P(8, 5) = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3}!}{\cancel{3}!} = 6,720$

41. <u>Proof</u>: Let n be an integer and $n \geq 2$. Then

$$
\begin{aligned}
&P(n+1, 2) - P(n, 2) \\
&\quad = \frac{(n+1)!}{[(n+1)-2]!} - \frac{n!}{(n-2)!} = \frac{(n+1)!}{(n-1)!} - \frac{n!}{(n-2)!} \\
&\quad = \frac{(n+1) \cdot n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} - \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)!}} \\
&\quad = n^2 + n - (n^2 - n) = 2n = 2 \cdot \frac{n \cdot (n-1)!}{(n-1)!} \\
&\quad = 2 \cdot \frac{n!}{(n-1)!} = 2P(n, 1).
\end{aligned}
$$

This is what was to be proved.

45. *Hint:* In the inductive step, suppose there exist $k!$ permutations of a set with k elements. Let X be a set with $k + 1$ elements. The process of forming a permutation of the elements of X can be considered a two-step operation where step 1 is to choose the element to write first. Step 2 is to write the remaining elements of X in some order.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$$= 8! - 10{,}080 = 40{,}320 - 10{,}080$$
$$= 30{,}240$$

14. number of variable names

$$= \begin{bmatrix}\text{number of numeric}\\ \text{variable names}\end{bmatrix} + \begin{bmatrix}\text{number of string}\\ \text{variable names}\end{bmatrix}$$
$$= (26 + 26\cdot 36) + (26 + 26\cdot 36) = 1{,}924$$

15. *Hint:* In exercise 14 note that

$$26 + 26\cdot 36 = 26\sum_{k=0}^{1} 36^k.$$

Generalize this idea here. Use Theorem 5.2.3 to evaluate the expression you obtain.

16. a. $10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4 = 604{,}800$

b. $\begin{bmatrix}\text{number of phone numbers with}\\ \text{at least one repeated digit}\end{bmatrix}$

$$= \begin{bmatrix}\text{total number of}\\ \text{phone numbers}\end{bmatrix} - \begin{bmatrix}\text{number of phone numbers}\\ \text{with no repeated digits}\end{bmatrix}$$
$$= 10^7 - 604{,}800 = 9{,}395{,}200$$

c. $9{,}395{,}200/10^7 \cong 93.95\%$

18. a. Proof: Let A and B be mutually disjoint events in a sample space S. By the addition rule, $N(A \cup B) = N(A) + N(B)$. Therefore, by the equally likely probability formula,

$$P(A \cup B) = \frac{N(A \cup B)}{N(S)} = \frac{N(A) + N(B)}{N(S)}$$
$$= \frac{N(A)}{N(S)} + \frac{N(B)}{N(S)} = P(A) + P(B).$$

19. *Hint:* Justify the following answer: $39\cdot 38\cdot 38$.

20. a. Identify the integers from 1 to 100,000 that contain the digit 6 exactly once with strings of five digits. Thus, for example, 306 would be identified with 00306. It is not necessary to use strings of six digits, because 100,000 does not contain the digit 6. Imagine the process of constructing a five-digit string that contains the digit 6 exactly once as a five-step operation that consists of filling in the five digit positions $\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}$.

Step 1: Choose one of the five positions for the 6.

Step 2: Choose a digit for the left-most remaining position.

Step 3: Choose a digit for the next remaining position to the right.

Step 4: Choose a digit for the next remaining position to the right.

Step 5: Choose a digit for the right-most position.

Since there are 5 choices for step 1 (any one of the five positions) and 9 choices for each of steps 2–5 (any digit except 6), by the multiplication rule, the number of ways to perform this operation is $5\cdot 9\cdot 9\cdot 9\cdot 9 = 32{,}805$. Hence there are 32,805 integers from 1 to 100,000 that contain the digit 6 exactly once.

21. *Hint:* The answer is $2/3$.

23. a. Let $A =$ the set of integers that are multiples of 4 and $B =$ the set of integers that are multiples of 7. Then $A \cap B =$ the set of integers that are multiples of 28.

But $n(A) = 250$ $\quad$ since $\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots & 999 & 1000, \\ & & & \updownarrow & & & & \updownarrow & & & \updownarrow \\ & & & 4\cdot 1 & & & & 4\cdot 2\ldots & & & 4\cdot 250 \end{array}$

or, equivalently, since $1{,}000 = 4\cdot 250$.

Also $n(B) = 142$ $\quad$ since $\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & 14 \ldots & 994 & 995\ldots 1000 \\ & & & & & & \updownarrow & & \updownarrow & \updownarrow & \\ & & & & & & 7\cdot 1 & & 7\cdot 2\ldots & 7\cdot 142 & \end{array}$

or, equivalently, since $1{,}000 = 7\cdot 142 + 6$.

and $n(A \cap B) = 35$ $\quad$ since $\begin{array}{cccccc} 1\ 2\ 3 \ldots & 28 & \ldots & 56 \ldots & 980 & \ldots 1000, \\ & \updownarrow & & \updownarrow & \updownarrow & \\ & 28\cdot 1 & & 28\cdot 2\ldots & 28\cdot 35 & \end{array}$

or, equivalently, since $1{,}000 = 28\cdot 35 + 20$.

So $n(A \cup B) = 250 + 142 - 35 = 357$.

25. a. Length 0: ϵ
Length 1: 0, 1
Length 2: 00, 01, 10, 11
Length 3: 000, 001, 010, 011, 100, 101, 110
Length 4: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101

b. By part (a), $d_0 = 1$, $d_1 = 2$, $d_2 = 4$, $d_3 = 7$, and $d_4 = 13$.

c. Let k be an integer with $k \geq 3$. Any string of length k that does not contain the bit pattern 111 starts either with a 0 or with a 1. If it starts with a 0, this can be followed by any string of $k - 1$ bits that does not contain the pattern 111. There are d_{k-1} of these. If the string starts with a 1, then the first two bits are 10 or 11. If the first two bits are 10, then these can be followed by any string of $k - 2$ bits that does not contain the pattern 111. There are d_{k-2} of these. If the string starts with a 11, then the third bit must be 0 (because the string does not contain 111), and these three bits can be followed by any string of $k - 3$ bits that does not contain the pattern 111. There are d_{k-3} of these. Therefore, for all integers $k \geq 3$, $d_k = d_{k-1} + d_{k-2} + d_{k-3}$.

d. By parts (b) and (c), $d_5 = d_4 + d_3 + d_2 = 13 + 7 + 4 = 24$. This is the number of bit strings of length five that do not contain the pattern 111.

26. c. *Hint:* $s_k = 2s_{k-1} + 2s_{k-2}$

e. *Hint:* For all integers $n \geq 0$,

$$s_n = \frac{\sqrt{3} + 2}{2\sqrt{3}}(1 + \sqrt{3})^n + \frac{\sqrt{3} - 2}{2\sqrt{3}}(1 - \sqrt{3})^n.$$

28. a. $a_3 = 3$ (The three permutations that do not move more than one place from their "natural" positions are 213, 132, and 123.)

29. a. $11001010_2 = 2 + 2^3 + 2^6 + 2^7 = 202$,
$00111000_2 = 2^3 + 2^4 + 2^5 = 56$,
$01101011_2 = 1 + 2 + 2^3 + 2^5 + 2^6 = 107$,
$11101110_2 = 2 + 2^2 + 2^3 + 2^5 + 2^6 + 2^7 = 238$
So the answer is 202.56.107.238.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

b. The network ID for a Class A network consists of 8 bits and begins with 0. If all possible combinations of eight 0's and 1's that start with a 0 were allowed, there would be 2 choices (0 or 1) for each of the 7 positions from the second through the eighth. This would give $2^7 = 128$ possible ID's. But because neither 00000000 nor 01111111 is allowed, the total is reduced by 2, so there are 126 possible Class A networks.

c. Let $w.x.y.z$ be the dotted decimal form of the IP address for a computer in a Class A network. Because the network IDs for a Class A network go from 00000001 $(= 1)$ through 01111110 $(= 126)$, w can be any integer from 1 through 126. In addition, each of x, y, and z can be any integer from 0 $(= 00000000)$ through 255 $(= 11111111)$, except that x, y, and z cannot all be 0 simultaneously and cannot all be 255 simultaneously.

d. Twenty-four positions are allocated for the host ID in a Class A network. If each could be either 0 or 1, there would be $2^{24} = 16{,}777{,}216$ possible host IDs. But neither all 0's nor all 1's is allowed, which reduces the total by 2. Thus there are 16,777,214 possible host IDs in a Class A network.

i. Observe that $140 = 128 + 8 + 4 = 10001100_2$, which begins with 10. Thus the IP address comes from a Class B network. An alternative solution uses the result of Example 9.3.5: Network IDs for Class B networks range from 128 through 191. Thus, since $128 \leq 140 \leq 191$, the given IP address is from a Class B network.

31. a. There are 12 possible birth months for A, 12 for B, 12 for C, and 12 for D, so the total is $12^4 = 20{,}736$.

b. If no two people share the same birth month, there are 12 possible birth months for A, 11 for B, 10 for C, and 9 for D. Thus the total is $12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880$.

c. If at least two people share the same birth month, the total number of ways birth months could be associated with A, B, C, and D is $20{,}736 - 11{,}880 = 8{,}856$.

d. The probability that at least two of the four people share the same birth month is $\dfrac{8856}{20736} \cong 42.7\%$.

e. When there are five people, the probability that at least two share the same birth month is $\dfrac{12^5 - 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{12^5} \cong 61.8\%$, and when there are more than five people, the probability is even greater. Thus, since the probability for four people is less than 50%, the group must contain five or more people for the propability to be at least 50% that two or more share the same birth month.

32. *Hint:* Analyze the solution to exercise 31.

33. a. The number of students who checked at least one of the statements is $N(H) + N(C) + N(D) - N(H \cap C) - N(N \cap D) - N(C \cap D) + N(H \cap C \cap D) = 28 + 26 + 14 - 14 - 4 - 8 + 2 = 45$

b. By the difference rule, the number of students who checked none of the statements is the total number of students minus the number who checked at least one statement. This is $100 - 45 = 55$.

d. The number of students who checked #1 and #2 but not #3 is $N(H \cap C) - N(N \cap C \cap D) = 14 - 2 = 12$.

35. Let

$$\begin{aligned} M &= \text{the set of married people in the sample,} \\ Y &= \text{the set of people between 20 and 30 in the sample, and} \\ F &= \text{the set of females in the sample.} \end{aligned}$$

Then the number of people in the set $M \cup Y \cup F$ is less than or equal to the size of the sample. And so

$$\begin{aligned} 1{,}200 \geq N(M \cup Y \cup F) & \\ &= N(M) + N(Y) + N(F) - N(M \cap Y) \\ &\quad - N(M \cap F) - N(Y \cap F) + N(M \cap Y \cap F) \\ &= 675 + 682 + 684 - 195 - 467 - 318 + 165 \\ &= 1{,}226. \end{aligned}$$

This is impossible since $1{,}200 < 1{,}226$, so the polltaker's figures are inconsistent. They could not have occurred as a result of an actual sample survey.

37. Let A be the set of all positive integers less than 1,000 that are not multiples of 2, and let B be the set of all positive integers less than 1,000 that are not multiples of 5. Since the only prime factors of 1,000 are 2 and 5, the number of positive integers that have no common factors with 1,000 is $N(A \cap B)$. Let the universe U be the set of all positive integers less than 1,000. Then A^c is the set of positive integers less than 1,000 that are multiples of 2, B^c is the set of positive integers less than 1,000 that are multiples of 5, and $A^c \cap B^c$ is the set of positive integers less than 1,000 that are multiples of 10. By one of the procedures discussed in Section 9.1 or 9.2, it is easily found that $N(A^c) = 499$, $N(B^c) = 199$, and $N(A^c \cap B^c) = 99$. Thus, by the inclusion/exclusion rule,

$$\begin{aligned} N(A^c \cup B^c) &= N(A^c) + N(B^c) - N(A^c \cap B^c) \\ &= 499 + 199 - 99 = 599. \end{aligned}$$

But by De Morgan's law, $N(A^c \cup B^c) = N((A \cap B)^c)$, and so

$$N((A \cap B)^c) = 599. \qquad (*)$$

Now since $(A \cap B)^c = U - (A \cap B)$, by the difference rule we have

$$N((A \cap B)^c) = N(U) - N(A \cap B). \qquad (**)$$

Equating the right-hand sides of $(*)$ and $(**)$ gives $N(U) - N(A \cap B) = 599$. And because $N(U) = 999$, we conclude that $999 - N(A \cap B) = 599$, or, equivalently, $N(A \cap B) = 999 - 599 = 400$. So there are 400 positive integers less than 1,000 that have no common factor with 1,000.

40. *Hint:* Let A and B be the sets of all positive integers less than or equal to n that are divisible by p and q, respectively. Then $\phi(n) = n - (N(A \cup B))$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

42. c. *Hint:* If $k \geq 6$, any sequence of k games must begin with W, LW, or LLW, where L stands for "lose" and W stands for "win."

43. c. *Hint:* Divide the set of all derangements into two subsets: one subset consists of all derangements in which the number 1 changes places with another number, and the other subset consists of all derangements in which the number 1 goes to position $i \neq 1$ but i does not go to position 1. The answer is $d_k = (k-1)d_{k-1} + (k-1)d_{k-2}$. Can you justify it?

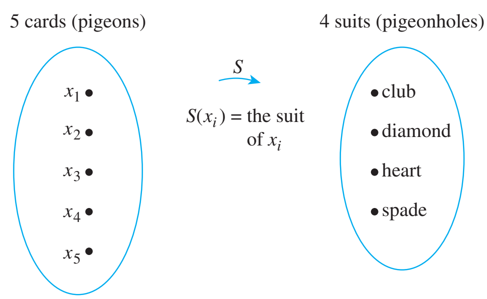
48. *Hint:* Use the associative law for sets and the generalized distributive law for sets from exercise 37, Section 6.2.

49. *Hint*: Use the solution method described in Section 5.8. The answer is $s_k = 2s_{k-1} + 3s_{k-2}$ for $k \geq 4$.

Section 9.4

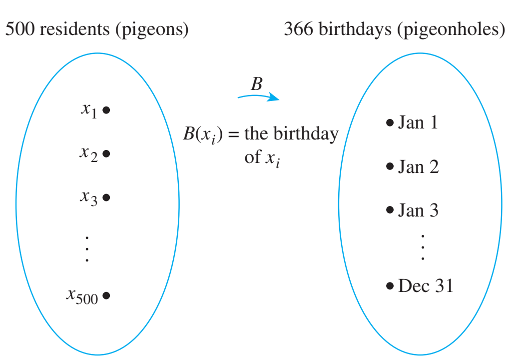
1. a. No. For instance, the aces of the four different suits could be selected.

b. Yes. Let x_1, x_2, x_3, x_4, x_5 be the five cards. Consider the function S that sends each card to its suit.



By the pigeonhole principle, S is not one-to-one: $S(x_i) = S(x_j)$ for some two cards x_i and x_j. Hence at least two cards have the same suit.

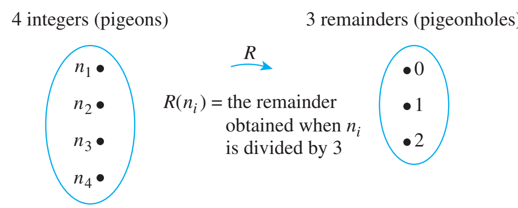
3. Yes. Denote the residents by $x_1, x_2, \ldots, x_{500}$. Consider the function B from residents to birthdays that sends each resident to his or her birthday:



By the pigeonhole principle, B is not one-to-one: $B(x_i) = B(x_j)$ for some two residents x_i and x_j. Hence at least two residents have the same birthday.

5. a. Yes. There are only three possible remainders that can be obtained when an integer is divided by 3: 0, 1, and 2. Thus, by the pigeonhole principle, if four integers are each divided by 3, then at least two of them must have the same remainder.

More formally, call the integers n_1, n_2, n_3, and n_4, and consider the function R that sends each integer to the remainder obtained when that integer is divided by 3:



By the pigeonhole principle, R is not one-to-one, $R(n_i) = R(n_j)$ for some two integers n_i and n_j. Hence at least two integers must have the same remainder.

b. No. For instance, $\{0, 1, 2\}$ is a set of three integers no two of which have the same remainder when divided by 3.

7. *Hint:* Look at Example 9.4.3.

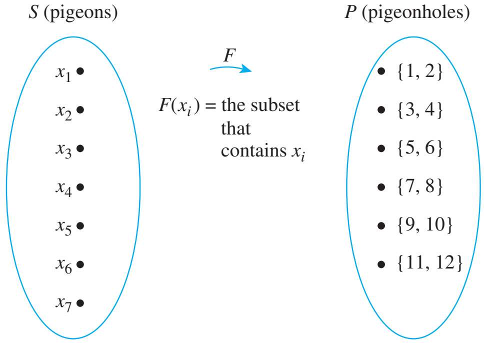
9. a. Yes.

Solution 1: Only six of the numbers from 1 to 12 are even (namely, 2, 4, 6, 8, 10, 12), so at most six even numbers can be chosen from between 1 and 12 inclusive. Hence if seven numbers are chosen, at least one must be odd.

Solution 2: Partition the set of all integers from 1 through 12 into six subsets (the pigeonholes), each consisting of an odd and an even number: $\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}$. If seven integers (the pigeons) are chosen from among 1 through 12, then, by the pigeonhole principle, at least two must be from the same subset. But each subset contains one odd and one even number. Hence at least one of the seven numbers is odd.

Solution 3: Let $S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be a set of seven numbers chosen from the set $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and let P be the following partition of T: $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$, $\{9, 10\}$, and $\{11, 12\}$. Since each element of S lies in exactly one subset of the partition, we can define a function F from S to P by letting $F(x_i)$ be the subset that contains x_i.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



Since S has 7 elements and P has 6 elements, by the pigeonhole principle, F is not one-to-one. Thus two distinct numbers of the seven are sent to the same subset, which implies that these two numbers are the two distinct elements of the subset. Therefore, since each pair consists of one odd and one even integer, one of the seven numbers is odd.

b. No. For instance, none of the 10 numbers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 is even.

10. Yes. There are n even integers in the set $\{1, 2, 3, \ldots, 2n\}$, namely $2(= 2 \cdot \underline{1}), 4(= 2 \cdot \underline{2}), 6(= 2 \cdot \underline{3}), \ldots, 2\underline{n}(= 2 \cdot n)$. So the maximum number of even integers that can be chosen is n. Thus if $n + 1$ integers are chosen, at least one of them must be odd.

12. The answer is 27. There are only 26 black cards in a standard 52-card deck, so at most 26 black cards can be chosen. Hence if 27 are taken, at least one must be red.

14. There are 61 integers from 0 to 60 inclusive. Of these, 31 are even ($0 = 2 \cdot \underline{0}, 2 = 2 \cdot \underline{1}, 4 = 2 \cdot \underline{2}, \ldots, 60 = 2 \cdot \underline{30}$) and so 30 are odd. Hence if 32 integers are chosen, at least one must be odd, and if 31 integers are chosen, at least one must be even.

17. The answer is 8. (There are only seven possible remainders for division by 7: 0, 1, 2, 3, 4, 5, 6.)

20. The answer is 20,483 *[namely,* $0, 1, 2, \ldots, 20482$*]*.

22. This number is irrational; the decimal expansion neither terminates nor repeats.

24. Let A be the set of the thirteen chosen numbers, and let B be the set of all prime numbers between 1 and 40. Note that $B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$. For each x in A, let $F(x)$ be the smallest prime number that divides x. Since A has 13 elements and B has 12 elements, by the pigeonhole principle F is not one-to-one. Thus $F(x_1) = F(x_2)$ for some $x_1 \neq x_2$ in A. By definition of F, this means that the smallest prime number that divides x_1 equals the smallest prime number that divides x_2. Therefore, two numbers in A, namely x_1 and x_2, have a common divisor greater than 1.

25. Yes. This follows from the generalized pigeonhole principle with 30 pigeons, 12 pigeonholes, and $k = 2$, using the fact that $30 > 2 \cdot 12$.

26. No. For instance, the birthdays of the 30 people could be distributed as follows: three birthdays in each of the six months January through June and two birthdays in each of the six months July through December.

29. The answer is $x = 3$. There are 18 years from 17 through 34. Now $40 > 18 \cdot 2$, so by the generalized pigeonhole principle, you can be sure that there are at least $x = 3$ students of the same age. However, since $18 \cdot 3 > 40$, you cannot be sure of having more than three students with the same age. (For instance, three students could be each of the ages 17 through 20, and two could be each of the ages from 21 through 34.) So x cannot be taken to be greater than 3.

31. *Hint:* Use the same type of reasoning as in Example 9.4.6.

32. *Hints:* (1) The number of subsets of the six integers is $2^6 = 64$. (2) Since each integer is less than 13, the largest possible sum is 57. (Why? What gives this sum?)

33. *Hint:* The power set of A has $2^6 = 64$ elements, and so there are 63 nonempty subsets of A. Let k be the smallest number in the set A. Then the sums over the elements in the nonempty subsets of A lie in the range from k through $k + 10 + 11 + 12 + 13 + 14 = k + 60$. How many numbers are in this range?

35. *Hint:* Let X be the set consisting of the given 52 positive integers, and let Y be the set containing the following elements: $\{00\}$, $\{50\}$, $\{01, 99\}$, $\{02, 98\}$, $\{03, 97\}$, $\ldots$, $\{48, 52\}$, $\{49, 51\}$. Define a function F from X to Y by the rule $F(x) =$ the set containing the last two digits of x. Use the pigeonhole principle to argue that F is not one-to-one, and show how the desired conclusion follows.

36. *Hint:* Represent each of the 101 integers x_i as $a_i 2^{k_i}$ where a_i is odd and $k_i \geq 0$. Now $1 \leq x_i \leq 200$, and so $1 \leq a_i \leq 199$ for all i. There are only 100 odd integers from 1 to 199 inclusive.

37. b. *Hint:* For each $k = 1, 2, \ldots, n$, let $a_k = x_1 + x_2 + \cdots + x_k$. If some a_k is divisible by n, then the problem is solved: the consecutive subsequence is $x_1, x_2, \ldots, x_k$. If no a_k is divisible by n, then $a_1, a_2, a_3, \ldots, a_n$ satisfies the hypothesis of part (a). Hence $a_j - a_i$ is divisible by n for some integers i and j with $j > i$. Write $a_j - a_i$ in terms of the x_i's to derive the given conclusion.

38. *Hint:* Let $a_1, a_2, \ldots, a_{n^2+1}$ be any sequence of $n^2 + 1$ distinct real numbers, and suppose that this sequence contains neither a strictly increasing subsequence of length $n + 1$ nor a strictly decreasing subsequence of length $n + 1$. Let S be the set of all ordered pairs of integers (i, d), where $1 \leq i \leq n$ and $1 \leq d \leq n$. For each term a_k in the sequence, let $F(a_k) = (i_k, d_k)$, where i_k is the length of the longest increasing sequence starting at a_k, and d_k is the length of the longest decreasing sequence starting at a_k. Suppose that F is one-to-one and derive a contradiction.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 9.5

1. a. 2-combinations: $\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}$.

Hence, $\binom{3}{2} = 3$.

b. Unordered selections: $\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}$.

Hence, $\binom{5}{4} = 5$.

3. $P(7, 2) = \binom{7}{2} \cdot 2!$

5. a. $\binom{6}{0} = \dfrac{6!}{0!(6-0)!} = \dfrac{\cancel{6!}}{1\cancel{6!}} = 1$

b. $\binom{6}{1} = \dfrac{6!}{1!(6-1)!} = \dfrac{6\cancel{5!}}{1\cancel{5!}} = 6$

6. a. number of committees of 6

$$= \binom{15}{6} = \frac{15!}{(15-6)!6!}$$

$$= \frac{\cancel{15}\cdot \cancelto{7}{14}\cdot 13 \cdot \cancel{12}\cdot 11 \cdot \cancelto{5}{10}\cdot \cancel{9!}}{\cancel{9!}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} = 5{,}005$$

b. $\begin{bmatrix}\text{number of committees}\\ \text{that don't contain } A\\ \text{and } B \text{ together}\end{bmatrix}$

$$= \begin{bmatrix}\text{number of}\\ \text{committees with } A\\ \text{and five others—}\\ \text{none of them } B\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{committees with } B\\ \text{and five others—}\\ \text{none of them } A\end{bmatrix} + \begin{bmatrix}\text{number of committees}\\ \text{with neither } A \text{ nor } B\end{bmatrix}$$

$$= \binom{13}{5} + \binom{13}{5} + \binom{13}{6}$$

$$= 1{,}287 + 1{,}287 + 1{,}716 = 4{,}290$$

Alternative solution:

$$\begin{bmatrix}\text{number of committees}\\ \text{that don't contain } A\\ \text{and } B \text{ together}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of committees}\end{bmatrix} - \begin{bmatrix}\text{number of committees}\\ \text{that contain both } A \text{ and } B\end{bmatrix}$$

$$= \binom{15}{6} - \binom{13}{4}$$

$$= 5{,}005 - 715 = 4{,}290$$

c. $\begin{bmatrix}\text{number of}\\ \text{committees with}\\ \text{both } A \text{ and } B\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{committees with}\\ \text{neither } A \text{ and } B\end{bmatrix}$

$$= \binom{13}{4} + \binom{13}{6} = 715 + 1{,}716 = 2{,}431$$

d. (i) $\begin{bmatrix}\text{number of subsets}\\ \text{of three men}\\ \text{chosen from eight}\end{bmatrix} \cdot \begin{bmatrix}\text{number of subsets}\\ \text{of three women}\\ \text{chosen from seven}\end{bmatrix}$

$$= \binom{8}{3}\binom{7}{3} = 56 \cdot 35 = 1{,}960$$

(ii) $\begin{bmatrix}\text{number of committees}\\ \text{with at least one woman}\end{bmatrix}$

$$= \begin{bmatrix}\text{total number of}\\ \text{committees}\end{bmatrix} - \begin{bmatrix}\text{number of all-male}\\ \text{committees}\end{bmatrix}$$

$$= \binom{15}{6} - \binom{8}{6} = 5{,}005 - 28$$

$$= 4{,}977$$

e. $\begin{bmatrix}\text{number of}\\ \text{ways to choose}\\ \text{two freshmen}\end{bmatrix} \cdot \begin{bmatrix}\text{number of}\\ \text{ways to choose two}\\ \text{sophomores}\end{bmatrix}$

$$\cdot \begin{bmatrix}\text{number of ways}\\ \text{to choose two juniors}\end{bmatrix} \cdot \begin{bmatrix}\text{number of ways}\\ \text{to choose two seniors}\end{bmatrix}$$

$$= \binom{3}{2}\binom{4}{2}\binom{3}{2}\binom{5}{2}$$

$$= 540$$

8. *Hint:* The answers are **a.** 1001, **b.** (i) 420, (ii) all 1001 require proof, (iii) 175, **c.** 506, **d.** 561

9. b. $\binom{24}{3}\binom{16}{3} + \binom{24}{4}\binom{16}{2} + \binom{24}{5}\binom{16}{1} + \binom{24}{6}\binom{16}{0} = 3{,}223{,}220$

11. a. (1) 4 (because there are as many royal flushes as there are suits)

(2) $\dfrac{4}{\binom{52}{5}} = \dfrac{4}{2{,}598{,}960} \cong 0.0000015$

c. (1) $13 \cdot \binom{48}{1} = 624$ (because one can first choose the denomination of the four-of-a-kind and then choose one additional card from the 48 remaining)

(2) $\dfrac{624}{\binom{52}{5}} = \dfrac{624}{2{,}598{,}960} = 0.00024$

f. (1) Imagine constructing a straight (including a straight flush and a royal flush) as a six-step process: step 1 is to choose the lowest denomination of any card of the five (which can be any one of $A, 2, \ldots, 10$), step 2 is to choose a card of that denomination, step 3 is to choose a card of the next higher denomination, and so forth until all five cards have been selected. By the multiplication rule, the number of ways to perform this process is

$$10 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10 \cdot 4^5 = 10{,}240.$$

By parts (a) and (b), 40 of these numbers represent royal or straight flushes, so there are $10{,}240 - 40 = 10{,}200$ straights in all.

