Graph Coloring

INTRODUCTION

Links

Problems related to the coloring of maps of regions, such as maps of parts of the world, have generated many results in graph theory. When a map* is colored, two regions with a common border are customarily assigned different colors. One way to ensure that two adjacent regions never have the same color is to use a different color for each region. However, this is inefficient, and on maps with many regions it would be hard to distinguish similar colors. Instead, a small number of colors should be used whenever possible. Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color. For instance, for the map shown on the left in Figure 1, four colors suffice, but three colors are not enough. (The reader should check this.) In the map on the right in Figure 1, three colors are sufficient (but two are not).

Each map in the plane can be represented by a graph. To set up this correspondence, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent. The resulting graph is called the **dual graph** of the map. By the way in which dual graphs of maps are constructed, it is clear that any map in the plane has a planar dual graph. Figure 2 displays the dual graphs that correspond to the maps shown in Figure 1.

The problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color. We now define a graph coloring.

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*We will assume that all regions in a map are connected. This eliminates any problems presented by such geographical entities as Michigan.

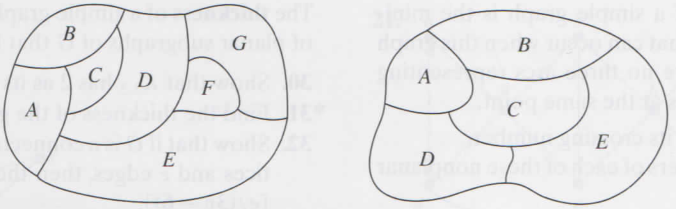

FIGURE 1 Two Maps.

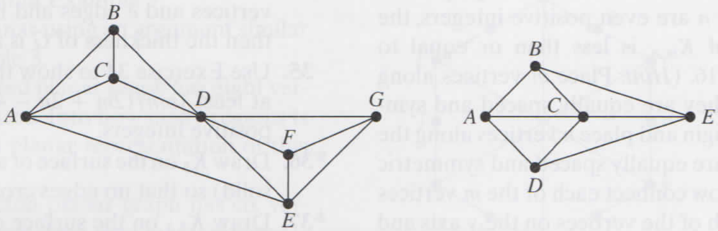

FIGURE 2 Dual Graphs of the Maps in Figure 1.

DEFINITION 1 A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

A graph can be colored by assigning a different color to each of its vertices. However, for most graphs a coloring can be found that uses fewer colors than the number of vertices in the graph. What is the least number of colors necessary?

DEFINITION 2 The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph.

Note that asking for the chromatic number of a planar graph is the same as asking for the minimum number of colors required to color a planar map so that no two adjacent regions are assigned the same color. This question has been studied for more than 100 years. The answer is provided by one of the most famous theorems in mathematics.

THEOREM 1 **THE FOUR COLOR THEOREM** The chromatic number of a planar graph is no greater than four.

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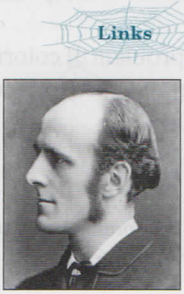


ALFRED BRAY KEMPE (1849–1922) Kempe was a barrister and a leading authority on ecclesiastical law. However, having studied mathematics at Cambridge University, he retained his interest in it, and later in life he devoted considerable time to mathematical research. Kempe made contributions to kinematics, the branch of mathematics dealing with motion, and to mathematical logic. However, Kempe is best remembered for his fallacious proof of the Four Color Theorem.

The Four Color Theorem was originally posed as a conjecture in the 1850s. It was finally proved by the American mathematicians Kenneth Appel and Wolfgang Haken in 1976. Prior to 1976, many incorrect proofs were published, often with hard-to-find errors. In addition, many futile attempts were made to construct counterexamples by drawing maps that require more than four colors. (Proving the Five Color Theorem is not that difficult; see Exercise 35.)

Perhaps the most notorious fallacious proof in all of mathematics is the incorrect proof of the Four Color Theorem published in 1879 by a London barrister and amateur mathematician, Alfred Kempe. Mathematicians accepted his proof as correct until 1890, when Percy Heawood found an error that made Kempe's argument incomplete. However, Kempe's line of reasoning turned out to be the basis of the successful proof given by Appel and Haken. Their proof relies on a careful case-by-case analysis carried out by computer. They showed that if the Four Color Theorem were false, there would have to be a counterexample of one of approximately 2000 different types, and they then showed that none of these types exists. They used over 1000 hours of computer time in their proof. This proof generated a large amount of controversy, since computers played such an important role in it. For example, could there be an error in a computer program that led to incorrect results? Was their argument really a proof if it depended on what could be unreliable computer output?

