Appendix C

Markov Chains

Eigenvalues and eigenvectors arise naturally in the study of matrix representations of linear transformations, but that is far from their only use. In this Appendix, we present an application to those probabilistic systems known as Markov chains.

An elementary understanding of Markov chains requires only a little knowledge of probabilities; in particular, that probabilities describe the likelihoods of different events occurring, that probabilities are numbers between 0 and 1, and that if the set of all possible events is limited to a finite number that are mutually exclusive then the sum of the probabilities of each event occurring is 1. Significantly more probability theory is needed to prove the relevant theorems about Markov chains, so we limit ourselves in this section to simply understanding the application.

> ▶ **Definition 1.** A finite *Markov chain* is a set of objects (perhaps people), a set of consecutive time periods (perhaps five-year intervals), and a finite set of different states (perhaps employed and unemployed) such that
>
> (i) during any given time period, each object is in only one state (although different objects can be in different states), and
>
> (ii) the probability that an object will move from one state to another state (or remain in the same state) over a time period depends only on the beginning and ending states. ◀

We denote the states as state 1, state 2, state 3, through state N, and let p_{ij} designate the probability of moving in one time period into state i from state $j (i, j = 1, 2, \ldots, N)$. The matrix $\mathbf{P} = [p_{ij}]$ is called a *transition matrix*.

A transition matrix for an N-state Markov chain is an $N \times N$ matrix with nonnegative entries; the sum of the entries in each column is 1.

Example 1

Construct a transition matrix for the following Markov chain. A traffic control administrator in the Midwest classifies each day as either clear or cloudy. Historical data show that the probability of a clear day following a cloudy day is 0.6, whereas the probability of a clear day following a clear day is 0.9.

Solution: Although one can conceive of many other classifications such as rainy, very cloudy, partly sunny, and so on, this particular administrator opted for only two, so we have just two states: clear and cloudy, and each day must fall into one and only one of these two states. Arbitrarily we take clear to be state 1 and cloudy to be state 2. The natural time unit is one day. We are given that $p_{12} = 0.6$, so it must follow that $p_{22} = 0.4$, because after a cloudy day the next day must be either clear or cloudy and the probability that one or the other of these two events occurring is 1. Similarly, we are given that $p_{11} = 0.9$, so it also follows that $p_{21} = 0.1$. The transition matrix is

$$\begin{array}{cc} & \begin{array}{cc} \textit{clear} & \textit{cloudy} \end{array} \\ \mathbf{P} = & \left[\begin{array}{cc} 0.9 & 0.6 \\ 0.1 & 0.4 \end{array}\right] \begin{array}{l} \textit{clear} \\ \textit{cloudy} \end{array} \end{array} \quad \blacksquare$$

Example 2

Construct a transition matrix for the following Markov chain. A medical survey lists individuals as thin, normal, or obese. A review of yearly check-ups from doctors' records showed that 80% of all thin people remained thin one year later while the other 20% gained enough weight to be reclassified as normal. For individuals of normal weight, 10% became thin, 60% remained normal, and 30% became obese the following year. Of all obese people, 90% remained obese one year later while the other 10% lost sufficient weight to fall into the normal range. Although some thin people became obese a year later, and vice versa, their numbers were insignificant when rounded to two decimals.

Solution: We take state 1 to be thin, state 2 to be normal, and state 3 to be obese. One time period equals one year. Converting each percent to its decimal representation so that it may also represent a probability, we have $p_{21} = 0.2$, the probability of an individual having normal weight after being thin the previous year, $p_{32} = 0.3$, the probability of an individual becoming obese one year after having a normal weight, and, in general,

$$\begin{array}{cc} & \begin{array}{ccc} \textit{thin} & \textit{normal} & \textit{obese} \end{array} \\ \mathbf{P} = & \left[\begin{array}{ccc} 0.8 & 0.1 & 0 \\ 0.2 & 0.6 & 0.1 \\ 0 & 0.3 & 0.9 \end{array}\right] \begin{array}{l} \textit{thin} \\ \textit{normal} \\ \textit{obese} \end{array} \end{array} \quad \blacksquare$$