(2) $\dfrac{10{,}200}{\binom{52}{5}} = \dfrac{10{,}200}{2{,}598{,}960} \cong 0.0039$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

13. a. $2^{10} = 1{,}024$

d. $$\begin{bmatrix}\text{number of outcomes}\\ \text{with at least one head}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of outcomes}\end{bmatrix} - \begin{bmatrix}\text{number of outcomes}\\ \text{with no heads}\end{bmatrix} = 1{,}024 - 1 = 1{,}023$$

15. a. 50 **b.** 50

c. To get an even sum, both numbers must be even or both must be odd. Hence

$$\begin{bmatrix}\text{number of subsets of two integers from}\\ \text{1 to 100 inclusive whose sum is even}\end{bmatrix} = \begin{bmatrix}\text{number of subsets}\\ \text{of two even}\\ \text{integers chosen from}\\ \text{the 50 possible}\end{bmatrix} + \begin{bmatrix}\text{number of subsets}\\ \text{of two odd}\\ \text{integers chosen from}\\ \text{the 50 possible}\end{bmatrix} = \binom{50}{2} + \binom{50}{2} = 2{,}450.$$

d. To obtain an odd sum, one of the numbers must be even and the other odd. Hence the answer is $\binom{50}{1}\cdot\binom{50}{1} = 2{,}500$. Alternatively, note that the answer equals the total number of subsets of two integers chosen from 1 through 100 minus the number of such subsets for which the sum of the elements is even. Thus the answer is $\binom{100}{2} - 2{,}450 = 2{,}500$.

17. a. Two points determine a line. Hence

$$\begin{bmatrix}\text{number of straight}\\ \text{lines determined}\\ \text{by the ten points}\end{bmatrix} = \begin{bmatrix}\text{number of subsets}\\ \text{of two points}\\ \text{chosen from ten}\end{bmatrix} = \binom{10}{2} = 45.$$

19. a. $\dfrac{10!}{2!1!1!3!2!1!} = 151{,}200$ since there are 2 A's, 1 B, 1 H, 3 L's, 2 O's, and 1 U

b. $\dfrac{8!}{2!1!1!2!2!} = 5{,}040$ **c.** $\dfrac{9!}{1!2!1!3!2!} = 15{,}120$

23. Rook must move seven squares to the right and seven squares up, so

$$\begin{bmatrix}\text{the number of}\\ \text{paths the rook}\\ \text{can take}\end{bmatrix} = \begin{bmatrix}\text{the number}\\ \text{of orderings}\\ \text{of seven R's}\\ \text{and seven U's}\end{bmatrix} = \frac{14!}{7!7!} = 3{,}432.$$

where R stands for "right" and U stands for "up"

24. b. *Solution 1:* One factor can be 1, and the other factor can be the product of all the primes. (This gives 1 factorization.) One factor can be one of the primes, and the other factor can be the product of the other three. (This gives $\binom{4}{1} = 4$ factorizations.) One factor can be a product of two of the primes, and the other factor can be a product of the two other primes. The number $\binom{4}{2} = 6$ counts all possible sets of two primes chosen from the four primes, and each set of two primes corresponds to a factorization. Note, however, that the set $\{p_1, p_2\}$ corresponds to the same factorization as the set $\{p_3, p_4\}$, namely, $p_1p_2p_3p_4$ (just written in a different order). In general, each choice of two primes corresponds to the same factorization as one other choice of two primes. Thus the number of factorizations in which each factor is a product of two primes is $\dfrac{\binom{4}{2}}{2} = 3$. (This gives 3 factorizations.) The foregoing cases account for all the possibilities, so the answer is $4 + 3 + 1 = 8$.

Solution 2: Let $S = \{p_1, p_2, p_3, p_4\}$. Let $p_1p_2p_3p_4 = P$, and let f_1f_2 be any factorization of P. The product of the numbers in any subset $A \subseteq S$ can be used for f_1, with the product of the numbers in A^c being f_2. There are as many ways to write f_1f_2 as there are subsets of S, namely $2^4 = 16$ (by Theorem 6.3.1). But given any factors f_1 and f_2, $f_1f_2 = f_2f_1$. Thus counting the number of ways to write f_1f_2 counts each factorization twice, so the answer is $\frac{16}{2} = 8$.

25. a. There are four choices for where to send the first element of the domain (any element of the co-domain may be chosen), three choices for where to send the second (since the function is one-to-one, the second element of the domain must go to a different element of the co-domain from the one to which the first element went), and two choices for where to send the third element (again since the function is one-to-one). Thus the answer is $4\cdot 3\cdot 2 = 24$.

b. none

e. *Hint:* The answer is $n(n-1)\cdots(n-m+1)$.

26. a. Let the elements of the domain be called a, b, and c and the elements of the co-domain be called u and v. In order for a function from $\{a, b, c\}$ to $\{u, v\}$ to be onto, two elements of the domain must be sent to u and one to v, or two elements must be sent to v and one to u. There are as many ways to send two elements of the domain to u and one to v as there are ways to choose which elements of $\{a, b, c\}$ to send to u, namely, $\binom{3}{2} = 3$. Similarly, there are $\binom{3}{2} = 3$ ways to send two elements of the domain to v and one to u. Therefore, there are $3 + 3 = 6$ onto functions from a set with three elements to a set with two elements.

c. *Hint:* The answer is 6.

d. Consider functions from a set with four elements to a set with two elements. Denote the set of four elements by $X = \{a, b, c, d\}$ and the set of two elements by $Y = \{u, v\}$. Divide the set of all onto functions from X to Y into two categories. The first category consists of all those that send the three elements in $\{a, b, c\}$ onto $\{u, v\}$ and that send d to either u or v. The functions in this category can be defined by the following two-step process:

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Step 1: Construct an onto function from $\{a, b, c\}$ to $\{u, v\}$.

Step 2: Choose whether to send d to u or to v.

By part (a), there are six ways to perform step 1, and, because there are two choices for where to send d, there are two ways to perform step 2. Thus, by the multiplication rule, there are $6 \cdot 2 = 12$ ways to define the functions in the first category.

The second category consists of all those onto functions from X to Y that send all three elements in $\{a, b, c\}$ to either u or v and that send d to whichever of u or v is not the image of the others. Because there are only two choices for where to send the elements in $\{a, b, c\}$, and because d is simply sent to wherever the others do not go, there are just two functions in the second category.

Every onto function from X to Y either sends at least two elements of X to $f(d)$ or it does not. If it sends at least two elements of X to $f(d)$ then it is in the second category. If it does not, then the image of $\{a, b, c\}$ is $\{u, v\}$ and so the "restriction" of the function to $\{a, b, c\}$ is onto. Therefore, the function is one of those included in the first category. Thus all onto functions from X to Y are in one of the two categories and no function is in both categories, and so the total number of onto functions is $12 + 2 = 14$.

Hints: **a.** (i) g is one-to-one (ii) g is not onto

b. G is onto. Proof: Suppose y is any element of $\mathbf{R}$. *[We must show that there is an element x in $\mathbf{R}$ such that $G(x) = y$. Use of scratch work to determine what x would have to be if it exists shows that x would have to equal $(y + 5)/4$. The proof must then show that x has the necessary properties.]* Let $x = (y + 5)/4$. Then (1) $x \in \mathbf{R}$, and (2) $G(x) = G((y + 5)/4) = 4[(y + 5)/4] - 5 = (y + 5) - 5 = y$ *[as was to be shown]*.

27. a. A relation on A is any subset of $A \times A$, and $A \times A$ has $8^2 = 64$ elements. So there are 2^{64} binary relations on A.

c. Form a symmetric relation by a two-step process: (1) pick a set of elements of the form (a, a) (there are eight such elements, so 2^8 sets); (2) pick a set of pairs of elements of the form (a, b) and (b, a) where $a \neq b$ (there are $(64 - 8)/2 = 28$ such pairs, so 2^{28} such sets). The answer is therefore $2^8 \cdot 2^{28} = 2^{36}$.

28. *Hint:* Use the difference rule and the generalization of the inclusion/exclusion rule for 4 sets. (See exercise 48 in Section 9.3.)

31. Call the set X, and suppose that $X = \{x_1, x_2, \ldots, x_n\}$. For each integer $i = 0, 1, 2, \ldots, n - 1$, we can consider the set of all partitions of X (let's call them *partitions of type* i) where one of the subsets of the partition is an $(i + 1)$-element set that contains x_n and i elements chosen from $\{x_1, \ldots, x_{n-1}\}$. The remaining subsets of the partition will be a partition of the remaining $(n - 1) - i$ elements of $\{x_1, \ldots, x_{n-1}\}$. For instance, if $X = \{x_1, x_2, x_3\}$, there are five partitions of the various types, namely,

Type 0: two partitions where one set is a 1-element set containing x_3: $[\{x_3\}, \{x_1\}, \{x_2\}]$, $[\{x_3\}, \{x_1, x_2\}]$

Type 1: two partitions where one set is a 2-element set containing x_3: $[\{x_1, x_3\}, \{x_2\}]$, $[\{x_2, x_3\}, \{x_1\}]$

Type 2: one partition where one set is a 3-element set containing x_3: $\{x_1, x_2, x_3\}$

In general, we can imagine constructing a partition of type i as a two-step process:

Step 1: Select out the i elements of $\{x_1, \ldots, x_{n-1}\}$ to put together with x_n,

Step 2: Choose any partition of the remaining $(n - 1) - i$ elements of $\{x_1, \ldots, x_{n-1}\}$ to put with the set formed in step 1.

There are $\binom{n-1}{i}$ ways to perform step 1 and $P_{(n-1)-i}$ ways to perform step 2. Therefore, by the multiplication rule, there are $\binom{n-1}{i} \cdot P_{(n-1)-i}$ partitions of type i. Because any partition of X is of type i for some $i = 0, 1, 2, \ldots, n - 1$, it follows from the addition rule that the total number of partitions is

$$\binom{n-1}{0} P_{n-1} + \binom{n-1}{1} P_{n-2} + \binom{n-1}{2} P_{n-3} + \cdots + \binom{n-1}{n-1} P_0.$$

33. $S_{5,2} = S_{4,1} + 2S_{4,2} = 1 + 2 \cdot 7 = 15$

36. Proof (by mathematical induction): Let the property $P(n)$ be the equation $S_{n,2} = 2^{n-1} - 1$.

***Show that P(2) is true*:**

We must show that $S_{2,2} = 2^{2-1} - 1$. By Example 9.5.13, $S_{2,2} = 1$, and $2^{2-1} - 1 = 2 - 1 = 1$ also. So $P(2)$ is true.

***Show that for all integers $k \geq 2$, if P(k) is true, then P(k+1) is true*:**

Let k be any integer with $k \geq 2$, and suppose that $S_{k,2} = 2^{k-1} - 1$. *[Inductive hypothesis.]* We must show that $S_{k+1,2} = 2^{(k+1)-1} - 1 = 2^k - 1$. But according to Example 9.5.13, $S_{k+1,2} = S_{k,1} + 2S_{k,2}$ and $S_{k,1} = 1$. So by substitution and the inductive hypothesis,

$$\begin{aligned} S_{k+1,2} &= 1 + 2S_{k,2} = 1 + 2(2^{k-1} - 1) \\ &= 1 + 2^k - 2 = 2^k - 1 \end{aligned}$$

[as was to be shown].

38. *Hint:* Observe that the number of onto functions from $X = \{x_1, x_2, x_3, x_4\}$ to $Y = \{y_1, y_2, y_3\}$ is $S_{4,3} \cdot 3!$ because the construction of an onto function can be thought of as a two-step process where step 1 is to choose a partition of X into three subsets and step 2 is to choose, for each subset of the partition, an element of Y for the elements of the subset to be sent to.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 9.6

1. a. $\binom{5+3-1}{5} = \binom{7}{5} = \frac{7 \cdot 6}{2} = 21.$

b. The three elements of the set are 1, 2 and 3. The 5-combinations are [1, 1, 1, 1, 1], [1, 1, 1, 1, 2], [1, 1, 1, 1, 3], [1, 1, 1, 2, 2], [1, 1, 1, 2, 3], [1, 1, 1, 3, 3], [1, 1, 2, 2, 2], [1, 1, 2, 2, 3], [1, 1, 2, 3, 3], [1, 1, 3, 3, 3], [1, 2, 2, 2, 2], [1, 2, 2, 2, 3], [1, 2, 2, 3, 3], [1, 2, 3, 3, 3], [1, 3, 3, 3, 3], [2, 2, 2, 2, 2], [2, 2, 2, 2, 3], [2, 2, 2, 3, 3], [2, 2, 3, 3, 3], [2, 3, 3, 3, 3], and [3, 3, 3, 3, 3].

2. a. $\binom{4+3-1}{4} = \binom{6}{4} = \frac{6 \cdot 5}{2} = 15$

3. a. $\binom{20+6-1}{20} = \binom{25}{20} = 53{,}130$

b. If at least three are eclairs, then 17 additional pastries are selected from six kinds. The number of selections is $\binom{17+6-1}{17} = \binom{22}{17} = 26{,}334.$

Note: In parts (a) and (b), it is assumed that the selections being counted are unordered.

c. Let T be the set of selections of pastry that may be any one of the six kinds, let $E_{\geq 3}$ be the set of selections containing three or more eclairs, and let $E_{\leq 2}$ be the set of selections containing two or fewer eclairs. Then

$$\begin{aligned} N(E_{\leq 2}) &= N(T) - N(E_{\geq 3}) && \text{because } T = E_{\leq 2} \cup E_{\geq 3} \text{ and} E_{\leq 2} \cap E_{\geq 3} = \emptyset \\ &= 53{,}130 - 26{,}334 && \text{by parts (a) and (b)} \\ &= 26{,}796. \end{aligned}$$

Thus there are 26,796 selections of pastry containing at most two eclairs.

5. The answer equals the number of 4-combinations with repetition allowed that can be formed from a set of n elements. It is

$$\begin{aligned} \binom{4+n-1}{4} &= \binom{n+3}{4} \\ &= \frac{(n+3)(n+2)(n+1)n\cancel{(n-1)!}}{4!\cancel{(n-1)!}} \\ &= \frac{n(n+1)(n+2)(n+3)}{24}. \end{aligned}$$

8. As in Example 9.6.4, the answer is the same as the number of quadruples of integers (i, j, k, m) for which $1 \leq i \leq j \leq k \leq m \leq n$. By exercise 5, this number is $\binom{n+3}{4} = \frac{n(n+1)(n+2)(n+3)}{24}$.

10. Think of the number 20 as divided into 20 individual units and the variables x_1, x_2, and x_3 as three categories into which these units are placed. The number of units in category x_i indicates the value of x_i in a solution of the equation. By Theorem 9.6.1, the number of ways to select 20 objects from the three categories is $\binom{20+3-1}{20} = \binom{22}{20} = \frac{22 \cdot 21}{2} = 231$, so there are 231 nonnegative integer solutions to the equation.

11. The analysis for this exercise is the same as for exercise 10 except that since each $x_i \geq 1$, we can imagine taking 3 of the 20 units, placing one in each category x_1, x_2, and x_3, and then distributing the remaining 17 units among the three categories. The number of ways to do this is $\binom{17+3-1}{17} = \binom{19}{17} = \frac{19 \cdot 18}{2} = 171$, so there are 171 positive integer solutions to the equation.

16. a. Let $L_{\geq 7}$ be the set of selections that include at least seven cans of lemonade. In this case an additional eight cans can be selected from the five types of soft drinks, and so

$$N(L_{\geq 7}) = \binom{8+5-1}{8} = \binom{12}{8} = 495.$$

Let T be the set of selections of cans in which the soft drink may be any one of the five types, and let $L_{\leq 6}$ be the set of selections that contain at most six cans of lemonade. Then

$$\begin{aligned} N(L_{\leq 6}) &= N(T) - N(L_{\geq 7}) && \text{because } T = L_{\leq 6} \cup L_{\geq 7} \text{ and } L_{\leq 6} \cap L_{\geq 7} = \emptyset \\ &= 3{,}876 - 495 && \text{by the above and part (a) of Example 9.6.2} \\ &= 3{,}381. \end{aligned}$$

Thus there are 3,381 selections of fifteen cans of soft drinks that contain at most six cans of lemonade.

b. Let $R_{\leq 5}$ be the set of selections containing at most five cans of root beer, and let $L_{\leq 6}$ be the set of selections containing at most six cans of lemonade. The answer to the question can be represented as $N(R_{\leq 5} \cap L_{\leq 6})$. As in part (a), let T be the set of all the selections of fifteen cans in which the soft drink may be any one of the five types. If you remove all the selections containing at least six cans of root beer or at least seven cans of lemonade from T, then you are left with all the selections containing at most five cans of root beer and at most six cans of lemonade. Thus, in the notation of part (a) and Example 9.6.2, $N(R_{\leq 5} \cap L_{\leq 6}) = N(T) - N(R_{\geq 6} \cup L_{\geq 7})$.

Use the inclusion/exclusion rule as follows to compute $N(R_{\geq 6} \cup L_{\geq 7})$:

$$N(R_{\geq 6} \cup L_{\geq 7}) = N(R_{\geq 6}) + N(L_{\geq 7}) - N(R_{\geq 6} \cap L_{\geq 7}).$$

To find $N(R_{\geq 6} \cap L_{\geq 7})$, observe that if at least six cans of root beer and at least seven cans of lemonade are selected, then at most two additional cans of soft drink can be chosen from the other three types to make up the total of fifteen cans. A selection of two such cans can be represented by a string of $2 \times$'s and $3|$'s, and a selection of one such can can be represented by a string of $1\times$ and $3|$'s. Hence

$$\begin{aligned} N(R_{\geq 6} \cap L_{\geq 7}) &= \binom{2+3-1}{2} = \binom{1+3-1}{1} \\ &= \binom{4}{2} + \binom{3}{1} = 6 + 3 = 9. \end{aligned}$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

It follows that

$$\begin{aligned} N(R_{\geq 6} \cup L_{\geq 7}) &= N(R_{\geq 6}) + N(L_{\geq 7}) - N(R_{\geq 6} \cap L_{\geq 7}) && \text{by the inclusion/exclusion rule} \\ &= 715 + 495 - 15 && \text{by part (a), the computation above, and part (b) of Example 9.6.2} \\ &= 1{,}201. \end{aligned}$$

Putting together the information from earlier in the solution gives that

$$\begin{aligned} N(R_{\leq 5} \cap L_{\leq 6}) &= N(T) - N(R_{\geq 6} \cup L_{\geq 7}) \\ &= 3{,}876 - 1{,}201 = 2{,}675. \end{aligned}$$

Thus there are 2,681 selections of fifteen soft drinks that contain at most five cans of root beer and at most six cans of lemonade.

17. *Hints*: **a.** The answer is 10,295,472. **b.** See the solution to part (c) of Example 9.6.2. The answer is 9,949,368. **c.** The answer is 9,111,432.

d. Let T denote the set of all the selections of thirty balloons, let $R_{\leq 12}$ denote the set of selections containing at most twelve red balloons, let $B_{\leq 8}$ denote the set of selections containing at most eight blue balloons, let $R_{\geq 13}$ denote the set of selections containing at least thirteen red balloons, and let $B_{\geq 9}$ denote the set of selections containing at least nine blue balloons.. Then the answer to the question can be represented as $N(R_{\leq 12} \cap B_{\leq 8})$. Out of the total of all the balloon selections, if you remove the selections containing at least thirteen red or at least nine blue balloons, then you are left with the selections containing at most twelve red and at most eight blue balloons. Thus $N(R_{\leq 12} \cap B_{\leq 8}) = N(T) - N(R_{\geq 13} \cup B_{\geq 9})$. Compute $N(R_{\geq 13} \cap B_{\geq 9})$, and use the inclusion/exclusion rule to find $N(R_{\geq 13} \cup B_{\geq 9})$.

19. *Hints*: The answers are **a.** 51,128 **b.** 46,761

Section 9.7

1. $\binom{n}{0} = \dfrac{n!}{0!(n-0)!} = \dfrac{\cancel{n!}}{1 \cdot \cancel{n!}} = 1$

3. $\binom{n}{2} = \dfrac{n!}{(n-2)! \cdot 2!} = \dfrac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)!} \cdot 2!} = \dfrac{n(n-1)}{2}$

5. Proof: Suppose n and r are nonnegative integers and $r \leq n$. Then

$$\begin{aligned} \binom{n}{r} &= \frac{n!}{r!(n-r)!} && \text{by Theorem 9.5.1} \\ &= \frac{n!}{(n-(n-r))!(n-r)!} && \text{since } n-(n-r) = n-n+r = r \\ &= \frac{n!}{(n-r)!(n-(n-r))!} && \text{by interchanging the factors in the denominator} \\ &= \binom{n}{n-r} && \text{by Theorem 9.5.1.} \end{aligned}$$

6. *Solution 1*: Apply formula (9.7.2) with $m+k$ in place of n. This is legal because $m+k \geq 1$.

Solution 2:

$$\begin{aligned} \binom{m+k}{m+k-1} &= \frac{(m+k)!}{(m+k-1)![(m+k)-(m+k-1)]!} \\ &= \frac{(m+k) \cdot (m+k-1)!}{(m+k-1)!(m+k-m-k+1)!} \\ &= \frac{(m+k) \cdot (m+k-1)!}{(m+k-1)! \cdot 1!} = m+k \end{aligned}$$

10. **a.** $\binom{6}{2} = \binom{5}{2} + \binom{5}{1} = 10 + 5 = 15,$

$\binom{6}{3} = \binom{5}{3} + \binom{5}{2} = 10 + 10 = 20$

b. $\binom{6}{4} = \binom{5}{4} + \binom{5}{3} = 5 + 10 = 15,$

$\binom{6}{5} = \binom{5}{5} + \binom{5}{4} = 1 + 5 = 6,$

$\binom{7}{3} = \binom{6}{3} + \binom{6}{2} = 20 + 15 = 35,$

$\binom{7}{4} = \binom{6}{4} + \binom{6}{3} = 15 + 20 = 35,$

$\binom{7}{5} = \binom{6}{5} + \binom{6}{4} = 6 + 15 = 21$

c. Row for $n = 7$: 1 7 21 35 35 21 7 1

13. Proof by mathematical induction: Let the property $P(n)$ be the formula

$$\sum_{i=2}^{n+1} \binom{i}{2} = \binom{n+2}{3}. \quad \leftarrow P(n)$$

Show that P(1) is true:

To prove $P(1)$ we must show that

$$\sum_{i=2}^{1+1} \binom{i}{2} = \binom{1+2}{3}. \quad \leftarrow P(1)$$

But

$$\sum_{i=2}^{1+1} \binom{i}{2} = \sum_{i=2}^{2} \binom{i}{2} = \binom{2}{2} = 1 = \binom{3}{3} = \binom{1+2}{3},$$

so $P(1)$ is true.

Show that for all integers k ≥ 1, P(k) is true, then P(k+1) is true:

Let k be any integer with $k \geq 1$, and suppose that

$$\sum_{i=2}^{k+1} \binom{i}{2} = \binom{k+2}{3} \quad \begin{array}{l} \leftarrow P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\sum_{i=2}^{(k+1)+1} \binom{i}{2} = \binom{(k+1)+2}{3},$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

or, equivalently,

$$\sum_{i=2}^{k+2} \binom{i}{2} = \binom{k+3}{3}. \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}\sum_{i=2}^{k+2} \binom{i}{2} &= \sum_{i=1}^{k+1} \binom{i}{2} + \binom{k+2}{2} && \text{by writing the last term separately}\\ &= \binom{k+2}{3} + \binom{k+2}{2} && \text{by inductive hypothesis}\\ &= \binom{(k+2)+1}{3} && \text{by Pascal's formula}\\ &= \binom{k+3}{3},\end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as was to be shown]. [Since we have proved the basis step and the inductive step, we conclude that $P(n)$ is true for all $n \geq 1$.]*

14. *Hint:* Use the results of exercises 3 and 13.

17. *Hint:* This follows by letting $m = n = r$ in exercise 16 and using the result of Example 9.7.2.

19. $1 + 7x + \binom{7}{2}x^2 + \binom{7}{3}x^3 + \binom{7}{4}x^4 + \binom{7}{5}x^5 + \binom{7}{6}x^6 + x^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

21. $1 + 6(-x) + \binom{6}{2}(-x)^2 + \binom{6}{3}(-x)^3 + \binom{6}{4}(-x)^4 + \binom{6}{5}(-x)^5 + (-x)^6 = 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

23.
$$\begin{aligned}(p-2q)^4 &= \sum_{k=0}^{4} \binom{4}{k} p^{4-k}(-2q)^k\\ &= \binom{4}{0} p^4(-2q)^0 + \binom{4}{1} p^3(-2q)^1\\ &\quad + \binom{4}{2} p^2(-2q)^2 + \binom{4}{3} p^1(-2q)^3\\ &\quad + \binom{4}{4} p^0(-2q)^4\\ &= p^4 - 8p^3q + 24p^2q^2 - 32pq^3 + 16q^4\end{aligned}$$

25.
$$\begin{aligned}\left(x + \frac{1}{x}\right)^5 &= \sum_{k=0}^{5} \binom{5}{k} x^{5-k} \left(\frac{1}{x}\right)^k\\ &= \binom{5}{0} x^5 \left(\frac{1}{x}\right)^0 + \binom{5}{1} x^4 \left(\frac{1}{x}\right)^1\\ &\quad + \binom{5}{2} x^3 \left(\frac{1}{x}\right)^2 + \binom{5}{3} x^2 \left(\frac{1}{x}\right)^3\\ &\quad + \binom{5}{4} x^1 \left(\frac{1}{x}\right)^4 + \binom{5}{5} x^0 \left(\frac{1}{x}\right)^5\\ &= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}\end{aligned}$$

29. The term is $\binom{9}{3}x^6y^3 = 84x^6y^3$, so the coefficient is 84.

31. The term is $\binom{12}{7}a^5(-2b)^7 = 792a^5(-128)b^7 = -101{,}376a^5b^7$, so the coefficient is $-101{,}376$.

33. The term is $\binom{15}{8}(3p^2)^8(-2q)^7 = \binom{15}{8}3^8(-2)^7p^{16}q^7$, so the coefficient is $\binom{15}{8}3^8(-2)^7 = -5{,}404{,}164{,}480$.

36. Proof: Let $a = 1$, let $b = -1$, and let n be a positive integer. Substitute into the binomial theorem to obtain

$$\begin{aligned}(1 + (-1))^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot (-1)^k\\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \qquad \text{since } 1^{n-k} = 1.\end{aligned}$$

But $(1 + (-1))^n = 0^n = 0$, so

$$\begin{aligned}0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k\\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}.\end{aligned}$$

37. *Hint:* $3 = 1 + 2$

38. Proof: Let m be any integer with $m \geq 0$, and apply the binomial theorem with $a = 2$ and $b = -1$. The result is

$$\begin{aligned}1 = 1^m = (2 + (-1))^m &= \sum_{i=0}^{m} \binom{m}{i} 2^{m-i}(-1)^i\\ &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} 2^{m-i}.\end{aligned}$$

41. *Hint:* Apply the binomial theorem with $a = -\frac{1}{2}$ and $b = 1$, and analyze the resulting equation when n is even and when n is odd.