Note that the Four Color Theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers, as will be shown in Example 2.

Two things are required to show that the chromatic number of a graph is k. First, we must show that the graph can be colored with k colors. This can be done by constructing such a coloring. Second, we must show that the graph cannot be colored using fewer than k colors. Examples 1–4 illustrate how chromatic numbers can be found.

EXAMPLE 1 What are the chromatic numbers of the graphs G and H shown in Figure 3?

Solution: The chromatic number of G is at least three, since the vertices a, b, and c must be assigned different colors. To see if G can be colored with three colors, assign red to a, blue to b, and green to c. Then, d can (and must) be colored red since it is adjacent to b and c. Furthermore, e can (and must) be colored green since it is adjacent only to vertices colored red and blue, and f can (and must) be colored blue since it is adjacent only to vertices colored red and green. Finally, g can (and must) be colored red since it is adjacent only to vertices colored blue and green. This produces a coloring of G using exactly three colors. Figure 4 displays such a coloring.

HISTORICAL NOTE In 1852, an ex-student of Augustus De Morgan, Francis Guthrie, noticed that the counties in England could be colored using four colors so that no adjacent counties were assigned the same color. On this evidence, he conjectured that the Four Color Theorem was true. Francis told his brother Frederick, at that time a student of De Morgan, about this problem. Frederick in turn asked his teacher De Morgan about his brother's conjecture. De Morgan was extremely interested in this problem and publicized it throughout the mathematical community. In fact, the first written reference to the conjecture can be found in a letter from De Morgan to Sir William Rowan Hamilton. Although De Morgan thought Hamilton would be interested in this problem, Hamilton apparently was not interested in it, since it had nothing to do with quaternions.

HISTORICAL NOTE Although a simpler proof of the Four Color Theorem was found by Robertson, Sanders, Seymour, and Thomas in 1996, reducing the computational part of the proof to examining 633 configurations, no proof that does not rely on extensive computation has yet been found.

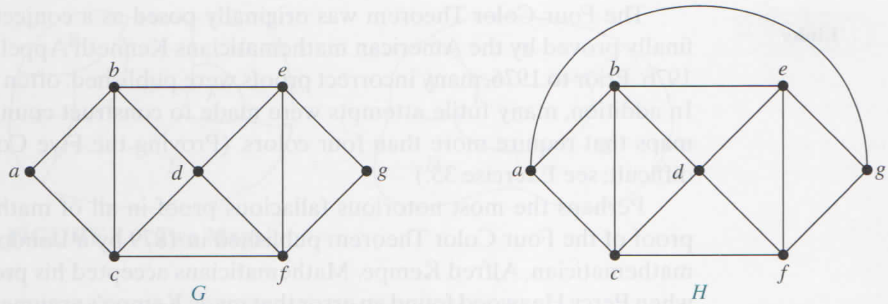


FIGURE 3 The Simple Graphs G and H.

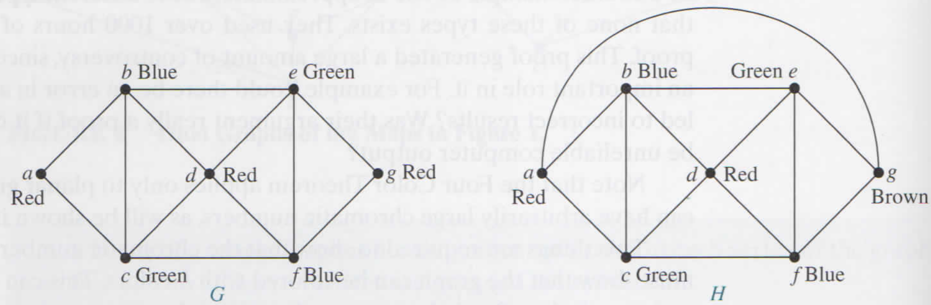


FIGURE 4 Colorings of the Graphs G and H.