Powers of a transition matrix have the same properties of a transition matrix: all elements are between 0 and 1, and every column sum equals 1 (see Problem 20). Furthermore,

> ▶ **Theorem 1.** *If* **P** *is a transition matrix for a finite Markov chain, and if* $p_{ij}^{(k)}$ *denotes the i-j element of* $\mathbf{P}^k$, *the kth power of* **P**, *then* $p_{ij}^{(k)}$ *is the probability of moving to state i from state j in k time periods.* ◀

For the transition matrix created in Example 2, we calculate the second and third powers as

$$\begin{array}{c} \\ \mathbf{P}^2 = \end{array} \begin{array}{c} \begin{array}{ccc} \textit{thin} & \textit{normal} & \textit{obese} \end{array} \\ \left[\begin{array}{ccc} 0.66 & 0.14 & 0.01 \\ 0.28 & 0.41 & 0.15 \\ 0.06 & 0.45 & 0.84 \end{array}\right] \end{array} \begin{array}{l} \\ \textit{thin} \\ \textit{normal} \\ \textit{obese} \end{array}$$

and

$$\begin{array}{c} \\ \mathbf{P}^3 = \end{array} \begin{array}{c} \begin{array}{ccc} \textit{thin} & \textit{normal} & \textit{obese} \end{array} \\ \left[\begin{array}{ccc} 0.556 & 0.153 & 0.023 \\ 0.306 & 0.319 & 0.176 \\ 0.138 & 0.528 & 0.801 \end{array}\right] \end{array} \begin{array}{l} \\ \textit{thin} \\ \textit{normal} \\ \textit{obese} \end{array} \quad \blacksquare$$

Here $p_{11}^{(2)}5 = 0.66$ is the probability of a thin person remaining thin two years later, $p_{32}^{(2)}6 = 0.45$ is the probability of a normal person becoming fat two years later, while $p_{13}^{(2)}7 = 0.023$ is the probability of a fat person becoming thin three years later.

For the transition matrix created in Example 1, we calculate the second power to be

$$\begin{array}{c} \\ \mathbf{P}^2 = \end{array} \begin{array}{c} \begin{array}{cc} \textit{clear} & \textit{cloudy} \end{array} \\ \left[\begin{array}{cc} 0.87 & 0.78 \\ 0.13 & 0.22 \end{array}\right] \end{array} \begin{array}{l} \\ \textit{clear} \\ \textit{cloudy} \end{array}$$

Consequently, $p_{12}^{(2)}9 = 0.78$ is the probability of a cloudy day being followed by a clear day two days later, while $p_{22}^{(2)}10 = 0.22$ is the probability of a cloudy day being followed by a cloudy day two days later. Calculating the tenth power of this same transition matrix and rounding all entries to four decimal places for presentation purposes, we have

$$\begin{array}{c} \\ P^{10} = \end{array} \begin{array}{c} \begin{array}{cc} \textit{clear} & \textit{cloudy} \end{array} \\ \left[\begin{array}{cc} 0.8571 & 0.8571 \\ 0.1429 & 0.1429 \end{array}\right] \end{array} \begin{array}{l} \\ \textit{clear} \\ \textit{cloudy} \end{array} \tag{C.1}$$

Since $p_{11}^{(10)}12 = p_{12}^{(10)}13 = 0.8571$, it follows that the probability of having a clear day 10 days after a cloudy day is the same as the probability of having a clear day 10 days after a clear day.

An object in a Markov chain must be in one and only one state at any time, but that state is not always known with certainty. Often, probabilities are provided to describe the likelihood of an object being in any one of the states at any given time. These probabilities can be combined into an n-tuple. A *distribution vector* **d** for an N-state Markov chain at a given time is an N-dimensional column matrix having as its components, one for each state, the probabilities that an object in the system is in each of the respective states at that time.

A distribution vector for an N-state Markov chain at a given time is a column matrix whose ith component is the probability that an object is in the ith state at that given time.

Example 3 Find the distribution vector for the Markov chain described in Example 1 if the current day is known to be cloudy.