43. $\sum_{k=0}^{n} \binom{n}{k} 5^k = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}5^k = (1+5)^n = 6^n$

45. $\sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i}x^i = (1+x)^n$

47. $\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} x^j = \sum_{j=0}^{2n} \binom{2n}{j} 1^{2n-j}(-x)^j = (1-x)^{2n}$

51.
$$\begin{aligned}\sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{1}{2^i} &= \sum_{i=0}^{m} \binom{m}{i} 1^{m-i} \left(-\frac{1}{2}\right)^i\\ &= \left(1 - \frac{1}{2}\right)^m = \frac{1}{2^m}\end{aligned}$$

53. $\sum_{i=0}^{n} (-1)^i \binom{n}{i} 5^{n-i}2^i = \sum_{i=0}^{n} \binom{n}{i} 5^{n-i}(-2)^i = (5-2)^n = 3^n$

55. b. $n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} kx^{k-1}$.

[The term corresponding to $k = 0$ is zero because $\frac{d}{dx}(x^0) = 0$.]

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

c. (i) Substitute $x = 1$ in part (b) above to obtain

$$n(1+1)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k \cdot 1^{k-1} = \sum_{k=1}^{n} \binom{n}{k} k$$
$$= \binom{n}{1} \cdot 1 + \binom{n}{2} \cdot 2 + \binom{n}{3} \cdot 3 + \cdots + \binom{n}{n} n.$$

Dividing both sides by n and simplifying gives

$$2^{n-1} = \frac{1}{n}\left[\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}\right].$$

Section 9.8

1. By probability axiom 2, $P(\emptyset) = 0$.

2. a. By probability axiom 3, $P(A \cup B) = P(A) + P(B) = 0.3 + 0.5 = 0.8$.

b. Because $A \cup B \cup C = S$, $C = S - (A \cup B)$. Thus, by the formula for the probability of the complement of an event, $P(C) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.8 = 0.2$.

4. By the formula for the probability of a general union of two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.7 - 0.6 = 0.9$.

7. a. $P(A \cup B) = 0.4 + 0.3 = 0.7$

b. $P(C) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$

c. $P(A \cup C) = 0.4 + 0.3 = 0.7$

d. $P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6$

e. $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$

f. $P(A^c \cup B^c) = P((A \cap B)^c) = P(\emptyset^c) = P(S) = 1$

9. a. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.2 = 0.7$

d. $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$

11. *Hint:* $V = (U \cup (V - U))$

12. *Hint:* Use the fact that for all sets U and V, $U \cup (V - U) = U \cup V$.

13. *Hint:* $(A_1 \cup A_2 \cup \cdots \cup A_k) \cap A_{k+1} = \emptyset$ and $A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1} = (A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}$.

14. *Solution 1:* The net gain of the grand prize winner is \$2,000,000 − \$2 = \$1,999,998. Each of the 10,000 second prize winners has a net gain of \$20 − \$2 = \$18, and each of the 50,000 third prize winners has a net gain of \$4 − \$2 = \$2. The number of people who do not win anything is 1,500,000 − 1 − 10,000 −50,000 = 1,439,999, and each of these people has a net loss of \$2. Because all of the 1,500,000 tickets have an equal chance of winning a prize, the expected gain or loss of a ticket is

$$\frac{1}{1500000}(\$1{,}999{,}998 \cdot 1 + \$18 \cdot 10000$$
$$+ \$2 \cdot 50000 + (-\$2) \cdot 1{,}439{,}999) = -\$0.40.$$

Solution 2: The total income to the lottery organizer is \$2 (per ticket) · 1,500,000 (tickets) = \$3,000,000. The payout the lottery organizer must make is \$2,000,000 + (\$20)(10,000) + (\$4)(50,000) = \$2,400,000, so the net gain to the lottery organizer is \$600,000, which amounts to $\frac{\$600{,}000}{1{,}500{,}000} = \$0.40$ per ticket. Thus the expected net loss to a purchaser of a ticket is \$0.40.

16. Let 2_1 and 2_2 denote the two balls with the number 2, and let 5 and 6 denote the other two balls. There are $\binom{6}{2} = 4$ subsets of 2 balls that can be chosen from the urn. The following table shows the sums of the numbers on the balls in each set and the corresponding probabilities:

Subset	Sum s	Probability that the sum = s
$\{2_1, 2_2\}$	4	1/6
$\{2_1, 5\}, \{2_2, 5\}$	7	2/6
$\{2_1, 6\}\{2_2, 6\}$	8	2/6
$\{5, 6\}$	11	1/6

So the expected value is $4 \cdot \frac{1}{6} + 7 \cdot \frac{2}{6} + 8 \cdot \frac{2}{6} + 11 \cdot \frac{1}{6} = 7.5$.

19. The following table displays the sum of the numbers showing face up on the dice:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Each cell in the table represents an outcome whose probability is $\frac{1}{36}$. Thus the expected value of the sum is

$$2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right)$$
$$+ 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) = \frac{252}{36} = 7.$$

20. *Hint:* The answer is about 7.7 cents.

22. *Hint:* The answer is 1.875.

23. *Hint:* To derive P_{20}, use the distinct roots theorem from Section 5.8. The answer is $P_{20} = \frac{5^{300} - 5^{20}}{5^{300} - 1} \cong 1$.

Section 9.9

1. $P(B) = \dfrac{P(A \cap B)}{P(A \mid B)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$

3. *Hint:* The answer is 60%.

4. a. Proof: Suppose S is any sample space and A and B are any events in S such that $P(B) \neq 0$. Note that

(1) $A \cup A^c = S$ by the complement law for $\cup$.

(2) $B \cap S = B$ by the identity law for $\cap$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

(3) $B \cap (A \cup A^c) = (A \cap B) \cup (A^c \cap B)$ by the distributive law and commutative laws for sets.

(4) $(A \cap B) \cap (A^c \cap B) = \emptyset$ by the complement law for $\cap$ and the commutative and associative laws for sets.

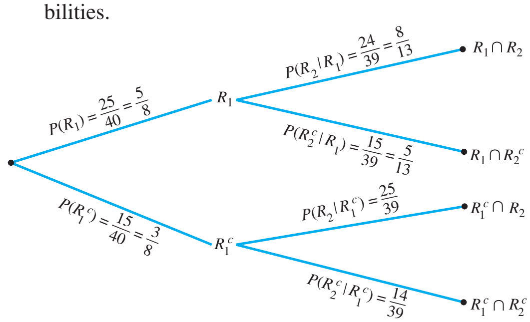
Thus $B = (A \cap B) \cup (A^c \cap B)$, and, by probability axiom 3, $P(B) = P(A \cap B) + P(A^c \cap B)$. Therefore, $P(A^c \cap B) = P(B) - P(A \cap B)$. By definition of conditional probability, it follows that

$$P(A^c \mid B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$$
$$= 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A|B).$$

5. *Hints:* (1) $A = (A \cap B) \cup (A \cap B^c)$.

(2) The answer is $P(A \mid B^c) = \dfrac{P(A) - P(A \mid B)P(B)}{1 - P(B)}$.

6. a. Let R_1 be the probability that the first ball is red, and let R_2 be the probability that the second ball is red. Then R_1^c is the probability that the first ball is not red, and R_2^c is the probability that the second ball is not red. The tree diagram shows the various relations among the probabilities.



Then

$$P(R_1 \cap R_2) = P(R_2 \mid R_1) \cdot P(R_1)$$
$$= \frac{8}{13} \cdot \frac{5}{8} = \frac{5}{13} \cong 38.5\%,$$
$$P(R_1 \cap R_2^c) = P(R_2^c \mid R_1) \cdot P(R_1)$$
$$= \frac{5}{13} \cdot \frac{5}{8} = \frac{25}{104} \cong 24\%,$$
$$P(R_1^c \cap R_2) = P(R_2 \mid R_1^c) \cdot P(R_1^c)$$
$$= \frac{25}{39} \cdot \frac{3}{8} = \frac{25}{104} \cong 24\%,$$
$$P(R_1^c \cap R_2^c) = P(R_2^c \mid R_1^c) \cdot P(R_1^c)$$
$$= \frac{14}{39} \cdot \frac{3}{8} = \frac{14}{104} \cong 13.5\%$$

So the probability that both balls are red is 5/13, the probability that the first ball is red and the second is not is 25/104, the probability that the first ball is not red and the second ball is red is 25/104, and the probability that neither ball is red is 14/104.

b. Note that

$$R_2 = (R_2 \cap R_1) \cup (R_2 \cap R_1^c) \quad \text{and}$$
$$(R_2 \cap R_1) \cap (R_2 \cap R_1^c) = \emptyset.$$

Thus the probability that the second ball is red is

$$P(R_2) = P(R_2 \cap R_1) + P(R_2 \cap R_1^c)$$
$$= \frac{5}{13} + \frac{25}{104} = \frac{65}{104} \cong 62.5\%.$$

c. If exactly one ball is red, then either the first ball is red and the second is not or the first ball is not red and the second is red, and these possibilities are mutually exclusive. Thus

$$P(\text{exactly one ball is red}) = P(R_1 \cap R_2^c) + P(R_1^c \cap R_2)$$
$$= \frac{25}{104} + \frac{25}{104} = \frac{50}{104}$$
$$= \frac{25}{52} \cong 48.1\%.$$

The probability that both balls are red is $P(R_1 \cap R_2) = \frac{5}{13} \cong 38.5\%$. Then

$$P(\text{at least one ball is red}) = P(\text{exactly one ball is red})$$
$$+ P(\text{both balls are red})$$
$$= \frac{25}{52} + \frac{5}{13}$$
$$= \frac{45}{52} \cong 86.5\%.$$

8. a. Let W_1 be the event that a woman is chosen on the first draw,
W_2 be the event that a woman is chosen on the second draw,
M_1 be the event that a man is chosen on the first draw,
M_2 be the event that a man is chosen on the second draw.
Then $P(W_1) = \frac{3}{10}$ and $P(W_2 \mid W_1) = \frac{2}{9}$, and thus
$P(W_1 \cap W_2) = P(W_2 \mid W_1)P(W_1) = \frac{2}{9} \cdot \frac{3}{10} = \frac{1}{15} = 6\frac{2}{3}\%$.

c. *Hint:* The answer is $\frac{7}{15} = 46\frac{2}{3}\ \%$.

9. *Hint:* Use the facts that $P(B_k \mid A) = \dfrac{P(B_k \cap A)}{P(A)}$ and that $(A \cap B_1) \cup (A \cap B_2) = A$.

11. a. Let U_1 be the event that the first urn is chosen, U_2 the event that the second urn is chosen, and B the event that the chosen ball is blue. Then

$$P(B \mid U_1) = \frac{12}{19} \quad \text{and} \quad P(B \mid U_2) = \frac{8}{27}.$$
$$P(B \cap U_1) = P(B \mid U_1)P(U_1) = \frac{12}{19} \cdot \frac{1}{2} = \frac{12}{38}.$$

Also

$$P(A \cap U_2) = P(B \mid U_2)P(U_2) = \frac{8}{27} \cdot \frac{1}{2} = \frac{8}{54}.$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Now B is the disjoint union of $B \cap U_1$ and $B \cap U_2$. So

$$P(B) = P(B \cap U_1) + P(B \cap U_2) = \frac{12}{38} + \frac{8}{54} \cong 46.4\%.$$

Thus the probability that the chosen ball is blue is approximately 46.4%.

b. Given that the chosen ball is blue, the probability that it came from the first urn is $P(U_1 \mid B)$. By Bayes' theorem and the computations in part (a),

$$\begin{aligned} P(U_1 \mid B) &= \frac{P(B \mid U_1)P(U_1)}{P(B \mid U_1)P(U_1) + P(B \mid U_2)P(U_2)} \\ &= \frac{(12/19)(0.5)}{(12/19)(0.5) + (8/27)(0.5)} \cong 68.1\% \end{aligned}$$

13. *Hint:* The answers to parts (a) and (b) are approximately 52.9% and 54.0%, respectively.

14. Let A be the event that a randomly chosen person tests positive for drugs, let B_1 be the event that a randomly chosen person uses drugs, and let B_2 be the event that a randomly chosen person does not use drugs. Then A^c is the event that a randomly chosen person does not test positive for drugs, and $P(B_1) = 0.04$, $P(B_2) = 0.96$, $P(A \mid B_2) = 0.03$, and $P(A^c \mid B_1) = 0.02$. Hence $P(A \mid B_1) = 0.97$ and $P(A^c \mid B_2) = 0.98$.

a. $$\begin{aligned} P(B_1 \mid A) &= \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)} \\ &= \frac{(0.97)(0.04)}{(0.97)(0.04) + (0.03)(0.96)} \cong 57.4\% \end{aligned}$$

b. $$\begin{aligned} P(B_2 \mid A^c) &= \frac{P(A^c \mid B_2)P(B_2)}{P(A^c \mid B_1)P(B_1) + P(A^c \mid B_2)P(B_2)} \\ &= \frac{(0.98)(0.96)}{(0.02)(0.04) + (0.98)(0.96)} \cong 99.9\% \end{aligned}$$

16. *Hint:* The answers to parts (a) and (b) are 11.25% and $21\frac{1}{3}\%$, respectively.

17. Proof: Suppose A and B are events in a sample space S, and $P(A|B) = P(A) \neq 0$. Then

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} \\ &= \frac{P(A)P(B)}{P(A)} = P(B). \end{aligned}$$

19. As in Example 6.9.1, the sample space is the set of all 36 outcomes obtained from rolling the two dice and noting the numbers showing face up on each. Let A be the event that the number on the blue die is 2 and B the event that the number on the gray die is 4 or 5. Then

$$A = \{21, 22, 23, 24, 25, 26\},$$
$$B = \{14, 24, 34, 44, 54, 64, 15, 25, 35, 45, 55, 65\}, \quad \text{and}$$
$$A \cap B = \{24, 25\}.$$

Since the dice are fair (so all outcomes are equally likely), $P(A) = \frac{6}{36}$, $P(B) = \frac{12}{36}$ and $P(A \cap B) = \frac{2}{36}$. By definition of conditional probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{12}{36}} = \frac{1}{6} \quad \text{and}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}.$$

But $P(A) = \frac{6}{36} = \frac{1}{6}$ and $P(B) = \frac{12}{36} = \frac{1}{3}$. Hence $P(A|B) = P(A)$ and $P(B \mid A) = P(B)$.

23. Let A be the event that the student answers the first question correctly, and let B be the event that the student answers the second answer correctly. Because two choices can be eliminated on the first question, $P(A) = \frac{1}{3}$, and because no choices can be eliminated on the second question, $P(B) = \frac{1}{5}$. Thus $P(A^c) = \frac{2}{3}$ and $P(B^c) = \frac{4}{5}$.

a. *Hint:* The probability that the student answers both questions correctly is

$$P(A \cap B) = P(A)P(B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15} = 6\frac{2}{3}\%.$$

b. The probability that the student answers exactly one question correctly is

$$\begin{aligned} &P((A \cap B^c) \cup (A^c \cap B)) \\ &\quad = P(A \cap B^c) + P(A^c \cap B) \\ &\quad = P(A)P(B^c) + P(A^c)P(B) \\ &\quad = \tfrac{1}{3} \cdot \tfrac{4}{5} + \tfrac{2}{3} \cdot \tfrac{1}{5} = \tfrac{6}{15} = \tfrac{2}{5} = 40\%. \end{aligned}$$

c. One solution is to say that the probability that the student answers both questions incorrectly is $P(A^c \cap B^c)$, and $P(A^c \cap B^c) = P(A^c)P(B^c)$ by the result of exercise 22. Thus the answer is

$$P(A^c)P(B^c) = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15} = 53\frac{1}{3}\%.$$

Another solution uses the fact that the event that the student answers both questions incorrectly is the complement of the event that the student answers at least one question correctly. Thus, by the results of parts (a) and (b), the answer is $1 - \left(\frac{1}{15} + \frac{2}{5}\right) = \frac{8}{15} = 53\frac{1}{3}\%$.

25. Let H_i be the event that the result of toss i is heads, and let T_i be the event that the result of toss i is tails. Then $P(H_i) = 0.7$ and $P(T_i) = 0.3$ for $i = 1, 2$.

b. The probability of obtaining exactly one head is

$$\begin{aligned} &P((H_1 \cap T_2) \cup (T_1 \cap H_2)) \\ &\quad = P(H_1 \cap T_2) + P(T_1 \cap H_2) \\ &\quad = P(H_1)P(T_2) + P(T_1)P(H_2) \\ &\quad = (0.7)(0.3) + (0.3)(0.7) = 42\%. \end{aligned}$$

27. *Hint:* The answer is $\frac{1}{2}$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

28. a. $P(\text{seven heads})$

$$= \begin{bmatrix}\text{the number of different} \\ \text{ways seven heads can} \\ \text{be obtained in ten tosses}\end{bmatrix}(0.7)^7(0.3)^3$$
$$= 120(0.7)^7(0.3)^3 \cong 0.267 = 26.7\%.$$

29. a. $P(\text{none is defective})$

$$= \begin{bmatrix}\text{the number of different} \\ \text{ways of having 0 defective} \\ \text{items in the sample of 10}\end{bmatrix}(0.03)^0(0.97)^{10}$$
$$= 1 \cdot (0.3.)^0(0.97)^{10} \cong 0.737 = 73.7\%$$

30. b. The probability that a woman will have at least one false positive result over a period of ten years is $1 - (0.96)^{10} \cong 33.5\%$.

31. a. $P(\text{none is male}) \cong 1.3\%$

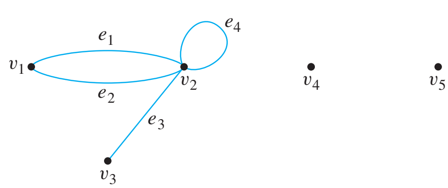
b. $P(\text{at least one is male}) = 1 - P(\text{none is male}) \cong 1 - 0.013 = 98.7\%$

Section 10.1

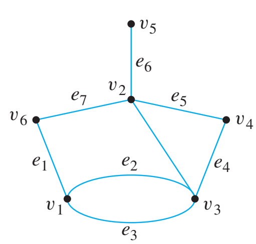
1. $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3\}$
Edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

3.



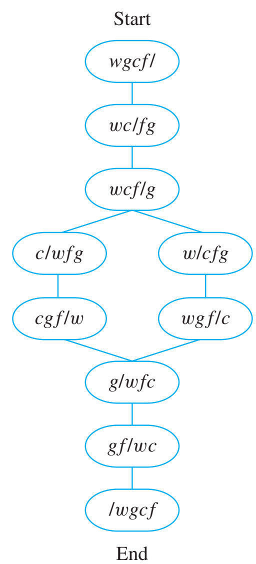
5. Imagine that the edges are strings and the vertices are knots. You can pick up the left-hand figure and lay it down again to form the right-hand figure as shown below.



8. (i) e_1, e_2, and e_3 are incident on v_1.
(ii) v_1, v_2, and v_3 are adjacent to v_3.
(iii) e_2, e_8, e_9, and e_3 are adjacent to e_1.
(iv) Loops are e_6 and e_7.
(v) e_8 and e_9 are parallel; e_4 and e_5 are parallel.
(vi) v_6 is an isolated vertex.
(vii) degree of $v_3 = 5$
(viii) total degree $= 20$

10. a. Yes. According to the graph, *Sports Illustrated* is an instance of a sports magazine, a sports magazine is a periodical, and a periodical contains printed writing.

12. To solve this puzzle using a graph, introduce a notation in which, for example, wc/fg means that the wolf and the cabbage are on the left bank of the river and the ferryman and the goat are on the right bank. Then draw those arrangements of wolf, cabbage, goat, and ferryman that can be reached from the initial arrangement $(wgcf/)$ and that are not arrangements to be avoided (such as (wg/fc)). At each stage ask yourself, "Where can I go from here?" and draw lines or arrows pointing to those arrangements. This method gives the graph shown at the top of the next column.



Examination of the diagram shows the solutions

$$(wgcf/) \to (wc/gf) \to (wcf/g) \to (w/gcf) \to (wgf/c) \to (g/wcf) \to (gf/wc) \to (/wgcf)$$

and

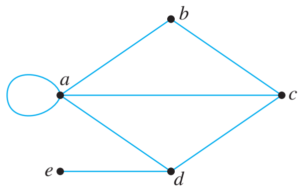
$$(wgcf/) \to (wc/gf) \to (wcf/g) \to (c/wgf) \to (gcf/w) \to (g/wcf) \to (gf/wc) \to (/wgcf)$$

14. *Hint:* The answer is yes. Represent possible amounts of water in jugs A and B by ordered pairs. For instance, the ordered pair $(1, 3)$ would indicate that there is one quart of water in jug A and three quarts in jug B. Starting with $(0, 0)$, draw arrows from one ordered pair to another if it is possible to go from the situation represented by one pair to that represented by the other by either filling a jug, emptying a jug, or transferring water from one jug to another. You need only draw arrows from states that have arrows pointing to them; the other states cannot be reached. Then find a directed path (sequence of directed edges) from the initial state $(0, 0)$ to a final state $(1, 0)$ or $(0, 1)$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

15. The total degree of the graph is $0 + 2 + 2 + 3 + 9 = 16$, so by Theorem 10.1.1, the number of edges is $16/2 = 8$.

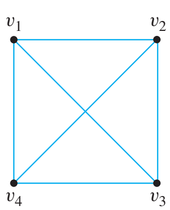
17. One such graph is



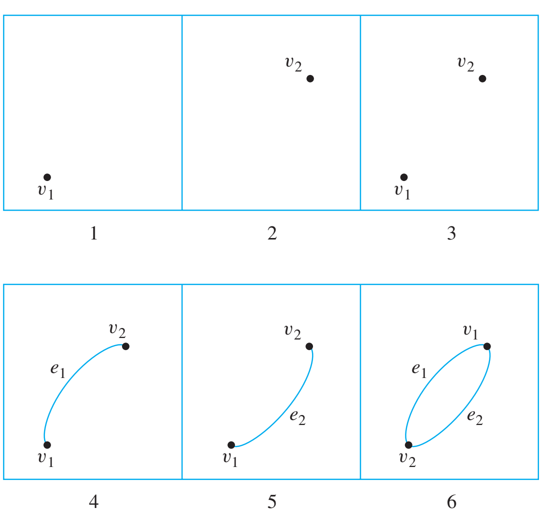
18. If there were a graph with four vertices of degrees 1, 2, 3, and 3, then its total degree would be 9, which is odd. But by Corollary 10.1.2, the total degree of the graph must be even. *[This is a contradiction.]* Hence there is no such graph. (Alternatively, if there were such a graph, it would have an odd number of vertices of odd degree. But by Proposition 10.1.3 this is impossible.)

21. Suppose there were a simple graph with four vertices of degrees 1, 2, 3, and 4. Then the vertex of degree 4 would have to be connected by edges to four distinct vertices other than itself because of the assumption that the graph is simple (and hence has no loops or parallel edges.) This contradicts the assumption that the graph has four vertices in total. Hence there is no simple graph with four vertices of degrees 1, 2, 3, and 4.

24.



26. **a.** The nonempty subgraphs are as follows:



27. **a.** Suppose that, in a group of 15 people, each person had exactly three friends. Then you could draw a graph representing each person by a vertex and connecting two vertices by an edge if the corresponding people were friends. But such a graph would have 15 vertices, each of degree 3, for a total degree of 45. This would contradict the fact that the total degree of any graph is even. Hence the supposition must be false, and in a group of 15 people it is not possible for each to have exactly three friends.

31. We give two proofs for the following statement, one less formal and the other more formal.

> For all integers $n \geq 0$, if $a_1, a_2, a_3, \ldots, a_{2n+1}$ are odd integers, then $\sum_{i=1}^{2n+1} a_i$ is odd.

Proof 1 (by mathematical induction): It is certainly true that the "sum" of one odd integer is odd. Suppose that for a certain positive odd integer r, the sum of r odd integers is odd. We must show that the sum of $r + 2$ odd integers is odd (because $r + 2$ is the next odd integer after r). But any sum of $r + 2$ odd integers equals a sum of r odd integers (which is odd by inductive hypothesis) plus a sum of two more odd integers (which is even). Thus the total sum is an odd integer plus an even integer, which is odd. *[This is what was to be shown.]*

Proof 2 (by mathematical induction): Let the property $P(n)$ be the following sentence: "If $a_1, a_2, a_3, \ldots, a_{2n+1}$ are odd integers, then $\sum_{i=1}^{2n+1} a_i$ is odd.

***Show that $P(0)$ is true*:**

Suppose a_1 is an odd integer. Then $\sum_{i=1}^{2\cdot 0+1} a_i = \sum_{i=1}^{1} a_i = a_1$, which is odd.

***Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is true*:**

Let k be an integer with $k \geq 0$, and suppose that

$$\text{if } a_1, a_2, \ldots, a_{2k+1} \text{ are odd integers, then } \sum_{i=1}^{2k+1} a_i \text{ is odd.}$$

[This is the inductive hypothesis $P(k)$.]

Suppose $a_1, a_2, a_3, \ldots, a_{2(k+1)+1}$ are odd integers. *[We must show $P(k + 1)$, namely that $\sum_{i=1}^{2(k+1)+1} a_i$ is odd, or, equivalently, that $\sum_{i=1}^{2k+3} a_i$ is odd.]* But

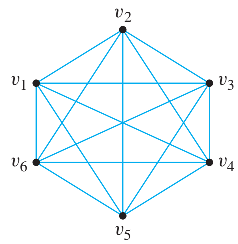
$$\sum_{i=1}^{2k+3} a_i = \sum_{i=1}^{2k+1} a_i + (a_{2k+2} + a_{2k+3}).$$

Since the sum of any two odd integers is even, $a_{2k+2} + a_{2k+3}$ is even, and, by inductive hypothesis, $\sum_{i=1}^{2k+1} a_i$ is odd. Therefore, $\sum_{i=1}^{2k+3} a_i$ is the sum of an odd integer and an even integer, which is odd. *[This is what was to be shown.]*

32. *Hint:* Use proof by contradiction.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

33. a. K_6:



b. A proof of this fact was given in Section 5.6 using recursion. Try to find a different proof.

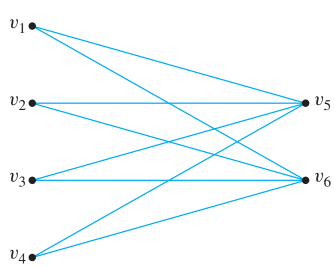
Hint for Proof 1: There are as many edges in K_n as there are subsets of two vertices (the endpoints) that can be chosen from a set of n vertices.

Hint for Proof 2: Use mathematical induction. A complete graph on $k+1$ vertices can be obtained from a complete graph on k vertices by adding one vertex and connecting this vertex by k edges to each of the other vertices.

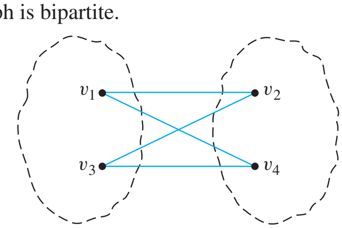
Hint for Proof 3: Use the fact that the number of edges of a graph is half the total degree. What is the degree of each vertex of K_n?

35. Suppose G is a simple graph with n vertices and $2n$ edges where n is a positive integer. By exercise 34, its number of edges cannot exceed $\frac{n(n-1)}{2}$. Thus $2n \le \frac{n(n-1)}{2}$, or $4n \le n^2 - n$. Equivalently, $n^2 - 5n \ge 0$, or $n(n-5) \ge 0$. This implies that $n \ge 5$ since $n > 0$. Hence a simple graph with twice as many edges as vertices must have at least five vertices. But a complete graph with five vertices has $\frac{5(5-1)}{2} = 10$ edges and $10 = 2 \cdot 5$. Consequently, the answer to the question is yes because K_5 is a graph with twice as many edges as vertices.

36. a. $K_{4,2}$:

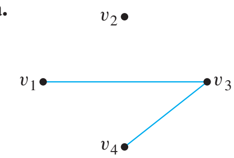


37. a. This graph is bipartite.