The graph H is made up of the graph G with an edge connecting a and g. Any attempt to color H using three colors must follow the same reasoning as that used to color G, except at the last stage, when all vertices other than g have been colored. Then, since g is adjacent (in H) to vertices colored red, blue, and green, a fourth color, say brown, needs to be used. Hence, H has a chromatic number equal to 4. A coloring of H is shown in Figure 4. ◀

EXAMPLE 2 What is the chromatic number of K_n?

Solution: A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. Is there a coloring using fewer colors? The answer is no. No two vertices can be assigned the same color, since every two vertices of this graph are adjacent. Hence, the chromatic number of $K_n = n$. (Recall that K_n is not planar when $n \geq 5$, so this result does not contradict the Four Color Theorem.) A coloring of K_5 using five colors is shown in Figure 5. ◀

EXAMPLE 3 What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers?

Solution: The number of colors needed may seem to depend on m and n. However, only two colors are needed. Color the set of m vertices with one color and the set of n vertices with a second color. Since edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color. A coloring of $K_{3,4}$ with two colors is displayed in Figure 6. ◀

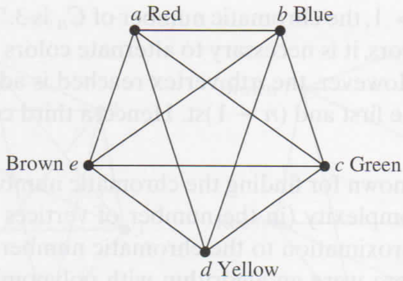


FIGURE 5 **A Coloring of K_5.**

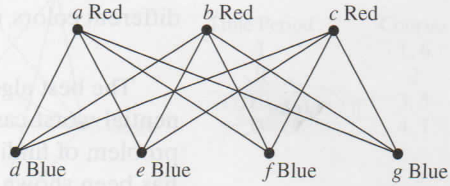


FIGURE 6 **A Coloring of $K_{3,4}$.**

Every connected bipartite simple graph has a chromatic number of 2, or 1, since the reasoning used in Example 3 applies to any such graph. Conversely, every graph with a chromatic number of 2 is bipartite. (See Exercises 25 and 26 at the end of this section.)

EXAMPLE 4 What is the chromatic number of the graph C_n? (Recall that C_n is the cycle with n vertices.)

Solution: We will first consider some individual cases. To begin, let $n = 6$. Pick a vertex and color it red. Proceed clockwise in the planar depiction of C_6 shown in Figure 7. It is necessary to assign a second color, say blue, to the next vertex reached. Continue in the clockwise direction; the third vertex can be colored red, the fourth vertex blue, and the fifth vertex red. Finally, the sixth vertex, which is adjacent to the first, can be colored blue. Hence, the chromatic number of C_6 is 2. Figure 7 displays the coloring constructed here.

Next, let $n = 5$ and consider C_5. Pick a vertex and color it red. Proceeding clockwise, it is necessary to assign a second color, say blue, to the next vertex reached. Continuing in the clockwise direction, the third vertex can be colored red, and the fourth vertex can be colored blue. The fifth vertex cannot be colored either red or blue, since it is adjacent to the fourth vertex and the first vertex. Consequently, a third color is required for this vertex. Note that we would have also needed three colors if we had colored vertices in the counterclockwise direction. Thus, the chromatic number of C_5 is 3. A coloring of C_5 using three colors is displayed in Figure 7.

In general, two colors are needed to color C_n when n is even. To construct such a coloring, simply pick a vertex and color it red. Proceed around the graph in a clockwise direction (using a planar representation of the graph) coloring the second vertex blue, the third vertex red, and so on. The nth vertex can be colored blue, since the two vertices adjacent to it, namely the $(n - 1)$st and the first vertices, are both colored red.