Solution: The objects in the system are days, which are classified as either clear, state 1, or cloudy, state 2. We are told with certainty that the current day is cloudy, so the probability that the day is cloudy is 1 and the probability that the day is clear is 0. Therefore,

$$\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \blacksquare$$

Example 4 Find the distribution vector for the Markov chain described in Example 2 if it is known that currently 7% of the population is thin, 31% of population is of normal weight, and 62% of the population is obese.

Solution: The objects in the system are people. Converting the stated percentages into their decimal representations, we have

$$\mathbf{d} = \begin{bmatrix} 0.07 \\ 0.31 \\ 0.62 \end{bmatrix} \quad \blacksquare$$

Different time periods can have different distribution vectors, so we let $\mathbf{d}^{(k)}$ denote a distribution vector *after* k time periods. In particular, $\mathbf{d}^{(1)}$ is a distribution vector after 1 time period, $\mathbf{d}^{(2)}$ is a distribution vector after 2 time periods, and $\mathbf{d}^{(10)}$ is a distribution vector after 10 time periods. An initial distribution vector for the beginning of a Markov chain is designated by $\mathbf{d}^{(0)}$. The distribution vectors for various time periods are related.

▶ **Theorem 2.** *If* **P** *is a transition matrix for a Markov chain, then*

$$\mathbf{d}^{(k)} = \mathbf{P}^k\mathbf{d}^{(0)} = \mathbf{P}\mathbf{d}^{(k-1)},$$

where $\mathbf{P}^k$ *denotes the* kth *power of* **P**. ◀

For the distribution vector and transition matrix created in Examples 1 and 3, we calculate

$$\mathbf{d}^{(1)} = \mathbf{P}\mathbf{d}^{(0)} = \begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\mathbf{d}^{(2)} = \mathbf{P}^2\mathbf{d}^{(0)} = \begin{bmatrix} 0.87 & 0.78 \\ 0.13 & 0.22 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.78 \\ 0.22 \end{bmatrix} \tag{C.2}$$

$$\mathbf{d}^{(10)} = \mathbf{P}^{10}\mathbf{d}^{(0)} = \begin{bmatrix} 0.8571 & 0.8571 \\ 0.1429 & 0.1429 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8571 \\ 0.1429 \end{bmatrix}$$

The probabilities of following a cloudy day with a cloudy day after 1 time period, 2 time periods, and 10 time periods, respectively, are 0.4, 0.22, and 0.1429.

For the distribution vector and transition matrix created in Examples 2 and 4, we calculate

$$\mathbf{d}^{(3)} = \mathbf{P}^3\mathbf{d}^{(0)} = \begin{bmatrix} 0.556 & 0.153 & 0.023 \\ 0.306 & 0.319 & 0.176 \\ 0.138 & 0.528 & 0.801 \end{bmatrix} \begin{bmatrix} 0.07 \\ 0.31 \\ 0.62 \end{bmatrix} = \begin{bmatrix} 0.10061 \\ 0.22943 \\ 0.66996 \end{bmatrix}$$

Rounding to three decimal places, we have that the probabilities of an arbitrarily chosen individual being thin, normal weight, or obese after three time periods (years) are, respectively, 0.101, 0.229, and 0.700.

The tenth power of the transition matrix created in Example 1 is given by equation (C.1) as

$$\mathbf{P}^{10} = \begin{bmatrix} 0.8571 & 0.8571 \\ 0.1429 & 0.1429 \end{bmatrix}$$

Continuing to calculate successively higher powers of $\mathbf{P}$ we find that each is identical to $\mathbf{P}^{10}$ when we round all entries to four decimal places. Convergence is a bit slower for the transition matrix associated with Example 3, but it also occurs. As we calculate successively higher powers of that matrix, we find that

$$\mathbf{P}^{10} = \begin{bmatrix} 0.2283 & 0.1287 & 0.0857 \\ 0.2575 & 0.2280 & 0.2144 \\ 0.5142 & 0.6433 & 0.6999 \end{bmatrix}$$

$$\mathbf{P}^{20} = \begin{bmatrix} 0.1294 & 0.1139 & 0.1072 \\ 0.2277 & 0.2230 & 0.2210 \\ 0.6429 & 0.6631 & 0.6718 \end{bmatrix} \tag{C.3}$$

and

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} 0.1111 & 0.1111 & 0.1111 \\ 0.2222 & 0.2222 & 0.2222 \\ 0.6667 & 0.6667 & 0.6667 \end{bmatrix}$$

where all entries have been rounded to four decimal places for presentation purposes.