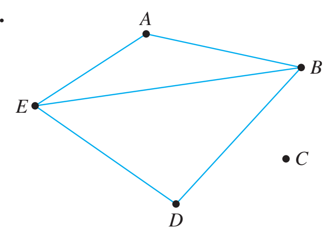


b. Suppose this graph is bipartite. Then the vertex set can be partitioned into two mutually disjoint subsets such that vertices in each subset are connected by edges only to vertices in the other subset and not to vertices in the same subset. Now v_1 is in one subset of the partition, say V_1. Since v_1 is connected by edges to v_2 and v_3, both v_2 and v_3 must be in the other subset, V_2. But v_2 and v_3 are connected by an edge to each other. This contradicts the fact that no vertices in V_2 are connected by edges to other vertices in V_2. Hence the supposition is false, and so the graph is not bipartite.

39. a.



41. b.

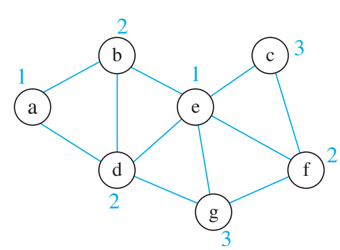


42. *Hint:* Consider the graph obtained by taking the vertices and edges of G plus all the edges of G'. Use exercise 33(b).

44. c. *Hint:* Suppose there were a simple graph with n vertices (where $n \ge 2$) each of which had a different degree. Then no vertex could have degree more than $n-1$ (why?), so the degrees of the n vertices must be $0, 1, 2, \ldots, n-1$ (why?). This is impossible (why?).

45. *Hint:* Use the result of exercise 44(c).

46.



Vertex e has maximal degree, so color it with color #1. Vertex a does not share an edge with e, and so color #1 may also be used for it. From the remaining uncolored vertices, all of d, g, and f have maximal degree. Choose any one of them, say d, and use color #2 for it. Observe that vertices b, c, and f do not share an edge with d, but c and f share an edge with each other, which means that color #2 may be used for only one of c or f. So color b with color #2, and choose to color f with color #2 because the degree of f is greater than the degree of c. From the remaining uncolored vertices, g has maximal degree, so color it with color #3. Then observe that because g does not share an edge with c, color #3 may also be used for c. At this point, all vertices have been colored.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

47. *Hint:* There are two solutions:
(1) Time 1: hiring, library
Time 2: personnel, undergraduate education, colloquium
Time 3: graduate education
(2) Time 1: hiring, library
Time 2: graduate education, colloquium
Time 3: personnel, undergraduate education

Section 10.2

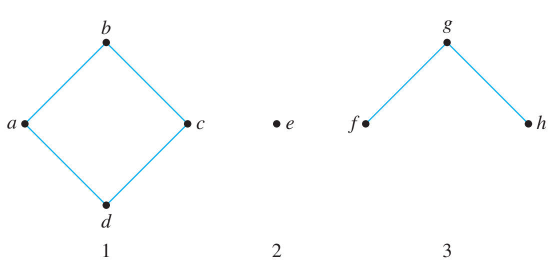
1. **a.** trail (no repeated edge), not a path (repeated vertex—v_1), not a circuit
b. walk, not a trail (has repeated edge—e_9), not a circuit
c. closed walk (starts and ends at the same vertex), trail (no repeated edge since no edge), not a path or a circuit (since no edge)
d. circuit, not a simple circuit (repeated vertex, v_4)
e. closed walk (starts and ends at the same vertex but has repeated edges $-\{v_2, v_3\}$ and $\{v_3, v_4\}$)
f. path

3. **a.** No. The notation $v_1 v_2 v_1$ could equally well refer to $v_1 e_1 v_2 e_2 v_1$ or to $v_1 e_2 v_2 e_1 v_1$, which are different walks.

4. **a.** Three (There are three ways to choose the middle edge.)
b. $3! + 3 = 9$ (In addition to the three paths, there are 3! with vertices $v_1, v_2, v_3, v_2, v_3, v_4$. The reason is that from v_2 there are three choices of an edge to go to v_3, then two choices of different edges to go back to v_2, and then one choice of different edge to return to v_3. This makes 3! trails from v_2 to v_3.)
c. Infinitely many (Since a walk may have repeated edges, a walk from v_1 to v_4 may contain an arbitrarily large number of repetitions of edges joining a pair of vertices along the way.)

6. **a.** $\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_3\}$, and $\{v_5, v_3\}$ are all the bridges.

8. **a.** Three connected components.



9. **a.** No. This graph has two vertices of odd degree, whereas all vertices of a graph with an Euler circuit have even degree.

12. One Euler circuit is $e_4 e_5 e_6 e_3 e_2 e_7 e_8 e_1$.

14. One Euler circuit is $iabihbchgcdgfdefi$.

19. There is an Euler path since deg(u) and deg(w) are odd, all other vertices have positive even degree, and the graph is connected. One Euler path is $uv_1v_0v_7uv_2v_3v_4v_2v_6v_4wv_5v_6w$.

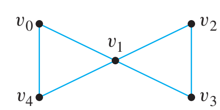
23. $v_0v_7v_1v_2v_3v_4v_5v_6v_0$

25. *Hint:* See the solution to Example 10.2.8.

26. Here is one sequence of reasoning you could use: Call the given graph G, and suppose G has a Hamiltonian circuit. Then G has a subgraph H that satisfies conditions (1)–(4) of Proposition 10.2.6. Since the degree of b in G is 4 and every vertex in H has degree 2, two edges incident on b must be removed from G to create H. Edge $\{a, b\}$ cannot be removed because doing so would result in vertex d having degree less than 2 in H. Similar reasoning shows that edge $\{b, c\}$ cannot be removed either. So edges $\{b, i\}$ and $\{b, e\}$ must be removed from G to create H. Because vertex e must have degree 2 in H and because edge $\{b, e\}$ is not in H, both edges $\{e, d\}$ and $\{e, f\}$ must be in H. Similarly, since both vertices c and g must have degree 2 in H, edges $\{c, d\}$ and $\{g, d\}$ must also be in H. But then three edges incident on d, namely $\{e, d\}$, $\{c, d\}$, and $\{g, d\}$, must be all in H, which contradicts the fact that vertex d must have degree 2 in H.

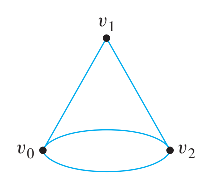
28. *Hint:* This graph does not have a Hamiltonian circuit.

32. *Partial answer:*



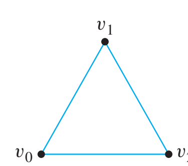
This graph has an Euler circuit $v_0v_1v_2v_3v_1v_4v_0$ but no Hamiltonian circuit.

33. *Partial answer:*



This graph has a Hamiltonian circuit $v_0v_1v_2v_0$ but no Euler circuit.

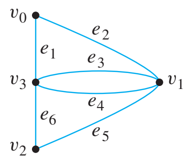
34. *Partial answer:*



The walk $v_0v_1v_2v_0$ is both an Euler circuit and a Hamiltonian circuit for this graph.

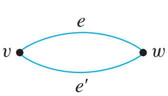
Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

35. *Partial answer:*



This graph has the Euler circuit $e_1e_2e_3e_4e_5e_6$ and the Hamiltonian circuit $v_0v_1v_2v_3v_0$. These are not the same.

37. a. Proof: Suppose G is a graph and W is a walk in G that contains a repeated edge e. Let v and w be the endpoints of e. In case $v = w$, then v is a repeated vertex of W. In case $v \neq w$, then one of the following must occur: (1) W contains two copies of vew or of wev (for instance, W might contain a section of the form $vewe'vew$, as illustrated below); (2) W contains separate sections of the form vew and wev (for instance, W might contain a section of the form $vewe'wev$, as illustrated below); or (3) W contains a section of the form $vewev$ or of the form $wevew$ (as illustrated below). In cases (1) and (2), both vertices v and w are repeated, and in case (3), one of v or w is repeated. In all cases, there is at least one vertex in W that is repeated.



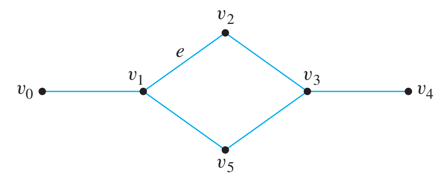
1

2

3

38. Proof: Suppose G is a connected graph and v and w are any particular but arbitrarily chosen vertices of G. *[We must show that u and v can be connected by a path.]* Since G is connected, there is a walk from v to w. If the walk contains a repeated vertex, then delete the portion of the walk from the first occurrence of the vertex to its next occurrence. (For example, in the walk $ve_1v_2e_5v_7e_6v_2e_3w$, the vertex v_2 occurs twice. Deleting the portion of the walk from one occurrence to the next gives $ve_1v_2e_3w$.) If the resulting walk still contains a repeated vertex, do the above deletion process another time. Then check again for a repeated vertex. Continue in this way until all repeated vertices have been deleted. (This must occur eventually, since the total number of vertices is finite.) The resulting walk connects v to w but has no repeated vertex. By exercise 37(b), it has no repeated edge either. Hence it is a path from v to w.

40. The graph to the right contains a circuit, any edge of which can be removed without disconnecting the graph. For instance, if edge e is removed, then the following walk can be used to go from v_1 to v_2: $v_1v_5v_3v_2$.



42. *Hint:* Look at the answer to exercise 40 and use the fact that all graphs have a finite number of edges.

44. Proof: Let G be a connected graph and let C be a circuit in G. Let G' be the subgraph obtained by removing all the edges of C from G and also any vertices that become isolated when the edges of C are removed. *[We must show that there exists a vertex v such that v is in both C and G'.]* Pick any vertex v of C and any vertex w of G'. Since G is connected, there is a path from v to w (by Lemma 10.2.1(a)):

$$\underset{\substack{\uparrow\\ \text{in } C}}{v} = v_0e_1v_1e_2v_2 \ldots v_{i-1}e_i\underset{\substack{\uparrow\\ \text{in } C}}{v_i}e_{i+1}\underset{\substack{\uparrow\\ \text{not in } C}}{v_{i+1}} \ldots v_{n-1}e_nv_n = \underset{\substack{\uparrow\\ \text{in } G'}}{w}.$$

Let i be the largest subscript such that v_i is in C. If $i = n$, then $v_n = w$ is in C and also in G', and we are done. If $i < n$, then v_i is in C and v_{i+1} is not in C. This implies that e_{i+1} is not in C (for if it were, both endpoints would be in C by definition of circuit). Hence when G' is formed by removing the edges and resulting isolated vertices from G, then e_{i+1} is not removed. That means that v_i does not become an isolated vertex, so v_i is not removed either. Hence v_i is in G'. Consequently, v_i is in both C and G' *[as was to be shown]*.

45. Proof: Suppose G is a graph with an Euler circuit. If G has only one vertex, then G is automatically connected. If v and w are any two vertices of G, then v and w each appear at least once in the Euler circuit (since an Euler circuit contains every vertex of the graph). The section of the circuit between the first occurrence of one of v or w and the first occurrence of the other is a walk from one of the two vertices to the other.

Section 10.3

1. a. Equating corresponding entries shows that

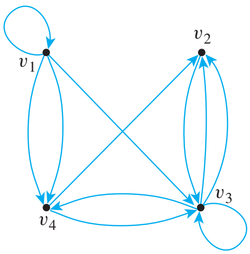
$$\begin{aligned} a + b &= 1, \\ a - c &= 0, \\ c &= -1, \\ b - a &= 3. \end{aligned}$$

Thus $a - c = a - (-1) = 0$, and so $a = -1$. Consequently, $a + b = (-1) + b = 1$, and hence $b = 2$. The last equation should be checked to make sure the answer is consistent: $b - a = 2 - (-1) = 3$, which agrees.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

2. a. $\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

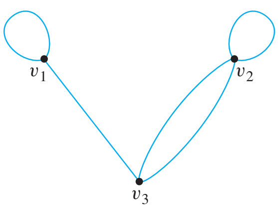
3. a.



Any labels may be applied to the edges because the adjacency matrix does not determine edge labels.

4. a. $\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$ **c.** $\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$

5. a.



Any labels may be applied to the edges because the adjacency matrix does not determine edge labels.

6. a. The graph is connected.

8. a. $2 \cdot 1 + (-1) \cdot 3 = -1$

9. a. $\begin{bmatrix} 3 & -3 & 12 \\ 1 & -5 & 2 \end{bmatrix}$

10. a. no product (**A** has three columns, and **B** has two rows.)

b. $BA = \begin{bmatrix} -2 & -2 & 2 \\ 1 & -5 & 2 \end{bmatrix}$ **f.** $B^2 = \begin{bmatrix} 4 & 0 \\ 1 & 9 \end{bmatrix}$

i. $AC = \begin{bmatrix} 2 & -1 \\ -5 & -2 \end{bmatrix}$

12. One among many possible examples is $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

14. *Hint:* If the entries of the $m \times m$ identity matrix are denoted by δ_{ik}, then $\delta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$. The ijth entry of **IA** is $\sum_{k=1}^{m} \delta_{ik} A_{kj}$.

15. Proof: Suppose **A** is an $m \times m$ symmetric matrix. Then for all integers i and j with $1 \leq i, j \leq m$,

$$(A^2)_{ij} = \sum_{k=1}^{m} A_{ik} A_{kj} \quad \text{and} \quad (A^2)_{ji} = \sum_{k=1}^{m} A_{jk} A_{ki}.$$

But since **A** is symmetric, $A_{ik} = A_{ki}$ and $A_{kj} = A_{jk}$ for all i, j, and k, and thus $A_{ik} A_{kj} = A_{jk} A_{ki}$ *[by the commutative law for multiplication of real numbers]*. Hence $(A^2)_{ij} = (A^2)_{ji}$ for all integers i and j with $1 \leq i, j \leq m$.

17. Proof (by mathematical induction): Let the property $P(n)$ be the equation $\mathbf{A}^n\mathbf{A} = \mathbf{A}\mathbf{A}^n$.

Show that $P(1)$ is true:

We must show that $\mathbf{A}^1\mathbf{A} = \mathbf{A}\mathbf{A}^1$. But this is true because $\mathbf{A}^1 = \mathbf{A}$ and $\mathbf{AA} = \mathbf{AA}$.

Show that for all integers $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true:

Let k be any integer such that $k \geq 1$, and suppose that $\mathbf{A}^k\mathbf{A} = \mathbf{A}\mathbf{A}^k$. *[This is the inductive hypothesis.]* We must show that $\mathbf{A}^{k+1}\mathbf{A} = \mathbf{A}\mathbf{A}^{k+1}$. But

$$\begin{aligned} \mathbf{A}^{k+1}\mathbf{A} &= (\mathbf{A}\mathbf{A}^k)\mathbf{A} && \text{by definition of matrix power} \\ &= \mathbf{A}(\mathbf{A}^k\mathbf{A}) && \text{by exercise 16} \\ &= \mathbf{A}(\mathbf{A}\mathbf{A}^k) && \text{by inductive hypothesis} \\ &= \mathbf{A}\,\mathbf{A}^{k+1} && \text{by definition of matrix power.} \end{aligned}$$

19. a.
$$A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$
$$A^3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix}$$

20. a. 2 since $(A^2)_{23} = 2$
b. 3 since $(A^2)_{34} = 3$
c. 6 since $(A^3)_{14} = 6$
d. 17 since $(A^3)_{23} = 17$

22. b. *Hint:* If G is bipartite, then its vertices can be partitioned into two sets V_1 and V_2 so that no vertices in V_1 are connected to each other by an edge and no vertices in V_2 are connected to each other by an edge. Label the vertices in V_1 as $v_1, v_2, \ldots, v_k$ and label the vertices in V_2 as $v_{k+1}, v_{k+2}, \ldots, v_n$. Now look at the matrix of G formed according to the given vertex labeling.

23. b. *Hint:* Consider the ijth entry of

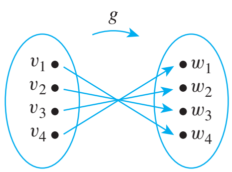
$$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^n.$$

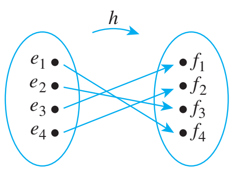
If G is connected, then given the vertices v_i and v_j, there is a walk connecting v_i and v_j. If this walk has length k, then by Theorem 10.3.2, the ijth entry of A^k is not equal to 0. Use the facts that all entries of each power of **A** are nonnegative and a sum of nonnegative numbers is positive provided that at least one of the numbers is positive.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Section 10.4

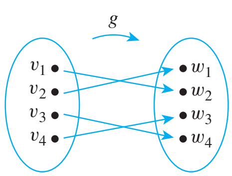
1. The graphs are isomorphic. One way to define the isomorphism is as follows:





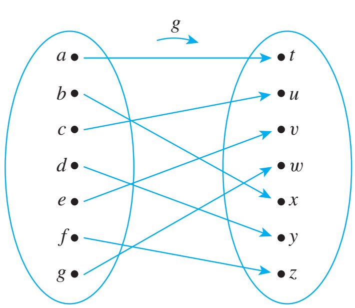
2. The graphs are not isomorphic. G has five vertices and G' has six.

6. The graphs are isomorphic. One isomorphism is the following:

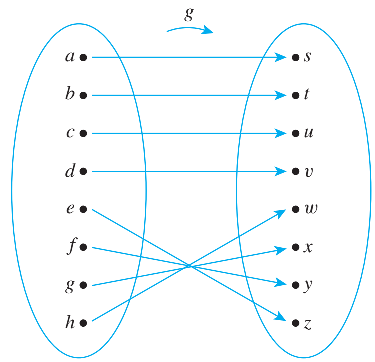


8. The graphs are not isomorphic. G has a simple circuit of length 3; G' does not.

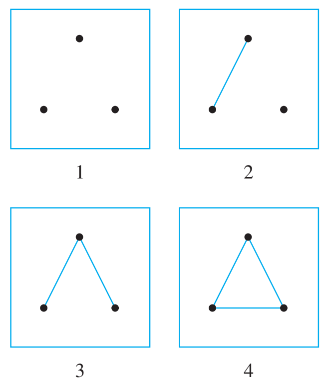
10. The graphs are isomorphic. One way to define the isomorphism is as follows:



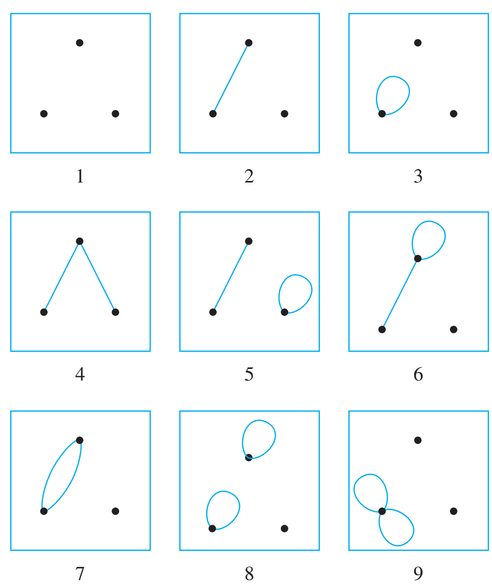
12. These graphs are isomorphic. One isomorphism is the following:



14.



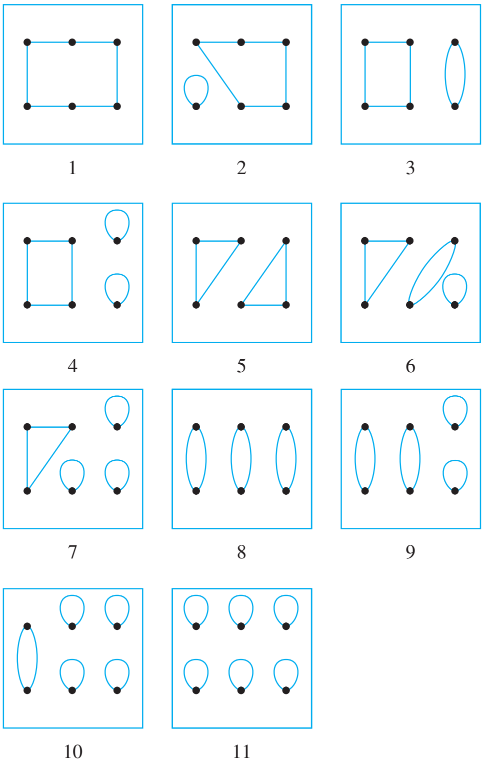
16.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

18. *Hint:* There are 20.

19.



21. Proof: Suppose G and G' are isomorphic graphs and G has n vertices, where n is a nonnegative integer. *[We must show that G' has n vertices.]* By definition of graph isomorphism, there is a one-to-one correspondence $g: V(G) \to V(G')$ sending vertices of G to vertices of G'. Since $V(G)$ is a finite set and g is a one-to-one correspondence, the number of vertices in $V(G')$ equals the number of vertices in $V(G)$. Hence G' has n vertices *[as was to be shown]*.

23. Proof: Suppose G and G' are isomorphic graphs and suppose G has a circuit C of length k, where k is a nonnegative integer. Let C be $v_0e_1v_1e_2 \dots e_kv_k (= v_0)$. By definition of graph isomorphism, there are one-to-one correspondences $g: V(G) \to V(G')$ and $h: E(G) \to E(G')$ that preserve the edge-endpoint functions in the sense that for all v in $V(G)$ and e in $E(G)$, v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$. Let C' be $g(v_0)h(e_1)g(v_1)h(e_2) \dots . h(e_k)g(v_k) (= g(v_0))$. Then C' is a circuit of length k in G'. The reason is that (1) because g and h preserve the edge-endpoint functions, for all $i = 0, 1, \dots, k-1$ both $g(v_i)$ and $g(v_{i+1})$ are incident on $h(e_{i+1})$ so that C' is a walk from $g(v_0)$ to $g(v_0)$, and (2) since C is a circuit, then $e_1, e_2, \dots, e_k$ are distinct, and since h is a one-to-one correspondence, $h(e_1), h(e_2), \dots, h(e_k)$ are also distinct, which implies that C' has k distinct edges. Therefore, G' has a circuit C of length k.

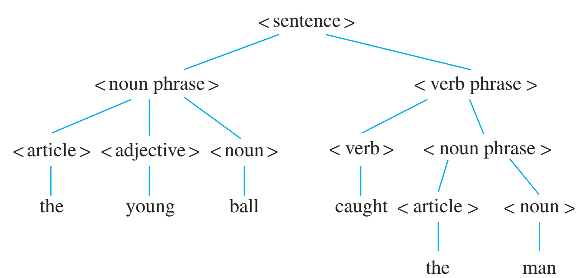
25. *Hint:* Suppose G and G' are isomorphic and G has m vertices of degree k; call them $v_1, v_2, \dots, v_m$. Since G and G' are isomorphic, there are one-to-one correspondences $g: V(G) \to V(G')$ and $h: E(G) \to E(G')$. Show that $g(v_1), g(v_2), \dots, g(v_m)$ are m distinct vertices of G' each of which has degree k.

27. *Hint:* Suppose G and G' are isomorphic and G is connected. To show that G' is connected, suppose w and x are any two vertices of G'. Show that there is a walk connecting w with x by finding a walk connecting the corresponding vertices in G.

Section 10.5

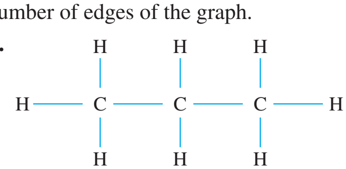
1. a. Math 110

2. a.



3. *Hint:* The answer is $2n - 2$. To obtain this result, use the relationship between the total degree of a graph and the number of edges of the graph.

4. a.



d. *Hint:* Each carbon atom in G is bonded to four other atoms in G, because otherwise an additional hydrogen atom could be bonded to it, and this would contradict the assumption that G has the maximum number of hydrogen atoms for its number of carbon atoms. Also each hydrogen atom is bonded to exactly one carbon atom in G, because otherwise G would not be connected.

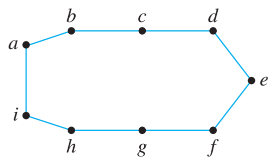
5. *Hint:* Revise the algorithm given in the proof of Lemma 10.5.1 to keep track of which vertex and edge were chosen in step 1 (by, say, labeling them v_0 and e_0). Then after one vertex of degree 1 is found, return to v_0 and search for another vertex of degree 1 by moving along a path outward from v_0 starting with e_0.

7. a. Internal vertices: v_2, v_3, v_4, v_6
Terminal vertices: v_1, v_5, v_7

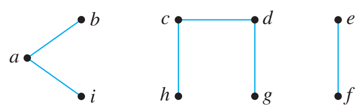
8. Any tree with nine vertices has eight edges, not nine. Thus there is no tree with nine vertices and nine edges.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

9. One such graph is

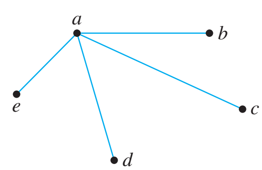


10. One such graph is



11. There is no tree with six vertices and a total degree of 14. Any tree with six vertices has five edges and hence (by Theorem 10.1.1) a total degree of 10, not 14.

12. One such tree is shown.



13. No such graph exists. By Theorem 10.5.4, a connected graph with six vertices and five edges is a tree. Hence such a graph cannot have a nontrivial circuit.

14.

22. Yes. Since it is connected and has 12 vertices and 11 edges, by Theorem 10.5.4 it is a tree. It follows from Lemma 10.5.1 that it has vertex of degree 1.

25. Suppose there were a connected graph with eight vertices and six edges. Either the graph itself would be a tree or edges could be eliminated from its circuits to obtain a tree. In either case, there would be a tree with eight vertices and six or fewer edges. But by Theorem 10.5.2, a tree with eight vertices has seven edges, not six or fewer. This contradiction shows that the supposition is false, so there is no connected graph with eight vertices and six edges.

26. *Hint:* See the answer to exercise 25.

27. Yes. Suppose G is a circuit-free graph with ten vertices and nine edges. Let $G_1, G_2, \ldots, G_k$ be the connected components of G *[To show that G is connected, we will show that* $k = 1$*.]* Each G_i is a tree since each G_i is connected and circuit-free. For each $i = 1, 2, \ldots, k$, let G_i have n_i vertices. Note that since G has ten vertices in all,

$$n_1 + n_2 + \cdots + n_k = 10.$$

By Theorem 10.5.2,

$$\begin{aligned} &G_1 \text{ has } n_1 - 1 \text{ edges,}\\ &G_2 \text{ has } n_2 - 1 \text{ edges,}\\ &\quad\vdots\\ &G_k \text{ has } n_k - 1 \text{ edges.} \end{aligned}$$

So the number of edges of G equals

$$\begin{aligned} &(n_i - 1) + (n_2 - 1) + \cdots + (n_k - 1)\\ &\quad = (n_1 + n_2 + \cdots + n_k) - \underbrace{(1 + 1 + \cdots + 1)}_{k \text{ 1's}}\\ &\quad = 10 - k. \end{aligned}$$

But we are given that G has nine edges. Hence $10 - k = 9$, and so $k = 1$. Thus G has just one connected component, G_1, and so G is connected.

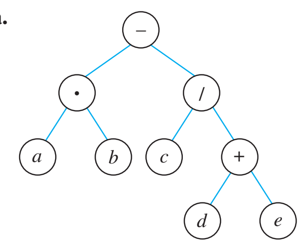
28. *Hint:* See the answer to exercise 27.

31. b. *Hint:* There are six.

Section 10.6

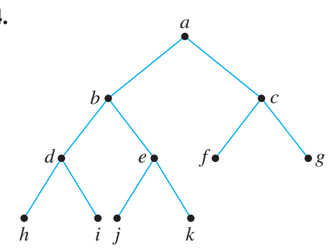
1. a. 3 **b.** 0 **c.** 5 **d.** u, v
e. d **f.** k, l **g.** m, s, t, x, y

3. a.