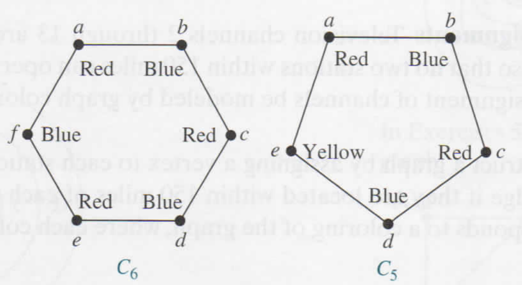


FIGURE 7 **Colorings of C_5 and C_6.**

When n is odd and $n > 1$, the chromatic number of C_n is 3. To see this, pick an initial vertex. To use only two colors, it is necessary to alternate colors as the graph is traversed in a clockwise direction. However, the nth vertex reached is adjacent to two vertices of different colors, namely, the first and $(n - 1)$st. Hence, a third color must be used. ◀

Links

The best algorithms known for finding the chromatic number of a graph have exponential worst-case time complexity (in the number of vertices of the graph). Even the problem of finding an approximation to the chromatic number of a graph is difficult. It has been shown that if there were an algorithm with polynomial worst-case time complexity that could approximate the chromatic number of a graph up to a factor of 2 (that is, construct a bound which was no more than double the chromatic number of the graph), then an algorithm with polynomial worst-case time complexity for finding the chromatic number of the graph would also exist.

APPLICATIONS OF GRAPH COLORINGS

Graph coloring has a variety of applications to problems involving scheduling and assignments. (Note that since no efficient algorithm is known for graph coloring, this does not lead to efficient algorithms for scheduling and assignments.) Examples of such applications will be given here. The first application deals with the scheduling of final exams.

EXAMPLE 5 **Scheduling Final Exams** How can the final exams at a university be scheduled so that no student has two exams at the same time?

Solution: This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for a final exam is represented by a different color. A scheduling of the exams corresponds to a coloring of the associated graph.

For instance, suppose there are seven finals to be scheduled. Suppose the courses are numbered 1 through 7. Suppose that the following pairs of courses have common students: 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, and 6 and 7. In Figure 8 the graph associated with this set of classes is shown. A scheduling consists of a coloring of this graph.

Since the chromatic number of this graph is 4 (the reader should verify this), four time slots are needed. A coloring of the graph using four colors and the associated schedule are shown in Figure 9. ◀

Now consider an application to the assignment of television channels.

EXAMPLE 6 **Frequency Assignments** Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Solution: Construct a graph by assigning a vertex to each station. Two vertices are connected by an edge if they are located within 150 miles of each other. An assignment of channels corresponds to a coloring of the graph, where each color represents a different channel. ◀

An application of graph coloring to compilers is considered in Example 7.

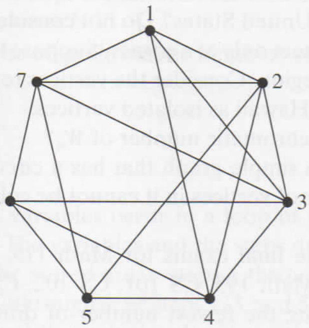
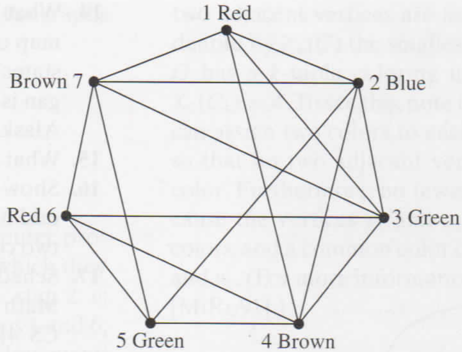


FIGURE 8 The Graph Representing the Scheduling of Final Exams.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

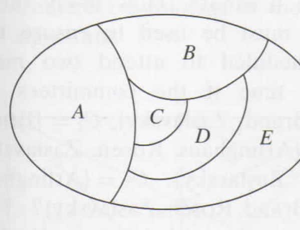
FIGURE 9 Using a Coloring to Schedule Final Exams.

EXAMPLE 7 Index Registers In efficient compilers the execution of loops is speeded up when frequently used variables are stored temporarily in index registers in the central processing unit, instead of in regular memory. For a given loop, how many index registers are needed? This problem can be addressed using a graph coloring model. To set up the model, let each vertex of a graph represent a variable in the loop. There is an edge between two vertices if the variables they represent must be stored in index registers at the same time during the execution of the loop. Thus, the chromatic number of the graph gives the number of index registers needed, since different registers must be assigned to variables when the vertices representing these variables are adjacent in the graph. ◀

Exercises

In Exercises 1–4 construct the dual graph for the map shown. Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

1.