A transition matrix is regular if one of its powers has only positive elements.

Not all transition matrices have powers that converge to a limiting matrix $\mathbf{L}$, but many do. A transition matrix for a finite Markov chain is *regular* if it or one of its powers contains only positive elements. Powers of a regular matrix always converge to a limiting matrix $\mathbf{L}$.

The transition matrix created in Example 1 is regular because all of its elements are positive. The transition matrix $\mathbf{P}$ created in Example 2 is also regular because all elements of $\mathbf{P}^2$, its second power, are positive. In contrast, the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is not regular because each of its powers is either itself or the 2×2 identity matrix, both of which contain zero entries.

By definition, some power of a regular matrix $\mathbf{P}$, say the mth, contains only positive elements. Since the elements of $\mathbf{P}$ are nonnegative, it follows from matrix multiplication that every power of $\mathbf{P}$ greater than m must also have all positive components. Furthermore, if $\mathbf{L} = \lim_{k\to\infty} \mathbf{P}^k$, then it is also true that $\mathbf{L} = \lim_{k\to\infty} \mathbf{P}^{k-1}$. Therefore,

$$\mathbf{L} = \lim_{k\to\infty} \mathbf{P}^k = \lim_{k\to\infty} (\mathbf{P}\mathbf{P}^{k-1}) = \mathbf{P}\left(\lim_{k\to\infty} \mathbf{P}^{k-1}\right) = \mathbf{PL} \tag{C.4}$$

Denote the columns of $\mathbf{L}$ as $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$, respectively, so that $\mathbf{L} = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_N]$. Then equation (C.4) becomes

$$[\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_N] = \mathbf{P}[\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_N]$$

where $\mathbf{x}_j = \mathbf{P}\mathbf{x}_j$ $(j = 1, 2, \ldots, N)$, or $\mathbf{P}\mathbf{x}_j = (1)\mathbf{x}_j$. Thus, each column of $\mathbf{L}$ is an eigenvector of $\mathbf{P}$ corresponding to the eigenvalue 1! We have proven part of the following important result:

► **Theorem 3.** *If an $N \times N$ transition matrix $\mathbf{P}$ is regular, then successive integral powers of $\mathbf{P}$ converge to a limiting matrix $\mathbf{L}$ whose columns are eigenvectors of $\mathbf{P}$ associated with eigenvalue $\lambda = 1$. The components of this eigenvector are positive and sum to unity.* ◄

Even more is true. If $\mathbf{P}$ is regular, then its eigenvalue $\lambda = 1$ has multiplicity 1, and there is only one linearly independent eigenvector associated with that eigenvalue. This eigenvector will be in terms of one arbitrary constant, which is uniquely determined by the requirement that the sum of the components is 1. Thus, each column of $\mathbf{L}$ is the *same* eigenvector.

We define the *limiting state distribution vector* for an N-state Markov chain as an N-dimensional column vector $\mathbf{d}^{(\infty)}$ having as its components the limiting probabilities that an object in the system is in each of the respective states after a large number of time periods. That is,

$$\mathbf{d}^{(\infty)} = \lim_{n\to\infty} \mathbf{d}^{(n)}$$

Consequently,

$$\mathbf{d}^{(\infty)} = \lim_{n\to\infty} \mathbf{d}^{(n)} = \lim_{n\to\infty} (\mathbf{P}^n\mathbf{d}^{(0)}) = \left(\lim_{n\to\infty} \mathbf{P}^n\right)\mathbf{d}^{(0)} = \mathbf{L}\mathbf{d}^{(0)}$$

Each column of $\mathbf{L}$ is identical to every other column, so each row of $\mathbf{L}$ contains a single number repeated N times. Combining this with the fact that $\mathbf{d}^{(0)}$ has components that sum to 1, it follows that the product $\mathbf{L}\mathbf{d}^{(0)}$ is equal to each of the identical columns of $\mathbf{L}$. That is, $\mathbf{d}^{(\infty)}$ is the eigenvector of $\mathbf{P}$ corresponding to $\lambda = 1$, having the sum of its components equal to 1.