Exercises 4 and 8–10 have other answers in addition to the ones shown.

4.



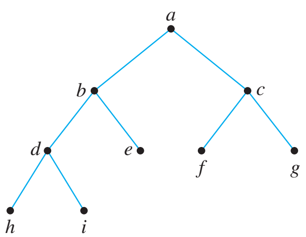
5. There is no full binary tree with the given properties because any full binary tree with five internal vertices has six terminal vertices, not seven.

6. Any full binary tree with four internal vertices has five terminal vertices for a total of nine, not seven, vertices in all. Thus there is no full binary tree with the given properties.

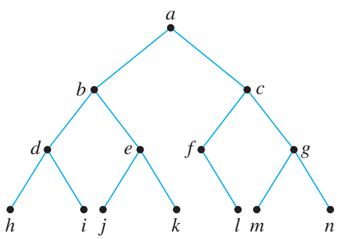
7. There is no full binary tree with 12 vertices because any full binary tree has $2k + 1$ vertices, where k is the number of internal vertices. But $2k + 1$ is always odd, and 12 is even.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

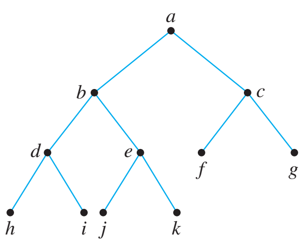
8.



9.



10.

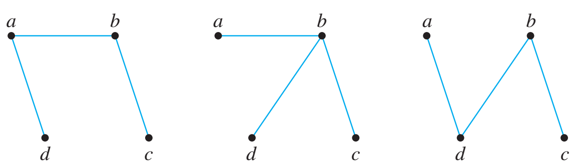


11. There is no binary tree that has height 3 and nine terminal vertices because any binary tree of height 3 has at most $2^3 = 8$ terminal vertices.

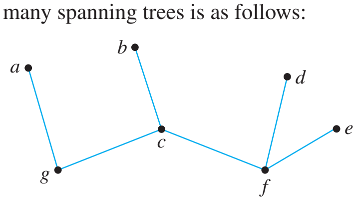
20. a. Height of tree $\geq \log_2 25 \cong 4.6$. Since the height of any tree is an integer, the height must be at least 5.

Section 10.7

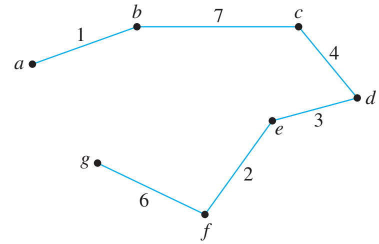
1.



3. One of many spanning trees is as follows:



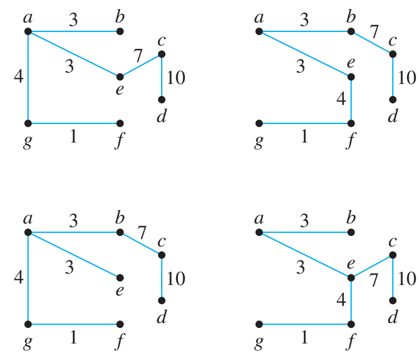
5. Minimum spanning tree:



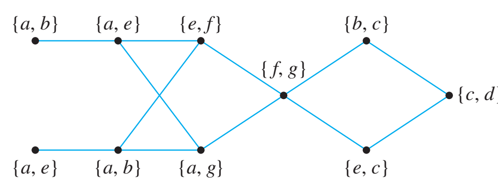
Order of adding the edges:
$\{a, b\}, \{e, f\}, \{e, d\}, \{d, c\}, \{g, f\}, \{b, c\}$

7. Minimum spanning tree: same as in exercise 5
Order of adding the edges:
$\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{f, g\}$

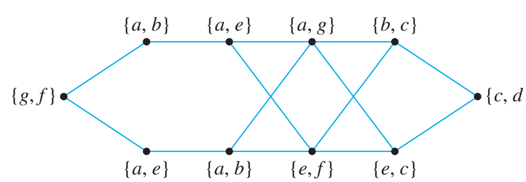
9. There are four minimum spanning trees:



When Prim's algorithm is used, edges are added in any of the orders obtained by following one of the eight paths from left to right across the diagram below.



When Kruskal's algorithm is used, edges are added in any of the orders obtained by following one of the eight paths from left to right across the diagram below.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

12. Let N = Nashville, S = St. Louis, Lv = Louisville, Ch = Chicago, Cn = Cincinnati, D = Detroit, Mw = Milwaukee, and Mn = Minneapolis.

Step	V(T)	E(T)	F
0	$\{N\}$	$\emptyset$	$\{N\}$
1	$\{N\}$	$\emptyset$	$\{Lv, Mn\}$
2	$\{N, Lv\}$	$\{\{N, Lv\}\}$	$\{Mn, S, Cn, Ch, D, Mw\}$
3	$\{N, Lv, Cn\}$	$\{\{N, Lv\}, \{Lv, Ci\}\}$	$\{Mn, S, Ch, D, Mw\}$
4	$\{N, Lv, Cn, S\}$	$\{\{N, Lv\}, \{Lv, Ci\}, \{Lv, S\}\}$	$\{Mn, Ch, D, Mw\}$
5	$\{N, Lv, Cn, S, Ch\}$	$\{\{N, Lv\}, \{Lv, Ci\}, \{Lv, S\}, \{Lv, Ch\}\}$	$\{Mn, D, Mw\}$
6	$\{N, Lv, Cn, S, Ch, D\}$	$\{\{N, Lv\}, \{Lv, Ci\}, \{Lv, S\}, \{Lv, Ch\}\{Lv, D\}\}$	$\{Mn, Mw\}$
7	$\{N, Lv, Cn, S, Ch, D, Mw\}$	$\{\{N, Lv\}, \{Lv, Ci\}, \{Lv, S\}, \{Lv, Ch\}\{Lv, D\}, \{Ch, Mw\}\}$	$\{Mn\}$
8	$\{N, Lv, Cn, S, Ch, D, Mw, Mn\}$		

Step	L(N)	L(S)	L(Lv)	L(Cn)	L(Ch)	L(D)	L(Mw)	L(Mn)
0	**0**	∞	∞	∞	∞	∞	∞	∞
1	0	∞	**151**	∞	∞	∞	∞	695
2	0	393	151	**234**	420	457	499	695
3	0	**393**	151	234	420	457	499	695
4	0	393	151	234	**420**	457	499	695
5	0	393	151	234	420	**457**	494	695
6	0	393	151	234	420	457	**494**	695
7	0	393	151	234	420	457	494	**695**

Thus the shortest path from Nashville to Minneapolis has length $L(Mn) = 695$ miles.

13.

Step	V(T)	E(T)	F	L(a)	L(b)	L(c)	L(d)	L(e)	L(z)
0	$\{a\}$	$\emptyset$	$\{a\}$	**0**	∞	∞	∞	∞	∞
1	$\{a\}$	$\emptyset$	$\{b, d\}$	0	2	∞	**1**	∞	∞
2	$\{a, d\}$	$\{\{a, d\}\}$	$\{b, c, e\}$	0	**2**	6	1	11	∞
3	$\{a, b, d\}$	$\{\{a, d\}, \{a, b\}\}$	$\{c, e\}$	0	2	**5**	1	6	∞
4	$\{a, b, c, d\}$	$\{\{a, d\}, \{a, b\}, \{b, c\}\}$	$\{e, z\}$	0	2	5	1	**6**	13
5	$\{a, b, c, d, e\}$	$\{\{a, d\}, \{a, b\}, \{b, c\}, \{c, e\}\}$	$\{z\}$	0	2	5	1	6	**8**
6	$\{a, b, c, d, e, z\}$	$\{\{a, d\}, \{a, b\}, \{b, c\}, \{c, e\}, \{e, z\}\}$							

Thus the shortest path from a to z has length $L(z) = 8$.

18. b. Proof: Suppose not. Suppose that for some tree T, u and v are distinct vertices of T, and P_1 and P_2 are two distinct paths joining u and v. *[We must deduce a contradiction. In fact, we will show that T contains a circuit.]* Let P_1 be denoted $u = v_0, v_1, v_2, \ldots, v_m = v$, and let P_2 be denoted $u = w_0, w_1, w_2, \ldots, w_n = v$. Because P_1 and P_2 are distinct, and T has no parallel edges, the sequence of vertices in P_1 must diverge from the sequence of vertices in P_2 at some point. Let i be the least integer such that $v_i \neq w_i$. Then $v_{i-1} = w_{i-1}$. Let j and k be the least integers greater than i so that $v_j = w_k$. (There must be such integers because $v_m = w_n$). Then

$$v_{i-1}v_iv_{i+1}\ldots v_j(= w_k)w_{k-1}\ldots w_iw_{i-1}(= v_{i-1})$$

is a circuit in T. The existence of such a circuit contradicts the fact that T is a tree. Hence the supposition must be false. That is, given any tree with vertices u and v, there is a unique path joining u and w.

20. Proof: Suppose G is a connected graph, T is a circuit free subgraph of G, and if any edge e of G not in T is added to T, the resulting graph contains a circuit. Suppose that T is not a spanning tree for G. *[We must derive a contradiction.]* ***Case 1 (T is not connected):*** In this case, there are vertices u and v in T such that there is no walk in T from u to v. Now, since G is connected, there is a walk in G from u to v, and hence, by Lemma 10.2.1, there is a path in G from u to v. Let $e_1, e_2, \ldots, e_k$ be the edges of this path that are not

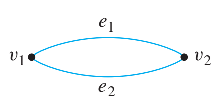
Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

in T. When these edges are added to T, the result is a graph T' in which u and v are connected by a path. In addition, by hypothesis, each of the edges e_i creates a circuit when added to T. Now remove these edges one by one from T'. By the same argument used in the proof of Lemma 10.5.3, each such removal leaves u and v connected since each e_i is an edge of a circuit when added to T. Hence, after all the e_i have been removed, u and v remain connected. But this contradicts the fact that there is no walk in T from u to v.

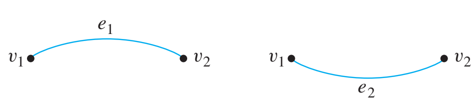
***Case 2* (*T is connected*):** In this case, since T is not a spanning tree and T is circuit-free, there is a vertex v in G such that v is not in T. *[For if T were connected, circuit-free, and contained every vertex in G, then T would be a spanning tree for G.]* Since G is connected, v is not isolated. Thus there is an edge e in G with v as an endpoint. Let T' be the graph obtained from T by adding e and v. *[Note that e is not already in T because if it were, its endpoint v would also be in T and it is not.]* Then T' contains a circuit because, by hypothesis, addition of any edge to T creates a circuit. Also T' is connected because T is and because when e is added to T, e becomes part of a circuit in T'. Now deletion of an edge from a circuit does not disconnect a graph, so if e is deleted from T' the result is a connected graph. But the resulting graph contains v, which means that there is an edge in T connecting v to another vertex of T. This implies that v is in T *[because both endpoints of any edge in a graph must be part of the vertex set of the graph],* which contradicts the fact that v is not in T.

Thus, in either case, the supposition that T is not a spanning tree leads to a contradiction. Hence the supposition is false, and T is a spanning tree for G.

21. a. No. Counterexample: Let G be the following graph.



Then G has the spanning trees shown below.



These trees have no edge in common.

22. *Hint:* Suppose e is contained in every spanning tree of G and the graph obtained by removing e from G is connected. Let G' be the subgraph of G obtained by removing e, and let T' be a spanning tree for G'. How is T' related to G?

24. Proof: Suppose that $w(e') > w(e)$. Form a new graph T' by adding e to T and deleting e'. By exercise 20, addition of an edge to a spanning tree creates a circuit, and by Lemma 10.5.3, deletion of an edge from a circuit does not disconnect a graph. Consequently, T' is also a spanning tree for G. Furthermore, $w(T') < w(T)$ because $w(T') = w(T) - w(e') + w(e) = w(T) - (w(e') - w(e)) < w(T)$ *[since $w(e') > w(e)$, which implies that $w(e') - w(e) > 0$].* But this contradicts the fact that T is a minimum spanning tree for G. Hence the supposition is false, and so $w(e') \le w(e)$.

25. *Hint:* Suppose e is an edge that has smaller weight than any other edge of G, and suppose T is a minimum spanning tree for G that does not contain e. Create a new spanning tree T' by adding e to T and removing another edge of T (which one?). Then $w(T') < w(T)$.

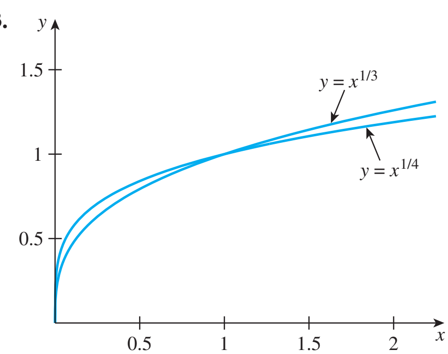
26. Yes. Proof by contradiction: Suppose G is a weighted graph in which all the weights of all the edges are distinct, and suppose G has two distinct minimum spanning trees T_1 and T_2. Let e be the edge of least weight that is in one of the trees but not the other. Without loss of generality, we may say that e is in T_1. Add e to T_2 to obtain a graph G'. By exercise 19, G' contains a nontrivial circuit. At least one other edge f of this circuit is not in T_1 because otherwise T_1 would contain the complete circuit, which would contradict the fact that T_1 is a tree. Now f has weight greater than e because all edges have distinct weights, f is in T_2 and not in T_1, and e is the edge of least weight that is in one of the trees and not the other. Remove f from G' to obtain a tree T_3. Then $w(T_3) < w(T_2)$ because T_3 is the same as T_2 except that it contains e rather than f and $w(e) < w(f)$. Consequently, T_3 is a spanning tree for G of smaller weight than T_2. This contradicts the supposition that T_2 is a minimum spanning tree for G. Thus G cannot have more than one minimum spanning tree.

28. The output will be a "minimum spanning forest" for the graph. It will contain a minimum spanning tree for each connected component of the input graph.

Section 11.1

1. a. $f(0)$ is positive.
b. $f(x) = 0$ when $x = -2$ and $x = 3$ (approximately)
c. $x_1 = -1$ and $x_2 = 2$ (approximately)
d. $x = 1$ or $x = -\frac{1}{2}$ (approximately)
e. increase
f. decrease

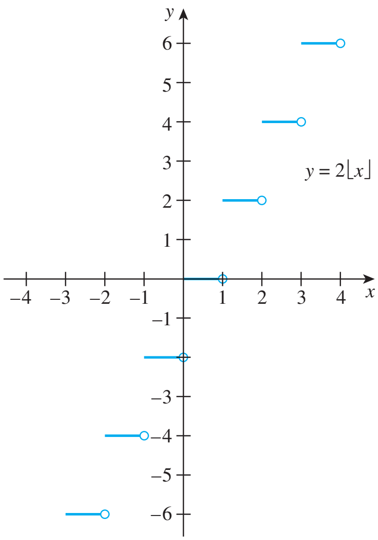
3.

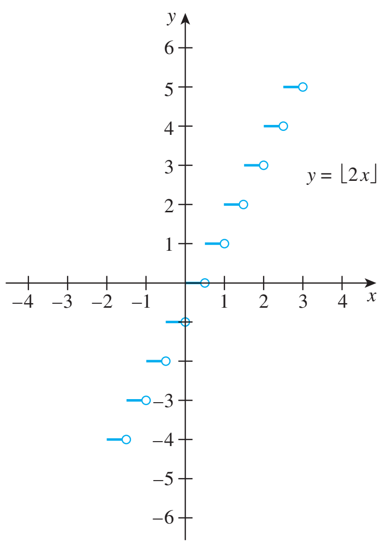


When $0 < x < 1$, $x^{1/3} < x^{1/4}$. When $x > 1$, $x^{1/3} > x^{1/4}$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

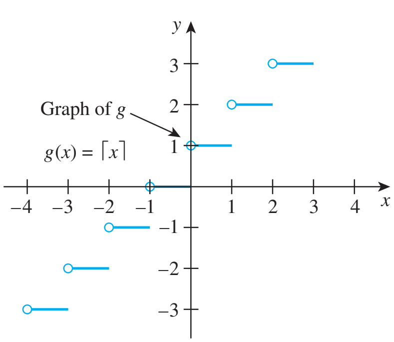
5.





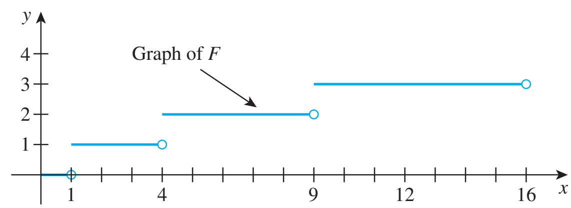
The graphs show that $2\lfloor x \rfloor \neq \lfloor 2x \rfloor$ for many values of x.

6.



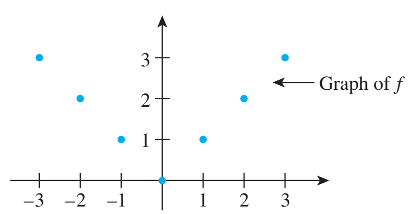
8.

x	$F(x) = \lfloor x^{1/2} \rfloor$
0	0
$\frac{1}{2}$	0
1	1
2	1
3	1
4	2



10.

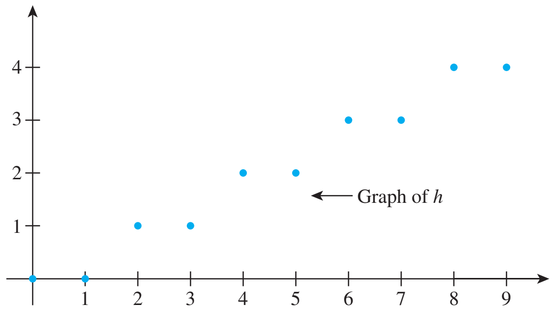
n	$f(n) = \lvert n \rvert$
0	0
1	1
2	2
3	3
−1	1
−2	2
−3	3



12.

n	$h(n) = \left\lfloor \frac{n}{2} \right\rfloor$
0	0
1	0
2	1
3	1
4	2
5	2
6	3
7	3
8	4
9	4

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



14. f is increasing on the intervals $\{x \in \mathbf{R} \mid -3 < x < -2\}$ and $\{x \in \mathbf{R} \mid 0 < x < 2.5\}$, and f is decreasing on $\{x \in \mathbf{R} \mid -2 < x < 0\}$ and $\{x \in \mathbf{R} \mid 2.5 < x < 4\}$ (approximately).

15. Proof: Suppose x_1 and x_2 are particular but arbitrarily chosen real numbers such that $x_1 < x_2$. *[We must show that $f(x_1) < f(x_2)$.]* Since

$$x_1 < x_2$$

then $$2x_1 < 2x_2$$

and $$2x_1 - 3 < 2x_2 - 3$$

by basic properties of inequalities. But then, by definition of f,

$$f(x_1) < f(x_2)$$

[as was to be shown]. Hence f is increasing on the set of all real numbers.

17. a. Proof: Suppose x_1 and x_2 are real numbers with $x_1 < x_2 < 0$. *[We must show that $h(x_1) > h(x_2)$.]* Multiply both sides of $x_1 < x_2$ by x_1 to obtain $(x_1)^2 > x_1x_2$ *[by T23 of Appendix A since $x_1 < 0$]*, and multiply both sides of $x_1 < x_2$ by x_2 to obtain $x_1x_2 > (x_2)^2$ *[by T23 of Appendix A since $x_2 < 0$]*. By transitivity of order *[Appendix A, T18]* $(x_2)^2 < (x_1)^2$, and so, by definition of h, $h(x_2) < h(x_1)$.

18. a. *Preliminaries:* If both x_1 and x_2 are positive, then by the rules for working with inequalities (see Appendix A),

$$\frac{x_1 - 1}{x_1} < \frac{x_2 - 1}{x_2} \Rightarrow x_2(x_1 - 1) < x_1(x_2 - 1)$$ by multiplying both sides by x_1x_2 (which is positive)

$$\Rightarrow x_1x_2 - x_2 < x_1x_2 - x_1$$ by multiplying out

$$\Rightarrow -x_2 < -x_1$$ by subtracting x_1x_2 from both sides

$$\Rightarrow x_2 > x_1$$ by multiplying by -1.

Are these steps reversible? Yes!

Proof: Suppose that x_1 and x_2 are positive real numbers and $x_1 < x_2$. *[We must show that $k(x_1) < k(x_2)$.]* Then

$x_1 < x_2$

$\Rightarrow -x_2 < -x_1$ by multiplying by -1

$\Rightarrow x_1x_2 - x_2 < x_1x_2 - x_1$ by adding x_1x_2 to both sides

$\Rightarrow x_2(x_1 - 1) < x_1(x_2 - 1)$ by factoring both sides

$\Rightarrow \dfrac{x_1 - 1}{x_1} < \dfrac{x_2 - 1}{x_2}$ by dividing both sides by the positive number x_1x_2

$\Rightarrow k(x_1) < k(x_2)$ by definition of k.

[This is what was to be shown.]

19. Proof: Suppose $f: \mathbf{R} \to \mathbf{R}$ is increasing. *[We must show that f is one-to-one. In other words, we must show that for all real numbers x_1 and x_2, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.]* Suppose x_1 and x_2 are real numbers and $x_1 \neq x_2$. By the trichotomy law *[Appendix A, T17]* $x_1 < x_2$, or $x_1 > x_2$. In case $x_1 < x_2$, then since f is increasing, $f(x_1) < f(x_2)$ and so $f(x_1) \neq f(x_2)$. Similarly in case $x_1 > x_2$, then $f(x_1) > f(x_2)$ and so $f(x_1) \neq f(x_2)$. Thus in either case, $f(x_1) \neq f(x_2)$ *[as was to be shown]*.

21. a. Proof: Suppose u and v are nonnegative real numbers with $u < v$. *[We must show that $f(u) < f(v)$.]* Note that $v = u + h$ for some positive real number h. By substitution and the binomial theorem,

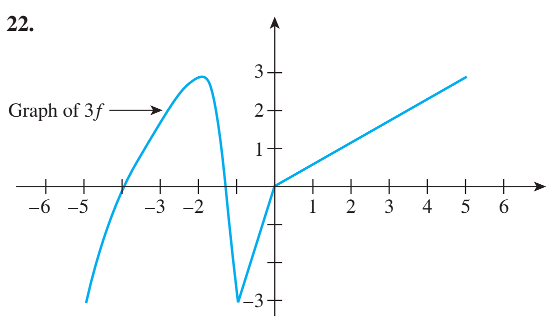
$$\begin{aligned} v^m &= (u+h)^m \\ &= u^m + \left[\binom{m}{1}u^{m-1}h + \binom{m}{2}u^{m-2}h^2 + \cdots \right. \\ &\qquad \left. + \binom{m}{m-1}uh^{m-1} + h^m\right]. \end{aligned}$$

The bracketed sum is positive because $u \geq 0$ and $h > 0$, and a sum of nonnegative terms that includes at least one positive term is positive. Hence

$$v^m = u^m + \text{a positive number},$$

and so $f(u) = u^m < v^m = f(v)$ *[as was to be shown]*.

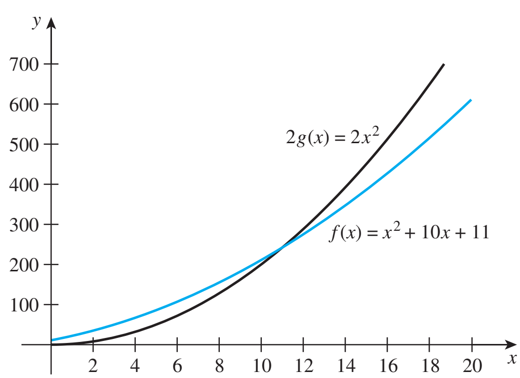
22.



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

24. Proof: Suppose that f is a real-valued function of a real variable, f is decreasing on a set S, and M is any positive real number. *[We must show that Mf is decreasing on S. In other words, we must show that for all x_1 and x_2 in S, if $x_1 < x_2$ then $(Mf)(x_1) > (Mf)(x_2)$.]* Suppose x_1 and x_2 are in S and $x_1 < x_2$. Since f is decreasing on S, $f(x_1) > f(x_2)$, and since M is positive, $Mf(x_1) > Mf(x_2)$ *[because when both sides of an inequality are multiplied by a positive number, the direction of the inequality is unchanged]*. It follows by definition of Mf that $(Mf)(x_1) > (Mf)(x_2)$ *[as was to be shown]*.

27. To find the answer algebraically, solve the equation $2x^2 = x^2 + 10x + 11$ for x. Subtracting x^2 from both sides gives $x^2 - 10x - 11 = 0$, and either factoring $x^2 - 10x - 11 = (x-11)(x+1)$ or using the quadratic formula gives $x = 11$ (since $x > 0$). To find an approximate answer with a graphing calculator, plot both $f(x) = x^2 + 10x + 11$ and $2g(x) = 2x^2$ for $x > 0$, as shown in the figure, and find that $2g(x) > f(x)$ when $x > 11$ (approximately). You can obtain only an approximate answer from a graphing calculator because the calculator computes values only to an accuracy of a finite number of decimal places.



Section 11.2

1. a. $\forall$ positive real numbers a and $A, \exists x > a$ such that $A|g(x)| > |f(x)|$.

b. No matter what positive real numbers a and A might be chosen, it is possible to find a number x greater than a with the property that $A|g(x)| > |f(x)|$.

4. $5x^8 - 9x^7 + 2x^5 + 3x - 1$ is $O(x^8)$

5. $\dfrac{(x^2-1)(12x+25)}{3x^2+4}$ is $\Theta(x)$

6. $\dfrac{(x^2-7)^2(10x^{1/2}+3)}{x+1}$ is $\Omega(x^{7/2})$

10. Proof: Suppose f and g are real-valued functions of a real variable that are defined on the same set of nonnegative real numbers, and suppose $g(x)$ is $O(f(x))$. By definition of O-notation, there exist positive real numbers b and B such that $|g(x)| \le B|f(x)|$ for all real numbers $x > b$. Divide both sides of the inequality by B to obtain $\frac{1}{B}|g(x)| \le |f(x)|$. Let $A = \frac{1}{B}$ and let $a = b$. Then $A|g(x)| \le |f(x)|$ for all real numbers $x > a$, and so, by definition of Ω-notation, $f(x)$ is $\Omega(g(x))$.

12. Proof: Suppose f, g, h, and k are real-valued functions of a real variable that are defined on the same set D of nonnegative real numbers, and suppose $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$. By definition of O-notation, there exist positive real numbers b_1, B_1, b_2, and B_2 such that $|f(x)| \le B_1|h(x)|$ for all real numbers $x > b_1$, and $|g(x)| \le B_2|k(x)|$ for all real numbers $x > b_2$. For each x in D, define $G(x) = \max(|h(x)|, |k(x)|)$, and let $b = \max(b_1, b_2)$ and $B = B_1 + B_2$. Note that the triangle inequality for absolute value (Theorem 4.4.6) implies that

$$|f(x) + g(x)| \le |f(x)| + |g(x)|$$

for all real numbers x in D. Suppose that $x > b$. Then because b is greater than both b_1 and b_2,

$$|f(x)| \le B_1|h(x)| \quad \text{and} \quad |g(x)| \le B_2|h(x)|,$$

and so, by adding the inequalities (Appendix A, T26), we get

$$|f(x)| + |g(x)| \le B_1|h(x)| + B_2|k(x)|.$$

Thus, by the transitive law for inequalities (Appendix A, T18),

$$|f(x) + g(x)| \le B_1|h(x)| + B_2|k(x)|.$$

Now, because each value of $G(x) = |G(x)|$ is greater than or equal to $|h(x)|$ and $|k(x)|$,

$$B_1|h(x)| + B_2|k(x)| \le B_1|G(x)| + B_2|G(x)| \le (B_1 + B_2)|G(x)|.$$

Hence, again by transitivity and because $B = B_1 + B_2$,

$$|f(x) + g(x)| \le B|G(x)| \quad \text{for all real numbers } x > b.$$

Therefore, by definition of O-notation, $f(x) + g(x)$ is $O(G(x))$.