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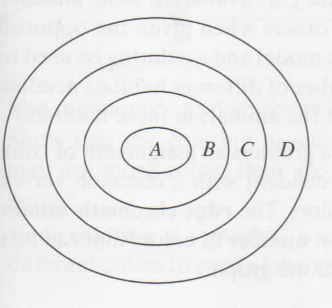

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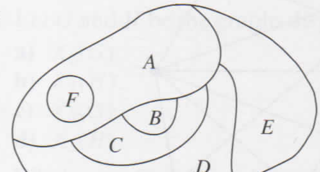

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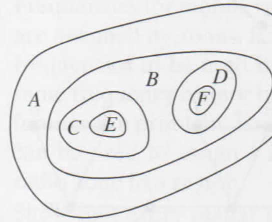
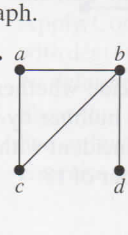


In Exercises 5–11 find the chromatic number of the given graph.

5.



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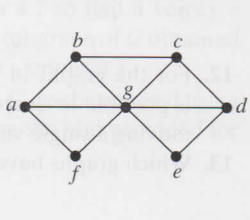

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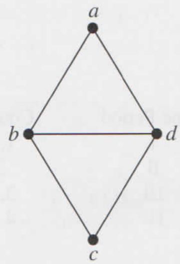

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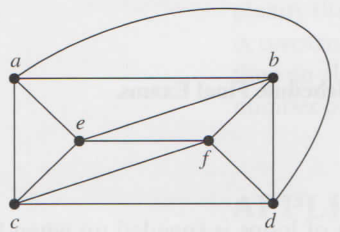

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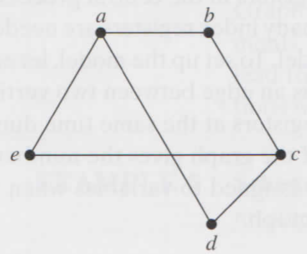

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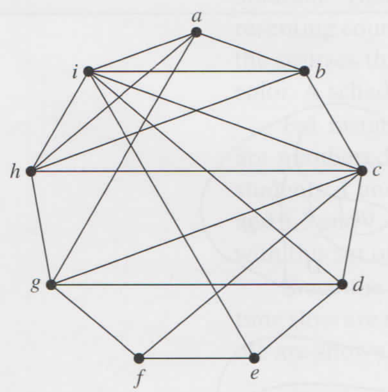

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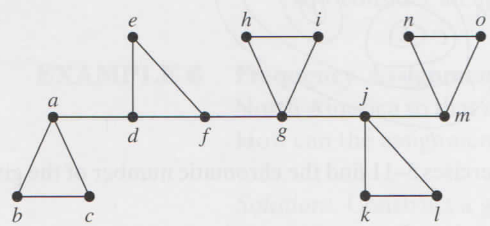

12. For the graphs in Exercises 5–11, decide whether it is possible to decrease the chromatic number by removing a single vertex and all edges incident with it.

13. Which graphs have a chromatic number of 1?

14. What is the least number of colors needed to color a map of the United States? Do not consider adjacent states that meet only at a corner. Suppose that Michigan is one region. Consider the vertices representing Alaska and Hawaii as isolated vertices.

15. What is the chromatic number of W_n?

16. Show that a simple graph that has a circuit with an odd number of vertices in it cannot be colored using two colors.

17. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other combination of courses.

18. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

	1	*2*	*3*	*4*	*5*	*6*
1	—	85	175	200	50	100
2	85	—	125	175	100	160
3	175	125	—	100	200	250
4	200	175	100	—	210	220
5	50	100	200	210	—	100
6	100	160	250	220	100	—

19. The mathematics department has six committees each meeting once a month. How many different meeting times must be used to ensure that no member is scheduled to attend two meetings at the same time if the committees are C_1 = {Arlinghaus, Brand, Zaslavsky}, C_2 = {Brand, Lee, Rosen}, C_3 = {Arlinghaus, Rosen, Zaslavsky}, C_4 = {Lee, Rosen, Zaslavsky}, C_5 = {Arlinghaus, Brand}, and C_6 = {Brand, Rosen, Zaslavsky}?

20. A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some animals will eat some of the others when given the opportunity. How can a graph model and a coloring be used to determine the number of different habitats needed and the placement of the animals in these habitats?

An **edge coloring** of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The **edge chromatic number** of a graph is the smallest number of colors that can be used in an edge coloring of the graph.