The limiting state distribution vector for a transition matrix $\mathbf{P}$ is the unique eigenvector of $\mathbf{P}$ corresponding to $\lambda = 1$, *having the sum of its components equal to 1.*

Example 5 Find the limiting state distribution vector for the Markov chain described in Example 1.

Solution: The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix}$$

which is regular. Eigenvectors for this matrix have the form

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvectors corresponding to $\lambda = 1$ satisfy the matrix equation $(\mathbf{P} - 1\mathbf{I})\mathbf{x} = \mathbf{0}$, or equivalently, the set of equations

$$\begin{aligned} -0.1x + 0.6y &= 0 \\ 0.1x - 0.6y &= 0 \end{aligned}$$

Solving by Gaussian elimination, we find $x = 6y$ with y arbitrary. Thus,

$$x = \begin{bmatrix} 6y \\ y \end{bmatrix}$$

If we choose y so that the sum of the components of **x** sum to 1, we have $7y = 1$, or $y = 1/7$. The resulting eigenvector is the limiting state distribution vector, namely

$$\mathbf{d}^{(\infty)} = \begin{bmatrix} 6/7 \\ 1/7 \end{bmatrix}$$

Furthermore,

$$\mathbf{L} = \begin{bmatrix} 6/7 & 6/7 \\ 1/7 & 1/7 \end{bmatrix}$$

Over the long run, six out of seven days will be clear and one out of seven days will be cloudy. We see from equations (C.1) and (C.2) that convergence to four decimal places for the limiting state distribution and **L** is achieved after 10 time periods. ■

Example 6 Find the limiting state distribution vector for the Markov chain described in Example 2.

Solution: The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.1 & 0 \\ 0.2 & 0.6 & 0.1 \\ 0 & 0.3 & 0.9 \end{bmatrix}$$

$\mathbf{P}^2$ has only positive elements, so **P** is regular. Eigenvectors for this matrix have the form

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvectors corresponding to $\lambda = 1$ satisfy the matrix equation $(\mathbf{P} - 1\mathbf{I})\mathbf{x} = \mathbf{0}$, or equivalently, the set of equations

$$\begin{aligned} -0.2x + 0.1y &= 0 \\ 0.2x - 0.4y + 0.1z &= 0 \\ 0.3y - 0.1z &= 0 \end{aligned}$$

Solving by Gaussian elimination, we find $x = (1/6)z$, $y = (1/3)z$, with z arbitrary. Thus,

$$x = \begin{bmatrix} z/6 \\ z/3 \\ z \end{bmatrix}$$

We choose z so that the sum of the components of $\mathbf{x}$ sum to 1, hence $(1/6)z + (1/3)z + z = 1$, or $z = 2/3$. The resulting eigenvector is the limiting state distribution vector, namely,

$$\mathbf{d}^{(\infty)} = \begin{bmatrix} 1/9 \\ 2/9 \\ 6/9 \end{bmatrix}$$

Furthermore,

$$\mathbf{L} = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 2/9 & 2/9 & 2/9 \\ 6/9 & 6/9 & 6/9 \end{bmatrix}$$

Compare $\mathbf{L}$ with equation (C.3). The components of $\mathbf{d}^{(\infty)}$ imply that, over the long run, one out of nine people will be thin, two out of nine people will be of normal weight, and six out of nine people will be obese. ■

Problems Appendix C

(1) Determine which of the following matrices cannot be transition matrices and explain why:

(a) $\begin{bmatrix} 0.15 & 0.57 \\ 0.85 & 0.43 \end{bmatrix}$, (b) $\begin{bmatrix} 0.27 & 0.74 \\ 0.63 & 0.16 \end{bmatrix}$,