14. *Start of proof:* Suppose f, g, h, and k are real-valued functions of a real variable that are defined on the same set D of nonnegative real numbers, and suppose $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$. By definition of O-notation, there exist positive real numbers b_1, B_1, b_2, and B_2 such that $|f(x)| \le B_1|h(x)|$ for all real numbers $x > b_1$, and $|g(x)| \le B_2|k(x)|$ for all real numbers $x > b_2$. Let $B = B_1B_2$ and let $b = \max(b_1, b_2)$.

15. b. *Hint:* By the laws of exponents, $x^{n-m} = \frac{x^n}{x^m}$. Thus if $x^{n-m} > 1$, then $\frac{x^n}{x^m} > 1$.

16. a. For all real numbers $x > 1$, $x^2 + 15x + 4 \ge 0$ because all terms are nonnegative. Adding x^2 to both sides gives $2x^2 + 15x + 4 \ge x^2$. By the nonnegativity of all terms when $x > 1$, absolute value signs may be added to both sides of the inequality. Thus $|x^2| \le |2x^2 + 15x + 4|$ for all real numbers $x > 1$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

b. For all real numbers $x > 1$,

$$
\begin{aligned}
&|2x^2+15x+4| = 2x^2+15x+4 && \text{because } 2x^2+15x+4 \text{ is positive (since } x>1)\\
\Rightarrow\ &|2x^2+15x+4| \le 2x^2+15x^2+4x^2 && \text{because since } x>1, \text{ then } x<x^2 \text{ and } 1<x^2\\
\Rightarrow\ &|2x^2+15x+4| \le 21x^2 && \text{because } 2+15+4=21\\
\Rightarrow\ &|2x^2+15x+4| \le 21|x^2| && \text{because } x^2 \text{ is positive.}
\end{aligned}
$$

c. Let $A = 1$ and $a = 1$. Then by part (a), $A|x^2| \le |2x^2 + 15x + 4|$ for all real numbers $x > a$, and so, by definition of Ω-notation, $2x^2 + 15x + 4$ is $\Omega(x^2)$.

Let $B = 21$ and $b = 1$. Then, by part (b), $|2x^2 + 15x + 4| \le B|x^2|$ for all real numbers $x > b$, and so, by definition of O-notation, $2x^2 + 15x + 4$ is $O(x^2)$.

d. Let $k = 1$, $A = 1$, and $B = 21$. By parts (a) and (b), for all real numbers $x > k$,

$$A|x^2| \le |2x^2 + 15x + 4| \le B|x^2|$$

and thus, by definition of Θ-notation, $2x^2 + 15x + 4$ is $\Theta(x^2)$. In other words, $2x^2 + 15x + 4$ has order x^2. (Alternatively, Theorem 11.2.1(1) could be used to derive this result.)

18. First observe that for all real numbers $x > 1$, $4x^3 + 65x + 30 \ge 0$ because all terms are nonnegative. Adding x^3 to both sides gives $5x^3 + 65x + 30 \ge x^3$. By the nonnegativity of the terms when $x > 1$, absolute value signs may be added to both sides of the inequality to obtain $|x^3| \le |5x^3 + 65x + 30|$ for all real numbers $x > 1$. Let $a = 1$ and $A = 1$. Then $A|x^3| \le |5x^3 + 65x + 30|$ (*) for all real numbers $x > a$.

Second, note that when $x > 1$,

$$
\begin{aligned}
&|5x^3+65x+30| \le 5x^3+65x+30 && \text{because all the terms are positive since } x>1.\\
\Rightarrow\ &|5x^3+65x+30| \le 5x^3+65x^3+30x^3 && \text{because since } x>1, \text{ then } 65x \le 65x^3 \text{ and } 30 \le 30x^3\\
\Rightarrow\ &|5x^3+65x+30| \le 100x^3 && \text{because } 5+65+30=100\\
\Rightarrow\ &|5x^3+65x+30| \le 100|x^3| && \text{because } x^3 \text{ is positive since } x>1.
\end{aligned}
$$

Let $b = 1$ and $B = 100$. Then $|5x^3 + 65x + 30| \le B|x^3|$ (**) for all real numbers $x > b$.

Let $k = \max(a, b)$. Putting inequalities (*) and (**) together gives that for all real numbers $x > k$,

$$A|x^3| \le |5x^3 + 65x + 30| \le B|x^3|.$$

Hence, by definition of Θ-notation, $5x^3 + 65x + 30$ is $\Theta(x^3)$; in other words, $5x^3 + 65x + 30$ has order x^3.

20. a. By definition of ceiling, for any real number x, $\lceil x^2 \rceil$ is that integer n such that $n - 1 < x^2 \le n$, and thus, by substitution, $x^2 \le \lceil x^2 \rceil$. Since $x > 1$, both sides of the inequality are positive, and so $|x^2| \le \left|\lceil x^2 \rceil\right|$.

b. As in part (a), $\lceil x^2 \rceil$ is that integer n such that $n - 1 < x^2 \le n$. Adding 1 to all parts of this inequality gives $n < x^2 + 1 \le n + 1$, so $\lceil x^2 \rceil < x^2 + 1$. Thus if x is any real number with $x > 1$, then

$$
\begin{aligned}
&\left|\lceil x^2 \rceil\right| \le \lceil x^2 \rceil && \text{because } \lceil x^2 \rceil \text{ is positive}\\
\Rightarrow\ &\left|\lceil x^2 \rceil\right| \le x^2+1 && \text{by the argument above}\\
\Rightarrow\ &\left|\lceil x^2 \rceil\right| \le x^2+x^2 && \text{because } 1<x^2 \text{ since } x>1\\
\Rightarrow\ &\left|\lceil x^2 \rceil\right| \le 2x^2 &&\\
\Rightarrow\ &\left|\lceil x^2 \rceil\right| \le 2\left|x^2\right| && \text{because } x^2 \text{ is positive.}
\end{aligned}
$$

c. Let $A = 1$ and $a = 1$. Then, by part (a), $\left|x^2\right| \le A|\lceil x^2 \rceil|$ for all real numbers $x > a$, and thus, by definition of Ω-notation, $\lceil x^2 \rceil$ is $\Omega(x^2)$.

Let $B = 2$ and $b = 1$. Then, by part (b), $\left|x^2\right| \le B\left|\lceil x^2 \rceil\right|$ for all real numbers $x > b$, and thus, by definition of O-notation, $\lceil x^2 \rceil$ is $O(x^2)$.

d. We conclude that $\lceil x^2 \rceil$ is $\Theta(x^2)$ by part (c) and Theorem 11.2.1(1). Alternatively, we can use the results of parts (a) and (b), letting $k = \max(a, b)$, to obtain the result that for all real numbers $x > k$,

$$A\left|x^2\right| \le \left|\lceil x^2 \rceil\right| \le B\left|x^2\right|$$

and conclude directly from the definition of Θ-notation that $\lceil x^2 \rceil$ is $\Theta(x^2)$.

22. a. For all real numbers $x > 1$,

$$
\begin{aligned}
&|7x^4-95x^3+3| \le |7x^4|+|95x^3|+|3| && \text{by the triangle inequality}\\
\Rightarrow\ &|7x^4-95x^3+3| \le 7x^4+95x^3+3 && \text{because all terms are positive since } x>1\\
\Rightarrow\ &|7x^4-95x^3+3| \le 7x^4+95x^4+3x^4 && \text{because } x>1 \text{ implies that } x^3 \le x^4 \text{ and } 1 \le x^4\\
\Rightarrow\ &|7x^4-95x^3+3| \le 105|x^4| && \text{because } 7+95+3=105 \text{ and } x^4>0.
\end{aligned}
$$

b. $7x^4 - 95x^3 + 3$ is $O(x^4)$

25. *Hint:* Use an argument by contradiction similar to the one in Example 11.2.8.

26. Proof: Suppose $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_n \ne 0$. By the generalized triangle inequality,

$$
\begin{aligned}
&|a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0|\\
&\qquad \le |a_nx^n| + |a_{n-1}x^{n-1}| + \cdots + |a_1x| + |a_0|,
\end{aligned}
$$

and because the absolute value of a product is the product of the absolute values (exercise 44, Section 4.4),

$$
\begin{aligned}
&|a_nx^n| + |a_{n-1}x^{n-1}| + \cdots + |a_1x| + |a_0|\\
&\qquad \le |a_n||x^n| + |a_{n-1}||x^{n-1}| + \cdots + |a_1||x| + |a_0|.
\end{aligned}
$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

In addition, when $x > 1$, property (11.2.1) implies that

$$x^n \le x^n, \quad x^{n-1} \le x^n, \ldots, x^2 \le x^n, \quad x \le x^n, \quad 1 \le x^n,$$

and also $x^n = |x^n|$ because $x > 1$. Thus

$$\begin{aligned}
&\left|a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right| \\
&\qquad \le |a_n||x^n| + |a_{n-1}||x^n| + \cdots + |a_1||x^n| + |a_0||x^n| \\
&\qquad \le (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|)|x^n|.
\end{aligned}$$

Let $b = 1$ and $B = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$. Then for all real numbers $x > b$,

$$\left|a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right| \le B\left|x^n\right|$$

and so, by definition of O-notation,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \text{ is } O(x^n).$$

28. Let $a = \left(\frac{95+3}{7}\right)\cdot 2 = 28$, and let $A = \frac{7}{2}$. If $x > a$, then

$$\begin{aligned}
& & x &\ge \left(\frac{95+3}{7}\right)\cdot 2 \\
&\Rightarrow & x &\ge \frac{95}{7}\cdot 2 + \frac{3}{7}\cdot 2 \\
&\Rightarrow & x &\ge \frac{95}{7}\cdot 2 + \frac{3}{7}\cdot 2\frac{1}{x^3} \\
& & & \qquad \text{because } \frac{1}{x^3} < 1 \text{ since } x > 28 \\
&\Rightarrow & \frac{7}{2}x^4 &\ge 95x^3 + 3 \\
& & & \qquad \text{by multiplying both sides by } \frac{7x^3}{2} \\
&\Rightarrow & \left(7 - \frac{7}{2}\right)x^4 &\ge 95x^3 - 3 \\
& & & \qquad \text{because } 95x^3 + 3 \ge 95x^3 - 3 \\
& & & \qquad \text{and } 7 - \frac{7}{2} = \frac{7}{2} \\
&\Rightarrow & 7x^4 - \frac{7}{2}x^4 &\ge 95x^3 - 3 \\
& & & \qquad \text{by multiplying out} \\
&\Rightarrow & 7x^4 - 95x^3 + 3 &\ge \frac{7}{2}x^4 \\
& & & \qquad \text{by adding } \tfrac{7}{2}x^4 - 95x^3 + 3 \\
& & & \qquad \text{to both sides} \\
&\Rightarrow & 7x^4 - 95x^3 + 3 &\ge Ax^4 \\
& & & \qquad \text{because } A = \frac{7}{2} \\
&\Rightarrow & |7x^4 - 95x^3 + 3| &\ge A|x^4| \\
& & & \qquad \text{because both sides are nonnegative.}
\end{aligned}$$

Hence, by definiton of Ω-notation, $7x^4 - 95x^3 + 3$ is $\Omega(x^4)$.

31. By exercise 22, $7x^4 - 95x^3 + 3$ is $O(x^4)$, and by exercise 28, $7x^4 - 95x^3 + 3$ is $\Omega(x^4)$. Thus, by Theorem 11.2.1(1), $7x^4 - 95x^3 + 3$ is $\Theta(x^4)$.

34. $\frac{(x+1)(x-2)}{4} = \frac{x^2-x-2}{4} = \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{2}$ is $\Theta(x^2)$ by the theorem on polynomial orders.

37. $\frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders.

40. By exercise 10 of Section 5.2, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, and, by exercise 37 above, $\frac{n(n+1)(2n+1)}{6}$ is $\Theta(n^3)$. Hence $1^2 + 2^2 + 3^2 + \cdots + n^2$ is $\Theta(n^3)$.

42. By Theorem 5.2.2, $2 + 4 + 6 + \cdots + 2n = 2\left(\frac{n(n+1)}{2}\right) = n^2 + n$, and by the theorem on polynomial orders, $n^2 + n$ is $\Theta(n^2)$. Thus $2 + 4 + 6 + \cdots + 2n$ is $\Theta(n^2)$.

44. By direct calculation or by Theorem 5.1.1, $\sum_{i=1}^{n}(4i - 9) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 9 = 4\left(\frac{n(n+1)}{2}\right) - 9n$. The last equality holds because of Theorem 5.2.2 and the fact that $\sum_{i=1}^{n} 9 = 9 + 9 + \cdots + 9$ (n summands) $= 9n$.
Then $4\left(\frac{n(n+1)}{2}\right) - 9n = 2n^2 + 2n - 9n = 2n^2 - 7n$, and hence $\sum_{i=1}^{n}(4i - 9) = 2n^2 - 7n$. But $2n^2 - 7n$ is $\Theta(n^2)$ by the theorem on polynomial orders. Thus $\sum_{i=1}^{n}(4i - 9)$ is $\Theta(n^2)$.

46. *Hint:* Use the result of exercise 13 from Section 5.2.

48. *Hints:*

a. $\dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{a_n x^n}$

$$= 1 + \frac{a_{n-1}}{a_n}\cdot\frac{1}{x} + \frac{a_{n-2}}{a_n}\cdot\frac{1}{x^2} + \cdots + \frac{a_1}{a_n}\cdot\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\cdot\frac{1}{x^n}.$$

b. $\lim_{n\to\infty} f(x) = L$ means that given any real number $\varepsilon > 0$, there is a real number $M > 0$ such that $L - \varepsilon < f(x) < L + \varepsilon$ for all real numbers $x > M$. Apply the definition of limit to the result of part (a), using $\varepsilon = \frac{1}{2}$.

49. a. Let f, g, and h be functions from **R** to **R**, and suppose $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$. Then there exist real numbers b_1, b_2, B_1, and B_2 such that $|f(x)| \le B_1|h(x)|$ for all $x > b_1$ and $|g(x)| \le B_2|h(x)|$ for all $x > b_2$. Let $B = B_1 + B_2$, and let b be the greater of b_1 and b_2. Then, for all $x > b$,

$$\begin{aligned}
& & |f(x) + g(x)| &< |f(x)| + |g(x)| \\
& & & \qquad \text{by the triangle inequality} \\
&\Rightarrow & |f(x) + g(x)| &\le B_1|h(x)| + B_2|h(x)| \\
& & & \qquad \text{by hypothesis} \\
&\Rightarrow & |f(x) + g(x)| &\le (B_1 + B_2)|h(x)| \\
& & & \qquad \text{by algebra} \\
&\Rightarrow & |f(x) + g(x)| &\le B|h(x)| \qquad \text{because } B = B_1 + B_2.
\end{aligned}$$

Hence, by definition of O-notation, $f(x) + g(x)$ is $O(h(x))$.

b. By exercise 15, for all $x > 1$, $x^2 < x^4$. Hence $|x^2| \le 1\cdot|x^4|$ for all $x > 1$. Thus, by definition of O-notation, x^2 is $O(x^4)$. Clearly also, $|x^4| \le 1\cdot|x^4|$ for all x, and so x^4 is $O(x^4)$. It follows by part (a) that $x^2 + x^4$ is $O(x^4)$.

50. d. *Hint:* If p, q, and s are positive integers, r is a nonnegative integer, and $\frac{p}{q} > \frac{r}{s}$, then $ps > qr$ and so $ps - qr > 0$. Also $\frac{x^{p/q}}{x^{r/s}} = x^{(p/q - r/s)} = x^{(pq-rs)/qs}$. Apply part (c) to $x^{1/qs}$, and use the fact that $ps - qr$ is an integer and $ps - qr > 0$ to make use of the result of exercise 15.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

51. By part (d) of exercise 50, for all $x > 1, x \le x^{4/3}$ and $1 = x^0 \le x^{4/3}$. Hence, by definition of O-notation (since all expressions are positive), x is $O(x^{4/3})$ and 1 is $O(x^{4/3})$. Also, by exercise 13, $x^{4/3}$ is $O(x^{4/3})$. By part (c) of exercise 49, then, $-15x = (-15)x$ is $O(x^{4/3})$ and $7 = 7 \cdot 1$ is $O(x^{4/3})$. It follows, by part (a) of exercise 49 (applied twice), that $4x^{4/3} - 15x + 7 = 4x^{4/3} + (-15x) + 7$ is $O(x^{4/3})$.

53. *Hint:* The proof is similar to the solution in Example 11.2.8. (Choose a real number x so that $x > B^{1/(r-s)}$, $x > 1$, and $x > b$.)

54. $f(x) = \dfrac{\sqrt{x}(3x+5)}{2x+1} = \dfrac{3x^{3/2} + 5x^{1/2}}{2x+1}$. The numerator of $f(x)$ is a sum of rational power functions with highest power 3/2, and the denominator is a sum of rational power functions with highest power 1. Because $3/2 - 1 = 1/2$, Theorem 11.2.4 implies that $f(x)$ is $\Theta(x^{1/2})$.

57. a. Proof (by mathematical induction): Let the property $P(n)$ be the inequality

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \le n^{3/2}.$$

Show that P(1) is true:

When $n = 1$, the left-hand side of the inequality is 1, and the right-hand side is $1^{3/2}$, which is also 1. Thus $P(1)$ is true.

Show that for all integers $k \ge 1$, if $P(k)$ is true, then $P(k+1)$ is true:

Let k be any integer with $k \ge 1$, and suppose

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k} \le k^{3/2}.$$

[inductive hypothesis]

We must show that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} \le (k+1)^{3/2}.$$

But

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k} + \sqrt{k+1}$$

by making the next-to-last term explicit

$$\Rightarrow \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} \le k^{3/2} + \sqrt{k+1}$$

by inductive hypothesis

$$\Rightarrow \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} \le k\sqrt{k} + \sqrt{k+1}$$

because $k^{3/2} = k\sqrt{k}$

$$\Rightarrow \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} \le k\sqrt{k+1} + \sqrt{k+1}$$

because $\sqrt{k} < \sqrt{k+1}$

$$\Rightarrow \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} \le (k+1)\sqrt{k+1}$$

by factoring out $\sqrt{k+1}$

$$\Rightarrow \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{k+1} \le (k+1)^{3/2}.$$

[This is what was to be shown.]

b. *Hint:* When $k \ge 1, k^2 \ge k^2 - 1$. Use the fact that $k^2 - 1 = (k-1)(k+1)$ and divide both sides by $k(k-1)$ to obtain $\frac{k}{k-1} \ge \frac{k+1}{k}$. But $\frac{k+1}{k} \ge 1$, and any number greater than or equal to 1 is greater than or equal to its own square root. Thus $\frac{k}{k-1} \ge \frac{k+1}{k} \ge \sqrt{\frac{k+1}{k}} = \frac{\sqrt{k+1}}{\sqrt{k}}$. Hence $k\sqrt{k} \ge (k-1)\sqrt{k+1} = (k+1-2)\sqrt{k+1} = (k+1)\sqrt{k+1} - 2\sqrt{k+1}$, and so $k\sqrt{k} + 2\sqrt{k+1} \ge (k+1)\sqrt{k+1}$.

c. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$ is $\Theta(x^{3/2})$.

59. Proof: Suppose $f(x)$ is $o(g(x))$. By definition of o-notation, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. By definition of limit, this implies that given any real number $\varepsilon > 0$, there exists a real number x_0 such that $\left|\frac{f(x)}{g(x)} - 0\right| < \varepsilon$ for all $x > x_0$. Let $b = \max(x_0, 1)$. Then $|f(x)| \le \varepsilon |g(x)|$ for all $x > b$. Choose $\varepsilon = 1$, and set $B = 1$. Then there exists a real number b such that $|f(x)| \le B|g(x)|$ for all $x > b$. Hence, by definition of O-notation, $f(x)$ is $O(g(x))$.

Section 11.3

1. a. $\log_2(200) = \frac{\ln 200}{\ln 2} \cong 7.6$ nanoseconds $=$ 0.0000000076 second

d. $200^2 = 40{,}000$ nanoseconds $= 0.00004$ second

e. $200^8 = 2.56 \times 10^{18}$ nanoseconds $\cong \frac{2.56 \times 10^{18}}{10^9 \cdot 60 \cdot 60 \cdot 24 \cdot (365.25)}$ years $\cong 81.1215$ years *[because there are 10^9 nanoseconds in a second, 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day and approximately 365.25 days in a year on average]*.

2. a. When the input size is increased from m to $2m$, the number of operations increases from cm^2 to $c(2m)^2 = 4cm^2$.

b. By part (a), the number of operations increases by a factor of $(4cm^2)/cm^2 = 4$.

c. When the input size is increased by a factor of 10 (from m to $10m$), the number of operations increases by a factor of $(c(10m)^2)/(cm^2) = (100cm^2)/cm^2 = 100$.

4. a. Algorithm A has order n^2 and algorithm B has order $n^{3/2}$.

b. Algorithm A is more efficient than algorithm B when $2n^2 < 80n^{3/2}$. This occurs exactly when

$$n^2 < 40n^{3/2} \Leftrightarrow \frac{n^2}{n^{3/2}} < 40 \Leftrightarrow n^{1/2} < 40 \Leftrightarrow n < 40^2.$$

Thus, algorithm A is more efficient than algorithm B when $n < 1{,}600$.

c. Algorithm B is at least 100 times more efficient than algorithm A for values of n with $100(80n^{3/2}) \le 2n^2$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

This occurs exactly when $8{,}000n^{3/2} \le 2n^2 \Leftrightarrow 4,000 \le \dfrac{n^2}{n^{3/2}} \Leftrightarrow 4{,}000 \le \sqrt{n} \Leftrightarrow 16{,}000{,}000 \le n$. Thus, algorithm B is at least 100 times more efficient than algorithm A when $n \ge 16{,}000{,}000$.

6. a. There are two multiplications, one addition, and one subtraction for each iteration of the loop, so there are four times as many operations as there are iterations of the loop. The loop is iterated $(n-1)-3+1 = n-3$ times (since the number of iterations equals the top minus the bottom index plus 1). Thus the total number of operations is $4(n-3) = 4n-12$.

b. By the theorem on polynomial orders, $4n-12$ is $\Theta(n)$, so the algorithm segment has order n.

8. a. There is one subtraction for each iteration of the loop, and there are $\lfloor n/2 \rfloor$ iterations of the loop.

b. $\lfloor n/2 \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$

is $\Theta(n)$ by theorem on polynomial orders, so the algorithm segment has order n.

9. a. For each iteration of the inner loop, there are two multiplications and one addition. There are $2n$ iterations of the inner loop for each iteration of the outer loop, and there are n iterations of the outer loop. Therefore, the number of iterations of the inner loop is $2n \cdot n = 2n^2$. It follows that the total number of elementary operations that must be performed when the algorithm is executed is $3 \cdot 2n^2 = 6n^2$.

b. Since $6n^2$ is $\Theta(n^2)$ (by the theorem on polynomial orders), the algorithm segment has order n^2.

11. a. There is one addition for each iteration of the inner loop. The number of iterations in the inner loop can be deduced from the table on the right, which shows the values of k and j for which the inner loop is executed.

Hence the total number of iterations of the inner loop is

$$\begin{aligned} 2+3+\cdots+n &= (1+2+3+\cdots+n)-1 \\ &= \frac{n(n+1)}{2} - 1 = \frac{1}{2}n^2 + \frac{1}{2}n - 1 \end{aligned}$$

(by Theorem 5.2.2). Because one operation is performed for each iteration of the inner loop, the total number of operations is $\frac{1}{2}n^2 + \frac{1}{2}n - 1$.

b. By the theorem on polynomial orders, $\frac{1}{2}n^2 + \frac{1}{2}n - 1$ is $\Theta(n^2)$, and so the algorithm segment has order n^2.

k	1		2			3				...	$n-1$				
j	1	2	1	2	3	1	2	3	4	...	1	2	3	...	n
	2		3			4					n				

14. a. There is one addition for each iteration of the inner loop, and there is one additional addition and one multiplication for each iteration of the outer loop. The number of iterations in the inner loop can be deduced from the following table, which shows the values of i and j for which the inner loop is executed.

i	1	2		3			...	n				
j	1	1	2	1	2	3	...	1	2	3	...	n
	1	2		3				n				

Hence the total number of iterations of the inner loop is

$$\begin{aligned} 1+2+3+\cdots+n &= (1+2+3+\cdots+n) \\ &= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n \end{aligned}$$

(by Theorem 5.2.2). Because one addition is performed for each iteration of the inner loop, the number of operations performed when the inner loop is executed is $\frac{1}{2}n^2 + \frac{1}{2}$. Now an additional two operations are performed each time the outer loop is executed, and because the outer loop is executed n times, this gives an additional $2n$ operations. Therefore, the total number of operations is

$$\frac{1}{2}n^2 + \frac{1}{2}n + 2n = \frac{1}{2}n^2 + \frac{5}{2}n.$$

b. By the theorem on polynomial orders, $\frac{1}{2}n^2 + \frac{5}{2}n$ is $\Theta(n^2)$, and so the algorithm segment has order n^2.

17. a. There are two subtractions and one multipliction for each iteration of the inner loop.

If n is odd, the number of iterations of the inner loop can be deduced from the following table, which shows the values of i and j for which the inner loop is executed.

i	1	2	3		4		5			6			...	$n-1$		...		n		...	
$\left\lfloor \frac{i+1}{2} \right\rfloor$	1	1	2		2		3			3			...	$\frac{n-1}{2}$		...		$\frac{n+1}{2}$		...	
j	1	1	1	2	1	2	1	2	3	1	2	3	...	1	2	...	$\frac{n-1}{2}$	1	2	...	$\frac{n+1}{2}$
	1	1	2		2		3			3				$\frac{n-1}{2}$				$\frac{n+1}{2}$			

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

Thus the number of iterations of the inner loop is

$$
\begin{aligned}
1+1+2+2+\cdots+\frac{n-1}{2}+\frac{n-1}{2}+\frac{n+1}{2}
&= 2\cdot\left(1+2+3+\cdots+\frac{n-1}{2}\right)+\frac{n+1}{2}\\
&= 2\cdot\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)}{2}+\frac{n+1}{2} \qquad \text{by Theorem 5.2.2}\\
&= \frac{n^2-2n+1}{4}+\frac{n-1}{2}+\frac{n+1}{2}\\
&= \frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}.
\end{aligned}
$$

By similar reasoning, if n is even, then the number of iterations of the inner loop is

$$
\begin{aligned}
1+1+2+2+3+3+\cdots+\frac{n}{2}+\frac{n}{2}
&= 2\cdot\left(1+2+3+\cdots+\frac{n}{2}\right)\\
&= 2\cdot\left(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\right) \qquad \text{by Theorem 5.2.2}\\
&= \frac{n^2}{4}+\frac{n}{2}.
\end{aligned}
$$

Because three operations are performed for each iteration of the inner loop, the answer is $3\left(\frac{n^2}{4}+\frac{n}{2}\right)$ when n is even and $3\left(\frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}\right)$ when n is odd.

b. Since $3\left(\frac{n^2}{4}+\frac{n}{2}\right)$ is $\Theta(n^2)$ and $3\left(\frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}\right)$ is also $\Theta(n^2)$ (by the theorem on polynomial orders), this algorithm segment has order n^2.

19. *Hint:* See Section 9.6 for a discussion of how to count the number of iterations of the innermost loop.

20.

	$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
Initial order	6	2	1	8	4
Result of step 1	2	6	1	8	4
Result of step 2	1	2	6	8	4
Result of step 3	1	2	6	8	4
Final order	1	2	4	6	8

22.

n	5												
***a*[1]**	6		2				1						
***a*[2]**	2	6				2							
***a*[3]**	1				6							4	
***a*[4]**	8							8			6		
***a*[5]**	4									8			
k	2			3			4		5				6
x	2			1			8		4				
j	1	0		2	1	0	3		4	3	2		

24. There are 14 comparisons. Each iteration of the while loop involves two comparisons, one to test whether $j \neq 0$ and one in the **if** statement to compare x and $a[j]$. When $k = 2$, the while loop executes one time, giving 2 comparisons; when $k = 3$, it executes twice, giving 4 comparisons, when $k = 4$, it executes once, giving 2 comparisons and when $k = 5$, it executes three times, giving 6 comparisons. Thus the total is $2 + 4 + 2 + 6 = 14$ comparisons.

27. *Hint:* The answer to part (a) is $E_n = 3 + 4 + \cdots + (n + 1)$, which equals $(1 + 2 + 3 + \cdots + (n + 1)) - (1 + 2)$.

28. The top row of the table below shows the initial values of the array, and the bottom row shows the final values. The result for each value of k is shown in a separate row.

$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
5	3	4	6	2
2	3	4	6	5
2	3	4	6	5
2	3	4	6	5
2	3	4	5	6

30.

n	5														
***a*[1]**	5					2									
***a*[2]**	3														
***a*[3]**	4														
***a*[4]**	6													5	
***a*[5]**	2					5								6	
k	1						2			3		4			5
IndexOfMin	1	2			5		2			3		4	5		
i	2	3	4	5			3	4	5	4	5	5			
temp					5								6		

32. There is one comparison for each combination of values of k and i: namely, $4 + 3 + 2 + 1 = 10$.

35. **b.** $n - 3 + 1 = n - 2$ **d.** *Hint:* The answer is n^2.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

36.

n	3									
***a*[0]**	2									
***a*[1]**	1									
***a*[2]**	−1									
***a*[3]**	3									
x	2									
polyval	2		4			0				24
i	1		2			3				
term	1	2	−1	−2	−4	3	6	12	24	
j	1		1	2		1	2	3		

38. Number of multiplications

$$\begin{aligned} &= \text{number of iterations of the inner loop} \\ &= 1 + 2 + 3 + \cdots + n \\ &= \frac{n(n+1)}{2} \qquad \text{by Theorem 5.2.2} \end{aligned}$$

number of additions

$$\begin{aligned} &= \text{number of iterations of the outer loop} \\ &= n \end{aligned}$$

Hence the total number of multiplications and additions is

$$\frac{n(n+1)}{2} + n = \frac{1}{2}n^2 + \frac{3}{2}n.$$

40.

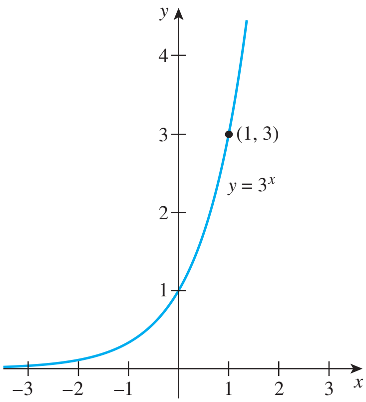
n	3			
***a*[0]**	2			
***a*[1]**	1			
***a*[2]**	−1			
***a*[3]**	3			
x	2			
polyval	3	5	11	24
i	1	2	3	

42. *Hint:* The answer is $t_n = 2n$.

Section 11.4

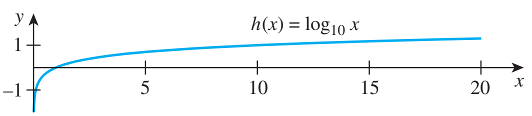
1.

x	$f(x) = 3^x$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$
−1	$3^{-1} = 1/3$
−2	$3^{-2} = 1/9$
1/2	$3^{1/2} \cong 1.7$
−(1/2)	$3^{-(1/2)} \cong 0.6$



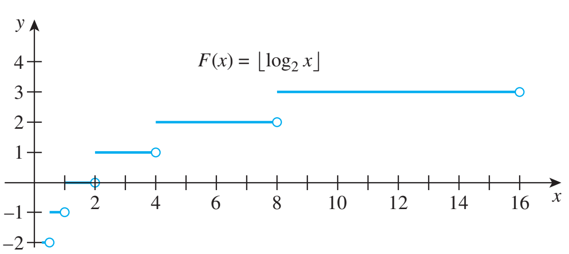
3.

x	$h(x) = \log_{10} x$
1	0
10	1
100	2
1/10	−1
1/100	−2



5.

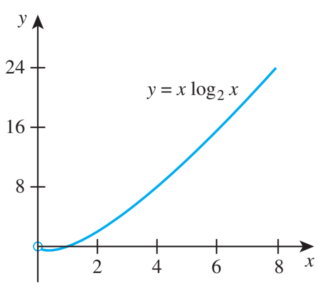
x	$\lfloor \log_2 x \rfloor$
$1 \le x < 2$	0
$2 \le x < 4$	1
$4 \le x < 8$	2
$8 \le x < 16$	3
$1/2 \le x < 1$	−1
$1/4 \le x < 1/2$	−2



Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

7.

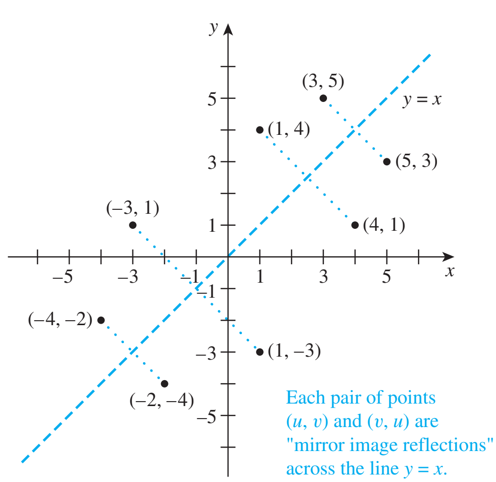
x	$x \log_2 x$
1	$1 \cdot 0 = 0$
2	$2 \cdot 1 = 2$
4	$4 \cdot 2 = 8$
8	$8 \cdot 3 = 24$
1/8	$(1/8) \cdot (-3) = -3/8$
1/4	$(1/4) \cdot (-2) = -1/2$
3/8	$(3/8) \cdot (\log_2(3/8)) \cong -0.53$



9. The distance above the axis is $(2^{64} \text{ units}) \cdot \left(\frac{1}{4}\frac{\text{inch}}{\text{unit}}\right) = \frac{2^{64}}{4}$ inches $= \frac{2^{64}}{4 \cdot 12 \cdot 5280}$ miles $\cong 72{,}785{,}448{,}520{,}000$ miles. The ratio of the height of the point to the average distance of the earth to the sun is approximately $72785448520000/93000000 \cong 782{,}639$. (If you perform the computation using metric units and the approximation $0.635\,\text{cm} \cong 1/4$ inch, the ratio comes out to be approximately 780,912.)

10. b. By definition of logarithm, $\log_b x$ is the exponent to which b must be raised to obtain x. Thus when b is actually raised to this exponent, x is obtained. That is, $b^{\log_b x} = x$.

11. b.



13. *Hints:* (1) $\lfloor \log_{10} x \rfloor = m$, (2) See Example 11.4.1.

15. No. Counterexample: Let $n = 2$. Then $\lceil \log_2(n-1) \rceil = \lceil \log_2 1 \rceil = \lceil 0 \rceil = 0$, whereas $\lceil \log_2 n \rceil = \lceil \log_2 2 \rceil = \lceil 1 \rceil = 1$.

16. *Hint:* The statement is true.

18. $\lfloor \log_2 148206 \rfloor + 1 = 18$

21. a. $a_1 = 1$

$$a_2 = a_{\lfloor 2/2 \rfloor} + 2 = a_1 + 2 = 1 + 2$$
$$a_3 = a_{\lfloor 3/2 \rfloor} + 2 = a_1 + 2 = 1 + 2$$
$$a_4 = a_{\lfloor 4/2 \rfloor} + 2 = a_2 + 2 = (1 + 2) + 2 = 1 + 2 \cdot 2$$
$$a_5 = a_{\lfloor 5/2 \rfloor} + 2 = a_2 + 2 = (1 + 2) + 2 = 1 + 2 \cdot 2$$
$$a_6 = a_{\lfloor 6/2 \rfloor} + 2 = a_3 + 2 = (1 + 2) + 2 = 1 + 2 \cdot 2$$
$$a_7 = a_{\lfloor 7/2 \rfloor} + 2 = a_3 + 2 = (1 + 2) + 2 = 1 + 2 \cdot 2$$
$$a_8 = a_{\lfloor 8/2 \rfloor} + 2 = a_4 + 2 = (1 + 2 \cdot 2) + 2 = 1 + 3 \cdot 2$$
$$a_9 = a_{\lfloor 9/2 \rfloor} + 2 = a_4 + 2 = (1 + 2 \cdot 2) + 2 = 1 + 3 \cdot 2$$
$$\vdots$$
$$a_{15} = a_{\lfloor 15/2 \rfloor} + 2 = a_7 + 2 = (1 + 2 \cdot 2) + 2 = 1 + 3 \cdot 2$$
$$a_{16} = a_{\lfloor 16/2 \rfloor} + 2 = a_8 + 2 = (1 + 3 \cdot 2) + 2 = 1 + 4 \cdot 2$$
$$\vdots$$

Guess:

$a_n = 1 + 2\lfloor \log_2 n \rfloor$

b. Proof: Suppose the sequence $a_1, a_2, a_3, \ldots$ is defined recursively as follows: $a_1 = 1$ and $a_k = a_{\lfloor k/2 \rfloor} + 2$ for all integers $k \geq 2$. We will show by strong mathematical induction that the following property, $P(n)$, is true for all integers $n \geq 2$: $a_n = 1 + 2\lfloor \log n \rfloor$.

Show that P(1) is true: $P(1)$ is the equation $1 + 2\lfloor \log_2 1 \rfloor = 1 + 2 \cdot 0 = 1$, which is the value of a_1.

Show that for any integer $k \geq 1$, if P(i) is true for all integers i from 1 through k, then P(k + 1) is true: Let k be any integer with $k \geq 1$, and suppose $a_i = 1 + 2\lfloor \log_2 i \rfloor$ for all integers i from 1 through k. *[This is the inductive hypothesis.]* We must show that $a_{k+1} = 1 + 2\lfloor \log_2(k + 1) \rfloor$.

Case 1 (k is odd): In this case $k + 1$ is even, and

$$a_{k+1} = a_{\lfloor (k+1)/2 \rfloor} + 2$$

by the recursive definition of $a_1, a_2, a_3, \ldots$

$$= a_{(k+1)/2} + 2$$

because k + 1 is even (Theorem 4.5.2)

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

$= 1 + 2\lfloor \log_2((k+1)/2) \rfloor + 2$
by inductive hypothesis
$= 3 + 2\lfloor \log_2(k+1) - \log_2 2 \rfloor$
by Theorem 7.2.1(b)
$= 3 + 2\lfloor \log_2(k+1) - 1 \rfloor$
because $\log_2 2 = 1$
$= 3 + 2(\lfloor \log_2(k+1) \rfloor - 1)$
because for all real numbers x, $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$
by exercise 15, Section 4.5
$= 1 + 2\lfloor \log_2(k+1) \rfloor$
by algebra.

Case 2 (k is even): In this case $k+1$ is odd, and

$a_{k+1} = a_{\lfloor (k+1)/2 \rfloor} + 2$
by the recursive definition of $a_1, a_2, a_3, \cdots$
$= a_{k/2} + 2$
by Theorem 4.5.2 because k + 1 is odd.
$= 1 + 2\lfloor \log_2(k/2) \rfloor + 2$
by inductive hypothesis
$= 3 + 2\lfloor \log_2 k - \log_2 2 \rfloor$
by Theorem 7.2.1(b)
$= 3 + 2\lfloor \log_2 k - 1 \rfloor$
because $\log_2 2 = 1$
$= 3 + 2(\lfloor \log_2 k \rfloor - 1)$
because for all real numbers x, $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$ by exercise 15, Section 4.5
$= 1 + 2\lfloor \log_2 k \rfloor$
by algebra.
$= 1 + 2\lfloor \log_2(k+1) \rfloor$
by property 11.4.3.

Thus in either case, $a_{k+1} = 1 + 2\lfloor \log_2(k+1) \rfloor$ *[as was to be shown].*

23. *Hint:* When $k \geq 2$, then $k^2 \geq 2k$, and so $k \leq \frac{k^2}{2}$. Hence $\frac{k^2}{2} + k \leq \frac{k^2}{2} + \frac{k^2}{2} = k^2$. Also when $k \geq 2$, then $k^2 > 1$, and so $\frac{1}{2} < \frac{k^2}{2}$. Consequently, $\frac{k^2}{2} + \frac{1}{2} < \frac{k^2}{2} + \frac{k^2}{2} = k^2$.

24. *Hint:* Here is the argument for the inductive step in the case where k is odd and $k+1$ is even.

$$\begin{array}{rrcl} & c_{k+1} & = & 2c_{\lfloor (k+1)/2 \rfloor} + (k+1) \\ & & & \text{by the recursive definition of } c_1, c_2, c_3, \cdots \\ \Rightarrow & c_{k+1} & = & c_{(k+1)/2} + (k+1) \\ & & & \text{by Theorem 4.5.2 because } k+1 \text{ is even} \\ \Rightarrow & & \leq & 2\left\lfloor \frac{k+1}{2} \log_2\left(\frac{k+1}{2}\right) \right\rfloor + (k+1) \\ & & & \text{by inductive hypothesis} \\ \Rightarrow & & \leq & (k+1)(\log_2(k+1) - \log_2 2) + (k+1) \\ & & & \text{by algebra and Theorem 7.2.1(b)} \\ \Rightarrow & & \leq & (k+1)(\log_2(k+1) - 1) + (k+1) \\ & & & \text{because } \log_2 2 = 1 \\ \Rightarrow & & \leq & (k+1)(\log_2(k+1)) \\ & & & \text{by algebra} \end{array}$$

25. *Solution 1:* One way to solve this problem is to compare values for $\log_2 x$ and $x^{1/10}$ for conveniently chosen, large values of x. For instance, if powers of 10 are used, the following results are obtained: $\log_2(10^{10}) = 10\log_2 10 \cong 33.2$ and $(10^{10})^{1/10} = 10^{10 \cdot (1/10)} = 10^1 = 10$. Thus the value $x = 10^{10}$ does not work.

However, since $\log_2(10^{20}) = 20\log_2 10 \cong 66.4$ and $(10^{20})^{1/10} = 10^{20 \cdot (1/10)} = 10^2 = 100$, and since $66.4 < 100$, the value $x = 10^{20}$ works.

Solution 2: Another approach is to use a graphing calculator or computer to sketch graphs of $y = \log_2 x$ and $y = x^{1/10}$, taking seriously the hint to "think big" in choosing the interval size for the x's. A few tries and use of the zoom and trace features make it appear that the graph of $y = x^{1/10}$ crosses above the graph of $y = \log_2 x$ at about 4.9155×10^{17}. Thus, for values of x larger than this, $x^{1/10} > \log_2 x$.

27. As with exercise 25, you can solve this problem either by numerical exploration or by exploring with a graphing calculator or computer. For instance, if you raise 1.0001 to successive large powers of 10, you can find the solution $x = 10^6 = 1{,}000{,}000$. That is,
$(1.0001)^{1000000} > 2.67 \times 10^{43} > 1{,}000{,}000$.
(This is the first power of 10 that works.)

Alternatively, you can use a graphing calculator or computer to sketch graphs of $y_1 = (1.0001)^x$ and $y_2 = x$ and look to see where the graph of $y_1 = (1.0001)^x$ rises above the graph of $y_2 = x$. You will need to zoom in carefully to obtain an accurate answer. If you use this method, you will find that if $x > 116703$, then $(1.0001)^x > x$.

29. $7x^2 + 3x\log_2 x$ is $\Theta(x^2)$.

30. *[To show that $2x + \log_2 x$ is $\Theta(x)$, we must find positive real numbers A, B, and k such that $A|x| \leq |2x + \log_2 x| \leq B|x|$ for all $x > k$.]* It is clear from the graphs of $y = \log_2 x$ and $y = x$ that for all $x > 0$, $\log_2 x \leq x$. Adding $2x$ to both sides gives $2x + \log_2 x \leq 3x$, or, because all terms are positive,

$$|2x + \log_2 x| \leq 3|x|.$$

Also, when $x > 1$, then $\log_2 x > 0$, and so $0 < x + \log_2 x$. Adding x to both sides gives $x < 2x + \log_2 x$. Thus when $x > 1$,

$$|x| \leq |2x + \log_2 x|$$

Therefore, let $k = 1$, $A = 1$, and $B = 3$. Then for all real numbers $x > k$,

$$A|x| \leq |2x + \log_2 x| \leq B|x|$$

and hence, by definition of Θ-notation, $2x + \log_2 x$ is $\Theta(x)$.

32. For all integers $n, 2^n \leq n^2 + 2^n$. Also, by property (11.4.10), there is a real number k such that $n^2 \leq 2^n$ for all $n > k$. Adding 2^n to both sides gives $n^2 + 2^n \leq 2^n + 2^n = 2 \cdot 2^n$. Because all quantities are nonnegative, we can write

$$|2^n| \leq |n^2 + 2^n| \leq 2 \cdot |2^n| \quad \text{for all integers } n > k.$$

Let $A = 1$ and $B = 2$. Then

$$A|2^n| \leq |n^2 + 2^n| \leq B|2^n| \quad \text{for all integers } n > k,$$

and hence, by definition of Θ-notation, $n^2 + 2^n$ is $\Theta(2^n)$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

33. *Hint:* $2^{n+1} = 2 \cdot 2^n$

34. *Hint:* Use a proof by contradiction. Start by supposing that there are positive real numbers B and b such that $4^n \le B \cdot 2^n$ for all real numbers $n > b$, and use the fact that $\frac{4^n}{2^n} = \left(\frac{4}{2}\right)^n = 2^n$ to obtain a contradiction.

35. By Theorem 5.2.3, for all integers $n \ge 0$,

$$1 + 2 + 2^2 + \cdots + 2^n = \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1.$$

Also

$$2^{n+1} - 1 \le 2^{n+1} = 2 \cdot 2^n.$$

Thus, by transitivity of order,

$$1 + 2 + 2^2 + \cdots + 2^n \le 2 \cdot 2^n. \qquad (*)$$

Moreover, if $n > 0$, then

$$2^n \le 1 + 2 + 2^2 + \cdots + 2^n. \qquad (**)$$

Combining (*) and (**) gives

$$1 \cdot 2^n \le 1 + 2 + 2^2 + \cdots + 2^n \le 2 \cdot 2^n,$$

and so, because all parts are positive,

$$1 \cdot |2^n| \le |1 + 2 + 2^2 + \cdots + 2^n| \le 2 \cdot |2^n|.$$

Let $A = 1$, $B = 2$, and $k = 1$. Then for all integers $n > k$,

$$A \cdot |2^n| \le |1 + 2 + 2^2 + \cdots + 2^n| \le B \cdot |2^n|.$$

Thus, by definition of Θ-notation, $1 + 2 + 2^2 + \cdots + 2^n$ is $\Theta(2^n)$.

36. *Hint:* This is similar to the solution for exercise 35. Use the fact that $4 + 4^2 + 4^3 + \cdots + 4^n = 4(1 + 4 + 4^2 + 4^3 + \cdots + 4^{n-1})$.

39. Factor out the n to obtain

$$\begin{aligned}
n + \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^n} \\
&= n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) \\
&= n\left(\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1}\right) && \text{by Theorem 5.2.3} \\
&= n\left(\frac{1 - 2^{n+1}}{2^n(1-2)}\right) && \text{by multiplying numerator and denominator by } 2^{n+1} \\
&= n\left(\frac{2^{n+1} - 1}{2^n}\right) \\
&= n\left(2 - \frac{1}{2^n}\right) && \text{by algebra.}
\end{aligned}$$

Now $1 \le 2 - \frac{1}{2^n} \le 2$ when $n > 1$. Thus

$$1 \cdot n \le n\left(2 - \frac{1}{2^n}\right) \le 2 \cdot n,$$

and so, by substitution,

$$1 \cdot n \le n + \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^n} \le 2 \cdot n.$$

Let $A = 1$, $B = 2$, and $k = 1$. Then, because all quantities are positive, for all integers $n > k$,

$$A \cdot |n| \le \left|n + \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^n}\right| \le B \cdot |n|.$$

Hence, by definition of Θ-notation, $n + \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^n}$ is $\Theta(n)$.

43. If n is any integer with $n \ge 3$, then

$$n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

By Example 11.4.7,

$$\ln(n) \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 2\ln(n).$$

If $n > 1$, then we may multiply through by n and use the fact that all quantities are positive to obtain

$$|n\ln(n)| \le \left|n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n}\right| \le 2\,|n\ln(n)|.$$

Let $A = 1$, $B = 2$, and $k = 1$. Then for all integers $n > k$,

$$A \cdot |n\ln(n)| \le \left|n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n}\right| \le B \cdot |n\ln(n)|$$

and so, by definition of Θ-notation, $n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n}$ is $\Theta(n\ln(n))$.

46. Proof (by mathematical induction): Let the property $P(n)$ be the inequality $n \le 10^n$.

Show that P(1) is true:

When $n = 1$, the inequality is $1 \le 10$, which is true.

Show that for all integers k ≥ 1, if P(k) is true, then P(k + 1) is true:

Let k be any integer with $k \ge 1$, and suppose $k \le 10^k$. *[This is the inductive hypothesis.]* We must show that $k + 1 \le 10^{k+1}$. By inductive hypothesis, $k \le 10^k$. Adding 1 to both sides gives $k + 1 \le 10^k + 1$. But when $k \ge 1$, $10^k + 1 \le 10^k + 9 \cdot 10^k = 10 \cdot 10^k = 10^{k+1}$. Thus, by transitivity of order, $k + 1 \le 10^{k+1}$ *[as was to be shown].*

47. *Hint:* To prove the inductive step, use the fact that if $k > 1$, then $k + 1 \le 2k$. Apply the logarithmic function with base 2 to both sides of this inequality, and use properties of logarithms.

48. *Hint:* $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ factors}} \le 2 \cdot (2 \cdot 3 \cdot 4 \cdots n) = 2 \cdot n!$

49. a. Proof: Suppose n is a variable that takes positive integer values. Then

$$\begin{aligned}
n! &= \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}_{n \text{ factors}} \\
&\le \underbrace{n \cdot n \cdot n \cdot n \cdot \ldots \cdot n}_{n \text{ factors}} = n^n
\end{aligned}$$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

because $(n-1) \le n, (n-2) \le n, \ldots,$ and $1 \le n$. Let $B = 1$ and $b = 1$. It follows from the displayed inequality and the fact that $n!$ and n^n are positive that $|n!| \le B \cdot |n^n|$ for all integers $n > b$. Hence, by definition of O-notation, $n!$ is $O(n^n)$.

c. *Hint:* $(n!)^2 = n! \cdot n! = (1 \cdot 2 \cdot 3 \cdots n)(n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1) = \left(\prod_{r=1}^{n} r\right)\left(\prod_{r=1}^{n}(n-r+1)\right) = \prod_{r=1}^{n} r(n-r+1)$. Show that for all integers $r = 1, 2, \ldots, n, nr - n^2 + r \ge n$.

50. a. Let n be a positive integer. For any real number $x > 1$, properties of exponents and logarithms (see Section 7.2) imply that $0 \le \log_2(x) = \log_2((x^{1/n})^n) = n\log_2(x^{1/n}) < nx^{1/n}$ (where the last inequality holds by substituting $x^{1/n}$ in place of u in $\log_2 u < u$).

b. Let $B = n$ and $b = 1$. Then if $x > x_0$, $|\log_2 x| = \log_2 x \le B \cdot |x^{1/n}|$, and so $\log_2 x$ is $O(x^{1/n})$.

52. Let n be a positive integer, and suppose that $x > (2n)^{2n}$. By properties of logarithms,

$$\log_2 x = (2n)\left(\frac{1}{2n}\right)(\log_2 x)$$
$$= (2n)\log_2\left(x^{\frac{1}{2n}}\right) < 2nx^{\frac{1}{2n}} \qquad (*)$$

(where the last inequality holds by substituting $x^{\frac{1}{2n}}$ in place of u in $\log_2 u < u$). But raising both sides of $x > (2n)^{2n}$ to the 1/2 power gives $x^{1/2} > ((2n)^{2n})^{1/2} = (2n)^n$. When both sides are multiplied by $x^{1/2}$, the result is $x = x^{1/2}x^{1/2} > x^{1/2}(2n)^n = x^{1/2}(2n)^n$, or, more compactly,

$$x^{1/2}(2n)^n < x.$$

Then, since the power function defined by $x \to x^{1/n}$ is increasing for all $x > 0$ (see exercise 21 of Section 11.1), we can take the nth root of both sides of the inequality and use the laws of exponents to obtain

$$(x^{1/2}(2n)^n)^{1/n} < x^{1/n}$$

or, equivalently,

$$2nx^{\frac{1}{2n}} < x^{1/n}. \qquad (**)$$

Now use transitivity of order (Appendix A, T18) to combine (*) and (**) and conclude that $\log_2 x < x^{1/n}$ *[as was to be shown].*

54. Proof (by mathematical induction): Let b be a real number with $b > 1$, and let the property $P(n)$ be the equation

$$\lim_{x\to\infty}\left(\frac{x^n}{b^x}\right) = 0.$$

***Show that $P(1)$ is true*:**

By L'Hôpital's rule, $\lim_{x\to\infty}\left(\frac{x^1}{b^x}\right) = \lim_{x\to\infty}\left(\frac{1}{b^x(\ln b)}\right) = 0$. Thus $P(1)$ is true.

***Show that for all integers $k \ge 1$, if $P(k)$ is true, then $P(k+1)$ is true*:**

Let k be any integer with $k \ge 1$, and suppose $\lim_{x\to\infty}\left(\frac{x^k}{b^x}\right) = 0$. *[This is the inductive hypothesis.]* We must show that $\lim_{x\to\infty}\left(\frac{x^{k+1}}{b^x}\right) = 0$. But by L'Hôpital's rule, $\lim_{x\to\infty}\frac{x^{k+1}}{b^x} = \lim_{x\to\infty}\frac{(k+1)x^k}{(\ln b)b^x} = \frac{(k+1)}{(\ln b)}\lim_{x\to\infty}\frac{x^k}{b^x} = \frac{(k+1)}{(\ln b)} \cdot 0$ *[by inductive hypothesis]* $= 0$. *[This is what was to be shown.]*

b. By the result of part (a) and the definition of limit, given any real number $\varepsilon > 0$, there exists an integer N such that $|\frac{x^n}{b^n} - 0| < \varepsilon$ for all $x > N$. In this case take $\varepsilon = 1$. It follows that for all $x > N$, $|\frac{x^n}{b^x}| = |\frac{x^n}{b^x}| < 1$. Multiply both sides by $|b^x|$ to obtain $|x^n| < |b^x|$. Let $B = 1$ and $b = N$. Then $|x^n| < B \cdot |b^x|$ for all $x > b$. Hence, by definition of O-notation, x^n is $O(b^x)$.