(c) $\begin{bmatrix} 0.45 & 0.53 \\ 0.65 & 0.57 \end{bmatrix}$, (d) $\begin{bmatrix} 1.27 & 0.23 \\ -0.27 & 0.77 \end{bmatrix}$,

(e) $\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/3 & 0 \\ 0 & 1/6 & 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1/2 & 1/2 & 1/3 \\ 1/4 & 1/3 & 1/4 \\ 1/4 & 1/6 & 7/12 \end{bmatrix}$,

(g) $\begin{bmatrix} 0.34 & 0.18 & 0.53 \\ 0.38 & 0.42 & 0.21 \\ 0.35 & 0.47 & 0.19 \end{bmatrix}$, (h) $\begin{bmatrix} 0.34 & 0.32 & -0.17 \\ 0.78 & 0.65 & 0.80 \\ -0.12 & 0.03 & 0.37 \end{bmatrix}$.

(2) Construct a transition matrix for the following Markov chain: Census figures show a population shift away from a large midwestern metropolitan city to its suburbs. Each year, 5% of all families living in the city move to the suburbs while during the same time period only 1% of those living in the suburbs move into the city. *Hint*: Take state 1 to represent families living in the city, state 2 to represent families living in the suburbs, and one year as one time period.

(3) Construct a transition matrix for the following Markov chain: Every four years, voters in a New England town elect a new mayor because a town ordinance prohibits mayors from succeeding themselves. Past data indicate that a Democratic mayor is succeeded by another Democrat 30% of the time and by a Republican 70% of the time. A Republican mayor, however, is succeeded by another Republican 60% of the time and by a Democrat 40% of the time. *Hint*: Take state 1 to represent a Republican mayor in office, state 2 to represent a Democratic mayor in office, and four years as one time period.

(4) Construct a transition matrix for the following Markov chain: The apple harvest in New York orchards is classified as poor, average, or good. Historical data indicates that if the harvest is poor one year then there is a 40% chance of having a good harvest the next year, a 50% chance of having an average harvest, and a 10% chance of having another poor harvest. If a harvest is average one year, the chance of a poor, average, or good harvest the next year is 20%, 60%, and 20%, respectively. If a harvest is good, then the chance of a poor, average, or good harvest the next year is 25%, 65%, and 10%, respectively. *Hint*: Take state 1 to be a poor harvest, state 2 to be an average harvest, state 3 to be a good harvest, and one year as one time period.

(5) Construct a transition matrix for the following Markov chain: Brand X and brand Y control the majority of the soap powder market in a particular region, and each has promoted its own product extensively. As a result of past advertising campaigns, it is known that over a two-year period of time, 10% of brand Y customers change to brand X and 25% of all other customers change to brand X. Furthermore, 15% of brand X customers change to brand Y and 30% of all other customers change to brand Y. The major brands also lose customers to smaller competitors, with 5% of brand X customers switching to a minor brand during a two-year time period and 2% of brand Y customers doing likewise. All other customers remain loyal to their past brand of soap powder. *Hint*: Take state 1 to be a brand X customer, state 2 a brand Y customer, state 3 another brand's customer, and two years as one time period.

(6) (a) Calculate $\mathbf{P}^2$ and $\mathbf{P}^3$ for the two-state transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix}$$

(b) Determine the probability of an object beginning in state 1 and remaining in state 1 after two time periods.

(c) Determine the probability of an object beginning in state 1 and ending in state 2 after two time periods.

(d) Determine the probability of an object beginning in state 1 and ending in state 2 after three time periods.

(e) Determine the probability of an object beginning in state 2 and remaining in state 2 after three time periods.

(7) Consider a two-state Markov chain. List the number of ways an object in state 1 can end in state 1 after three time periods.

(8) Consider the Markov chain described in Problem 2. Determine (a) the probability a family living in the city will find themselves in the suburbs after two years, and (b) the probability a family living in the suburbs will find themselves living in the city after two years.

(9) Consider the Markov chain described in Problem 3. Determine (a) the probability there will be a Republican mayor eight years after a Republican mayor serves, and (b) the probability there will be a Republican mayor 12 years after a Republican mayor serves.