Section 11.5

1. $\log_2 1000 = \log_2(10^3) = 3\log_2 10 \cong 3(3.32) \cong 9.96$
$\log_2(1{,}000{,}000) = \log_2(10^6) = 6\log_2 10 \cong 6(3.32) \cong 19.92$
$\log_2(1{,}000{,}000{,}000{,}000) = \log_2(10^{12}) = 12\log_2 10 \cong 12(3.32) = 39.84$

2. a. If $m = 2^k$, where k is a positive integer, then the algorithm requires $c\lfloor\log_2(2^k)\rfloor = c\lfloor k\rfloor = ck$ operations. If the input size is increased to $m^2 = (2^k)^2 = 2^{2k}$, then the number of operations required is $c\lfloor\log_2(2^{2k})\rfloor = c\lfloor 2k\rfloor = 2(ck)$. Hence the number of operations doubles.

b. As in part (a), for an input of size $m = 2^k$, where k is a positive integer, the algorithm requires ck operations. If the input size is increased to $m^{10} = (2^k)^{10} = 2^{10k}$, then the number of operations required is $c\lfloor\log_2(2^{10k})\rfloor = c\lfloor 10k\rfloor = 10(ck)$. Thus the number of operations increases by a factor of 10.

c. When the input size is increased from 2^7 to 2^{28}, the factor by which the number of operations increases is $\frac{c\lfloor\log_2(2^{28})\rfloor}{c\lfloor\log_2(2^7)\rfloor} = \frac{28c}{7c} = 4$.

3. A little numerical exploration can help find an initial window to use to draw the graphs of $y = x$ and $y = \lfloor 50\log_2 x\rfloor$. Note that when $x = 2^8 = 256$, $\lfloor 50\log_2 x\rfloor = \lfloor 50\log_2(2^8)\rfloor = \lfloor 50 \cdot 8\rfloor = \lfloor 400\rfloor = 400 > 256 = x$. But when $x = 2^9 = 512$, $\lfloor 50\log_2 x\rfloor = \lfloor 50\log_2(2^9)\rfloor = \lfloor 50 \cdot 9\rfloor = \lfloor 450\rfloor = 450 < 512 = x$. So a good choice of initial window would be the interval from 256 to 512. Drawing the graphs, zooming if necessary, and using the trace feature reveal that when $n < 438$, $n < \lfloor 50\log_2 n\rfloor$.

5. a.

index	0			1
bot	1			
top	10	4	1	
mid		5	2	1

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

b.

index	0				
bot	1	6		7	
top	10		7		6
mid		5	8	6	7

7. a. $top - bot + 1$

b. Proof: Suppose *top* and *bot* are particular but arbitrarily chosen positive integers such that $top - bot + 1$ is an odd number. Then, by definition of odd, there is an integer k such that

$$top - bot + 1 = 2k + 1$$

Adding $2 \cdot bot - 1$ to both sides gives

$$\begin{aligned} bot + top &= 2 \cdot bot - 1 + 2k + 1 \\ &= 2(bot + k). \end{aligned}$$

But $bot + k$ is an integer. Hence, by definition of even, $bot + top$ is even.

8.

n	27	13	6	3	1	0

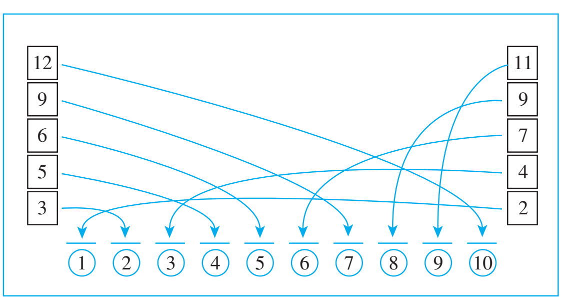
9. For each positive integer n, $n \; div \; 2 = \lfloor n/2 \rfloor$. Thus when the algorithm segment is run for a particular n and the **while** loop has iterated one time, the input to the next iteration is $\lfloor n/2 \rfloor$. It follows that the number of iterations of the loop for n is one more than the number of iterations for $\lfloor n/2 \rfloor$. That is, $a_n = 1 + a_{\lfloor n/2 \rfloor}$. Also $a_1 = 1$.

10. The recurrence relation and initial condition of $a_1, a_2, a_3, \ldots$ derived in exercise 9 are the same as those for the sequence $w_1, w_2, w_3, \ldots$ discussed in the worst-case analysis of the binary search algorithm. Thus the general formulas for the two sequences are the same. That is, $a_n = 1 + \lfloor \log_2 n \rfloor$, for all integers $n \geq 1$.

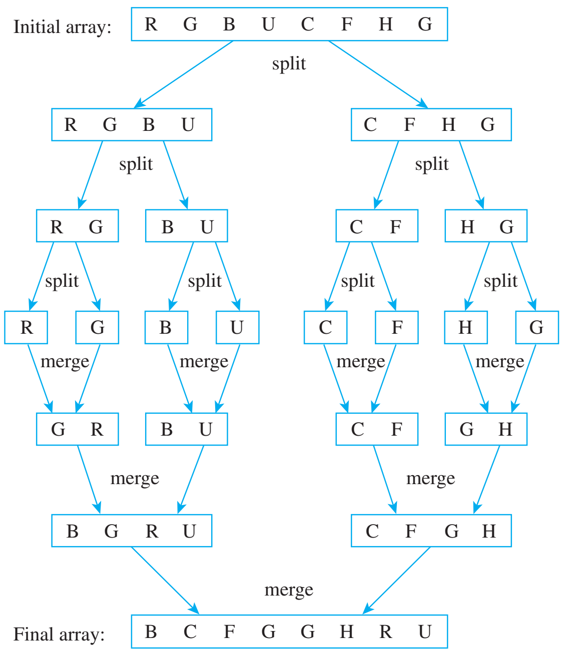
11. In the analysis of the binary search algorithm, it was shown that $1 + \lfloor \log_2 n \rfloor$ is $\Theta(\log_2 n)$. Thus the algorithm segment has order $\log_2 n$.

14. *Hint:* The formula is $b_n = 1 + \lfloor \log_3 n \rfloor$.

20.



22.



24. b. Refer to Figure 11.5.3 and observe that when k is odd, the subarray $a[bot], a[bot+1], \ldots, a[mid]$ has length $(k+1)/2 = \lceil k/2 \rceil$ and that when k is even, it also has length $k/2 = \lceil k/2 \rceil$.

25. *Hint:* The following are the steps for part (a) in the case where k is odd and $k+1$ is even:

$$\begin{aligned}
& m_{k+1} &=& \; m_{\lfloor (k+1)/2 \rfloor} + m_{\lceil (k+1)/2 \rceil} + (k+1) - 1 \\
\Rightarrow \; & m_{k+1} &=& \; m_{(k+1)/2} + m_{(k+1)/2} + (k+1) - 1 \\
& & & \text{by Theorem 4.5.2 and exercise 19 in} \\
& & & \text{Section 4.5 because } k+1 \text{ is even} \\
\Rightarrow \; & m_{k+1} &=& \; 2m_{(k+1)/2} + k \\
\Rightarrow \; & m_{k+1} &\geq& \; 2 \cdot \left[\tfrac{1}{2} \cdot \left(\tfrac{k+1}{2} \right) \log_2 \left(\tfrac{k+1}{2} \right) \right] + k \\
& & & \text{by inductive hypothesis} \\
\Rightarrow \; & m_{k+1} &\geq& \; \left(\tfrac{k+1}{2} \right) [\log_2 (k+1) - \log_2 2] + k \\
\Rightarrow \; & m_{k+1} &\geq& \; \tfrac{1}{2}(k+1)[\log_2 (k+1) - 1] + k \\
\Rightarrow \; & m_{k+1} &\geq& \; \tfrac{1}{2}(k+1) \log_2 (k+1) - \left(\tfrac{k+1}{2} \right) + \tfrac{2k}{2} \\
\Rightarrow \; & m_{k+1} &\geq& \; \tfrac{1}{2}(k+1) \log_2 (k+1) + \tfrac{k-1}{2} \\
\Rightarrow \; & m_{k+1} &\geq& \; \tfrac{1}{2}(k+1) \log_2 (k+1)
\end{aligned}$$

Section 12.1

1. a. $L_1 = \{\epsilon, x, y, xx, yy, xxx, xyx, yxy, yyy, xxxx, xyyx, yxxy, yyyy\}$

b. $L_2 = \{x, xx, xy, xxx, xxy, xyx, xyy\}$

3. a. $(a+b) \cdot (c+d)$

b. *Partial answer*: $11* = 1 \cdot 1 = 1$, $\quad 12* = 1 \cdot 2 = 2$, $21/ = 2/1 = 2$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

4. L_1L_2 is the set of all strings of a's and b's that start with an a and contain an odd number of a's.

$L_1 \cup L_2$ is the set of all strings of a's and b's that contain an even number of a's or that start with an a and contain only that one a. (Note that because 0 is an even number, both ϵ and b are in $L_1 \cup L_2$.)

$(L_1 \cup L_2)^*$ is the set of all strings of a's and b's. The reason is that a and b are both in $L_1 \cup L_2$, and thus every string in a and b is in $(L_1 \cup L_2)^*$.

7. $(a \mid ((b^*)b))((a^*) \mid (ab))$

10. $(ab^* \mid cb^*)(ac \mid bc)$

13.
$$\begin{aligned} L(\epsilon \mid ab) &= L(\epsilon) \cup L(ab) = \{\epsilon\} \cup L(a)L(b) \\ &= \{\epsilon\} \cup \{xy \mid x \in L(a) \text{ and } y \in L(b)\} \\ &= \{\epsilon\} \cup \{xy \mid x \in \{a\} \text{ and } y \in \{b\}\} \\ &= \{\epsilon\} \cup \{ab\} = \{\epsilon, ab\} \end{aligned}$$

16. Here are five strings out of infinitely many: 0101, 1, 01, 10000, and 011100.

19. The language consists of all strings of a's and b's that contain exactly three a's and end in an a.

22. *aaaba* is in the language but *baabb* is not because if a string in the language contains a b to the right of the left-most a, then it must contain another a to the right of the all b's.

25. One solution is $0^*10^*(0^*10^*10^*)^*$.

28.
$$\begin{aligned} L((r \mid s)t) &= L(r \mid s)L(t) = (L(r) \cup L(s))L(t) \\ &= \{xy \mid x \in (L(r) \cup L(s)) \text{ and } y \in L(t)\} \\ &= \{xy \mid (x \in L(r) \text{ or } x \in L(s)) \text{ and } y \in L(t)\} \\ &= \{xy \mid (x \in L(r) \text{ and } y \in L(t)) \text{ or } \\ &\qquad (x \in L(s) \text{ and } y \in L(t))\} \\ &= \{xy \mid xy \in L(rt) \text{ or } xy \in L(st)\} \\ &= L(rt) \cup L(st) \\ &= L(rt \mid st) \end{aligned}$$

31. $pre[a-z]^+$

34. $[a-z]^*(a \mid e \mid i \mid o \mid u)[a-z]^*$

37. $[0-9]\{3\}$ - $[0-9]\{2\}$ - $[0-9]\{4\}$

39. $([+-] \mid \epsilon)[0-9]^*(\backslash. \mid \epsilon)[0-9]^*$

40. *Hint:* Leap years from 1980 to 2079 are 1980, 1984, 1988, 1992, 1996, 2000, 2004, etc. Note that the fourth digit is 0, 4, or 8 for the ones whose third digit is even and that the fourth digit is 2 or 6 for those whose third digit is odd.

Section 12.2

1. a. \$1 or more deposited

2. a. s_0, s_1, s_2 **b.** 0, 1 **c.** s_0 **d.** s_2

e. Annotated next-state table:

	State	Input 0	Input 1
→	s_0	s_1	s_0
	s_1	s_1	s_2
◎	s_2	s_2	s_2

5. a. A, B, C, D, E, F **b.** x, y **c.** A **d.** D, E

e. Annotated next-state table:

	State	Input x	Input y
→	A	C	B
	B	F	D
	C	E	F
◎	D	F	D
◎	E	E	F
	F	F	F

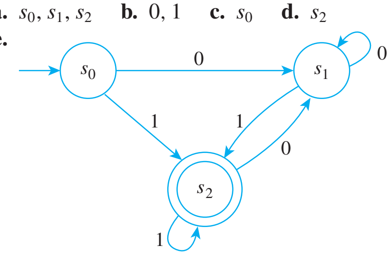
7. a. s_0, s_1, s_2, s_3 **b.** 0, 1 **c.** s_0 **d.** s_0, s_2

e. Annotated next-state table:

	State	Input 0	Input 1
→ ◎	s_0	s_0	s_1
	s_1	s_1	s_2
◎	s_2	s_2	s_3
	s_3	s_3	s_0

8. a. s_0, s_1, s_2 **b.** 0, 1 **c.** s_0 **d.** s_2

e.



10. a. $N(s_1, 1) = s_2$, $N(s_0, 1) = s_3$

c. $N^*(s_0, 10011) = s_2$, $N^*(s_1, 01001) = s_2$

11. a. $N(s_3, 0) = s_4$, $N(s_2, 1) = s_4$

c. $N^*(s_0, 010011) = s_3$, $N^*(s_3, 01101) = s_4$

Note that multiple correct answers exist for part (d) of exercises 12 and 13, part (b) of exercises 14–19, and for exercises 20–48.

12. a. (i) s_2 (ii) s_2 (iii) s_1

b. those in (i) and (ii) but not (iii)

c. The language accepted by this automaton is the set of all strings of 0's and 1's that contain at least one 0 followed (not necessarily immediately) by at least one 1.

d. $1^*00^*1(0 \mid 1)^*$

14. a. The language accepted by this automaton is the set of all strings of 0's and 1's that end 00.

b. $(0 \mid 1)^*00$

15. a. The language accepted by this automaton is the set of all strings of x's and y's of length at least two that consist either entirely of x's or entirely of y's.

b. $xxx^* \mid yyy^*$

17. a. The language accepted by this automaton is the set of all strings of 0's and 1's with the following property: If n is the number of 1's in the string, then $n \bmod 4 = 0$ or $n \bmod 4 = 2$. This is equivalent to saying that n is even.

b. $0^* \mid (0^*10^*10^*)^*$

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

18. a. The language accepted by this automaton is the set of all strings of 0's and 1's that end in 1.

b. $(0 \mid 1)^*1$

20. a. Call the automaton being constructed A. Acceptance of a string by A depends on the values of three consecutive inputs. Thus A requires at least four states:

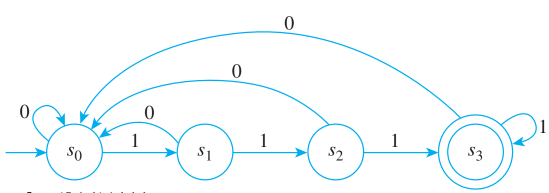
s_0: initial state

s_1: state indicating that the last input character was a 1

s_2: state indicating that the last two input characters were 1's

s_3: state indicating that the last three input characters were 1's, the acceptance state

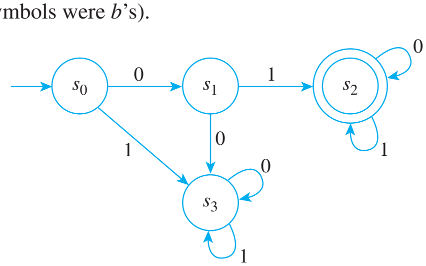
If a 0 is input to A when it is in state s_0, no progress is made toward achieving a string of three consecutive 1's. Hence A should remain in state s_0. If a 1 is input to A when it is in state s_0, it goes to state s_1, which indicates that the last input character of the string is a 1. From state s_1, A goes to state s_2 if a 1 is input. This indicates that the last two characters of the string are 1's. But if a 0 is input, A should return to s_0 because the wait for a string of three consecutive 1's must start all over again. When A is in state s_2 and a 1 is input, then a string of three consecutive 1's is achieved, so A should go to state s_3. If a 0 is input when A is in state s_2, then progress toward accumulating a sequence of three consecutive 1's is lost, so A should return to s_0. When A is in a state s_3 and a 1 is input, then the final three symbols of the input string are 1's, and so A should stay in state s_3. If a 0 is input when A is in state s_3, then A should return to state s_0 to await the input of more 1's. Thus the transition diagram is as follows:



b. $(0 \mid 1)^*111$

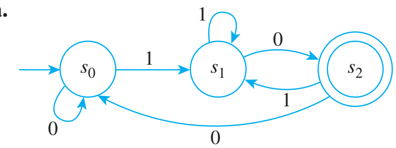
21. *Hint:* Use five states: s_0 (the initial state), s_1 (the state indicating that the previous input symbol was an a), s_2 (the state indicating that the previous input symbol was a b), s_3 (the state indicating that the previous two input symbols were a's), and s_4 (the state indicating that the previous two input symbols were b's).

23. a.



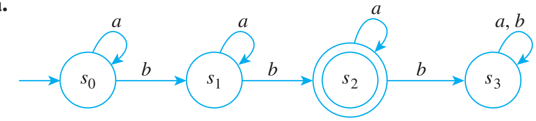
b. $01(0 \mid 1)^*$

25. a.



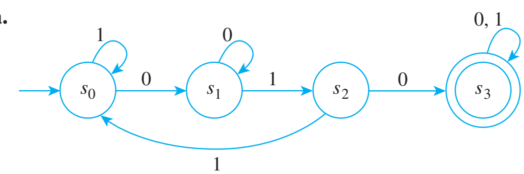
b. $(0 \mid 1)^*10$

26. a.



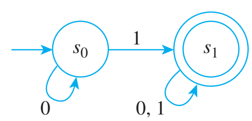
b. $a^*ba^*ba^*$

28. a.

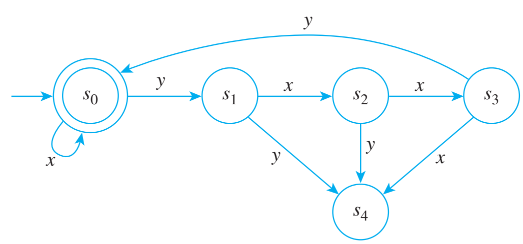


b. $(0 \mid 1)^*010(0 \mid 1)^*$

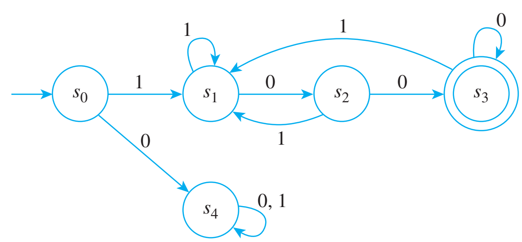
29.



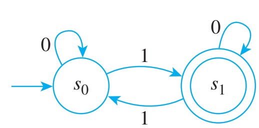
31.



33.

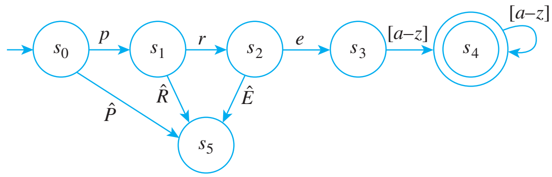


36.

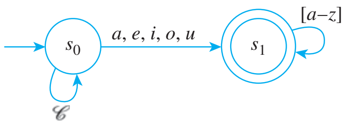


39. Let $\hat{P}$ denote a list of all letters of a lower-case alphabet except p, $\hat{R}$ denote a list of all the letters of a lower-case alphabet except r, and $\hat{E}$ denote a list of all the letters of a lower-case alphabet except e.

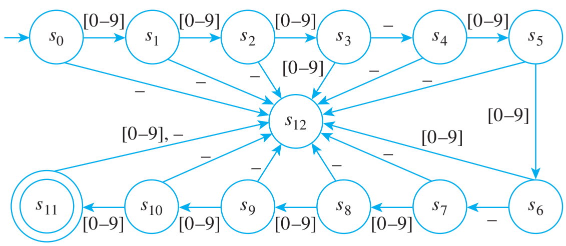
Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s).
Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.



42. Let $\mathscr{C}$ denote a list of all the consonants in a lower-case alphabet.

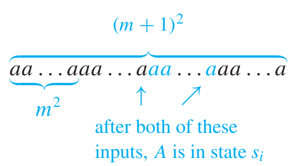


45.



51. *Hint:* This proof is virtually identical to that of Example 12.2.8. Just take p and q in that proof so that $p > q$. From the fact that A accepts $a^p b^p$, you can deduce that A accepts $a^q b^p$. Since $p > q$, this string is not in L.

53. *Hint:* Suppose the automaton A has N states. Choose an integer m such that $(m+1)^2 - m^2 > N$. Consider strings of a's of lengths between m^2 and $(m+1)^2$.
Since there are more strings than states, at least two strings must send A to the same state s_i:

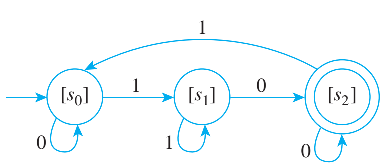


It follows (by removing the a's shown in color) that the automaton must accept a string of the form a^k, where $m^2 < k < (m+1)^2$.

Section 12.3

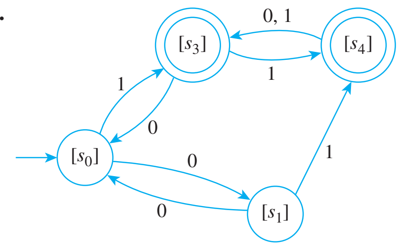
1. a. 0-equivalence classes: $\{s_0, s_1, s_3, s_4\}, \{s_2, s_5\}$
1-equivalence classes: $\{s_0, s_3\}, \{s_1, s_4\}, \{s_2, s_5\}$
2-equivalence classes: $\{s_0, s_3\}, \{s_1, s_4\}, \{s_2, s_5\}$

b.



4. a. 0-equivalence classes: $\{s_0, s_1, s_2\}, \{s_3, s_4, s_5\}$
1-equivalence classes: $\{s_0, s_1, s_2\}, \{s_3, s_5\}, \{s_4\}$
2-equivalence classes: $\{s_0, s_2\}, \{s_1\}, \{s_3, s_5\}\{s_4\}$
3-equivalence classes: $\{s_0, s_2\}, \{s_1\}, \{s_3, s_5\}, \{s_4\}$

b.

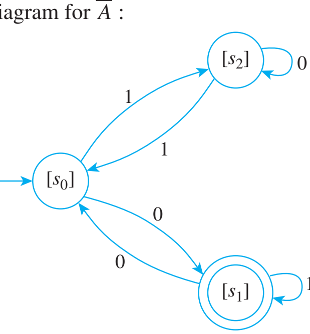


6. a. *Hint:* The 3-equivalence classes are $\{s_0\}, \{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}, \{s_5\}$, and $\{s_6\}$.

7. Yes. For A:

0-equivalence classes: $\{s_0, s_2\}, \{s_1, s_3\}$
1-equivalence classes: $\{s_0\}, \{s_2\}, \{s_1, s_3\}$
2-equivalence classes: $\{s_0\}, \{s_2\}, \{s_1, s_3\}$

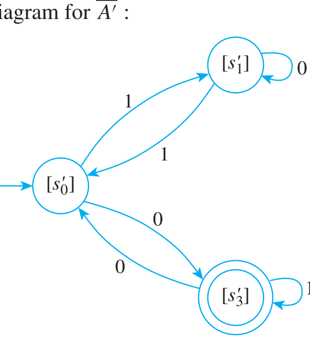
Transition diagram for $\overline{A}$:



For A':

0-equivalence classes: $\{s'_0, s'_1, s'_2\}, \{s'_3\}$
1-equivalence classes: $\{s'_0, s'_2\}, \{s'_1\}, \{s'_3\}$
2-equivalence classes: $\{s'_0, s'_2\}, \{s'_1\}, \{s'_3\}$

Transition diagram for $\overline{A'}$:



Except for the labeling of the states, the transition diagrams for $\overline{A}$ and $\overline{A'}$ are identical. Hence $\overline{A}$ and $\overline{A'}$ accept the same language, and so, by Theorem 12.3.3, A and A' also accept the same language. Thus A and A' are equivalent automata.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.

9. For A:

0-equivalence classes: $\{s_1, s_2, s_4, s_5\}, \{s_0, s_3\}$
1-equivalence classes: $\{s_1, s_2\}, \{s_4, s_5\}, \{s_0, s_3\}$
2-equivalence classes: $\{s_1\}, \{s_2\}, \{s_4, s_5\}, \{s_0, s_3\}$
3-equivalence classes: $\{s_1\}, \{s_2\}, \{s_4, s_5\}, \{s_0, s_3\}$

Therefore, the states of $\overline{A}$ are the 3-equivalence classes of A.

For A':

0-equivalence classes: $\{s'_2, s'_3, s'_4, s'_5\}, \{s'_0, s'_1\}$
1-equivalence classes: $\{s'_2, s'_3, s'_4, s'_5\}, \{s'_0, s'_1\}$

Therefore, the states of $\overline{A'}$ are the 1-equivalence classes of A'.

According to the text, two automata are equivalent if, and only if, their quotient automata are isomorphic, provided inaccessible states have first been removed. Now A and A' have no inaccessible states, and $\overline{A}$ has four states whereas $\overline{A'}$ has only two states. Therefore, A and A' are not equivalent.

This result can also be obtained by noting, for example, that the string 11 is accepted by A' but not by A.

11. *Partial answer:* Suppose A is a finite-state automaton with set of states S and relation R_* of $*$-equivalence of states. *[To show that R_* is an equivalence relation, we must show that R is reflexive, symmetric, and transitive.]*

Proof that R_* is symmetric:

[We must show that for all states s and t, if $s\ R_*\ t$ then $t\ R_*\ s$.] Suppose that s and t are states of A such that sR_*t. *[We must show that $t\ R_*\ s$.]* Since $s\ R_*\ t$, then for all input strings w,

$$\begin{bmatrix}N^*(s, w) \text{ is an}\\ \text{accepting state}\end{bmatrix} \Leftrightarrow \begin{bmatrix}N^*(t, w) \text{ is an}\\ \text{accepting state}\end{bmatrix}$$

where N^* is the eventual-state function on A. But then, by symmetry of the $\Leftrightarrow$ relation, it is true that for all input strings w,

$$\begin{bmatrix}N^*(t, w) \text{ is an}\\ \text{accepting state}\end{bmatrix} \Leftrightarrow \begin{bmatrix}N^*(s, w) \text{ is an}\\ \text{accepting state}\end{bmatrix}$$

Hence $t\ R_*\ s$ *[as was to be shown]*, so R_* is symmetric.

12. The proof is identical to the proof of property (12.3.1) given in the solution to exercise 11 provided each occurrence of "for all input strings w" is replaced by "for all input strings w of length less than or equal to k."

13. Proof: By property (12.3.2), for each integer $k \geq 0$, k-equivalence is an equivalence relation. But by Theorem 10.3.4, the distinct equivalence classes of an equivalence relation form a partition of the set on which the relation is defined. In this case, the relation is defined on the states of the automaton. So the k-equivalence classes form a partition of the set of all states of the automaton.

15. *Hint 1:* Suppose C_k is a particular but arbitrarily chosen k-equivalence class. You must show that there is a $(k-1)$-equivalence class C_{k-1} such that $C_k \subseteq C_{k-1}$.

Hint 2: If s is any element in C_k, then s is a state of the automaton. Now the $(k-1)$-equivalence classes partition the set of all states of the automaton into a union of mutually disjoint subsets, so $s \in C_{k-1}$ for *some* $(k-1)$-equivalence class C_{k-1}.

Hint 3: To show that $C_k \subseteq C_{k-1}$, you must show that for any state t, if $t \in C_k$, then $t \in C_{k-1}$.

17. *Hint:* If $m < k$, then every input string of length less than or equal to m has length less than or equal to k.

19. *Hint:* Suppose two states s and t are equivalent. You must show that for any input symbol m, the next-states $N(s, m)$ and $N(t, m)$ are equivalent. To do this, use the definition of equivalence and the fact that for any string w', input symbol m, and state s, $N^*(N(s, m), w') = N^*(s, mw')$.

Copyright 2010 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.