(10) Consider the Markov chain described in Problem 4. It is known that this year that the apple harvest was poor. Determine (a) the probability next year's harvest will be poor, and (b) the probability that the harvest in two years will be poor.

(11) Consider the Markov chain described in Problem 5. Determine (a) the probability that a brand X customer will remain a brand X customer after 4 years, (b) after 6 years, and (c) the probability that a brand X customer will become a brand Y customer after 4 years.

(12) Consider the Markov chain described in Problem 2. (a) Explain the significance of each component of $\mathbf{d}^{(0)} = [\,0.6 \quad 0.4\,]^{\mathrm{T}}$. (b) Use this vector to find $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$.

(13) Consider the Markov chain described in Problem 3. (a) Explain the significance of each component of $\mathbf{d}^{(0)} = [\,0.4 \quad 0.5 \quad 0.1\,]^{\mathrm{T}}$. (b) Use this vector to find $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$.

(14) Consider the Markov chain described in Problem 4. (a) Determine an initial distribution vector if the town currently has a Democratic mayor, and (b) show that the components of $\mathbf{d}^{(1)}$ are the probabilities that the next mayor will be a Republican and a Democrat, respectively.

(15) Consider the Markov chain described in Problem 5. (a) Determine an initial distribution vector if this year's crop is known to be poor. (b) Calculate $\mathbf{d}^{(2)}$ and use it to determine the probability that the harvest will be good in two years.

(16) Find the limiting distribution vector for the Markov chain described in Problem 2, and use it to determine the probability that a family eventually will reside in the city.

(17) Find the limiting distribution vector for the Markov chain described in Problem 3, and use it to determine the probability of having a Republican mayor over the long run.

(18) Find the limiting distribution vector for the Markov chain described in Problem 4, and use it to determine the probability of having a good harvest over the long run.

(19) Find the limiting distribution vector for the Markov chain described in Problem 5, and use it to determine the probability that a person will become a Brand Y customer over the long run.

(20) Use mathematical induction to prove that if $\mathbf{P}$ is a transition matrix for an n-state Markov chain, then any integral power of $\mathbf{P}$ has the properties that (a) all elements are nonnegative numbers between zero and 1, and (b) the sum of the elements in each column is 1.

(21) A nonzero row vector $\mathbf{y}$ is a *left eigenvector* for a matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{yA} = \lambda\mathbf{y}$. Prove that if $\mathbf{x}$ and λ are a corresponding pair of eigenvectors and eigenvalues for a matrix $\mathbf{B}$, then $\mathbf{x}^{\mathrm{T}}$ and λ are a corresponding pair of left eigenvectors and eigenvalues for $\mathbf{B}^{\mathrm{T}}$.

(22) Show directly that the n-dimensional row vector $\mathbf{y} = [\,1 \quad 1 \quad 1 \ldots 1\,]$ is a left eigenvector for any $N \times N$ transition matrix $\mathbf{P}$. Then, using the results of Problem 20, deduce that $\lambda = 1$ is an eigenvalue for any transition matrix.

(23) Prove that every eigenvalue λ of a transition matrix $\mathbf{P}$ satisfies the inequality $|\lambda| \leq 1$. *Hint:* Let $\mathbf{x} = [\,x_1 \quad x_2 \quad \ldots \quad x_{\mathrm{N}}]^{\mathrm{T}}$ be an eigenvector of $\mathbf{P}$ corresponding to the eigenvalue λ, and let $x_{\mathrm{i}} = \max\{x_1, x_2, \ \ldots, x_{\mathrm{N}}\}$. Consider the ith component of the vector equation $\mathbf{Px} = \lambda\mathbf{x}$, and show that $|\lambda||x_{\mathrm{i}}| \leq |x_{\mathrm{i}}|$.

(24) A state in a Markov chain is *absorbing* if no objects in the system can leave the state after they enter it. Describe the ith column of a transition matrix for a Markov chain in which the ith state is absorbing.

(25) Prove that a transition matrix for a Markov chain with one or more absorbing states cannot be regular.