Last time:

- Introduction to Optimization Models
  Linear Programming (LP)
- Simple modeling

Today:

- Important concepts/definitions in LP
- Presenting/solving LP's in graphical form
- Classification of LP problems
- Converting LP's to Standard form

# Important concepts/definitions

$$\max\ 12X_T + 5X_C + 15X_D + 10X_B$$

$$\begin{aligned} \text{s.t.}\quad & 5X_T + X_C + 9X_D + 12X_B \le 1500 && \text{(pine)} \\ & 2X_T + 3X_C + 4X_D + X_B \le 1000 && \text{(oak)} \\ & 3X_T + 2X_C + 5X_D + 10X_B \le 800 && \text{(labor)} \\ & X_T \ge 40 \\ & X_C \ge 130 \\ & X_D \ge 30 \\ & X_B \ge 0 \end{aligned}$$

- A point $\bar{x} \in \mathbb{R}^n$ is called a <u>solution</u>.
- $\bar{x}$ is called a <u>feasible</u> solution if it satisfies <u>all</u> constraints.

  <u>E.g.</u>, $\bar{x} = \begin{pmatrix} 40 \\ 130 \\ 30 \\ 0 \end{pmatrix}$ in our example
- $\bar{x}$ is called an <u>infeasible</u> solution if there is a constraint that it doesn't satisfy (that is violated).

  <u>E.g.</u>, $\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} 40 \\ 130 \\ 30 \\ -1 \end{pmatrix}$
- <u>feasible region</u> = collection of all feasible sol-ns.
- $\bar{x}$ is called an <u>optimal solution</u> if
  - a) it is feasible
  - b) there is no other feasible solution $\bar{X}'$ with a better objective function value

<u>E.g.</u>, optimal sol-n in our ex. = production schedule that maximizes profit

# Important concepts/definitions (cont.)

solutions

feasible    infeasible

optimal    nonoptimal

- The value of a solution $\bar{x}$ is the objective function value at $\bar{x}$.

  E.g., value at $\bar{x} = \begin{pmatrix} 40 \\ 130 \\ 30 \\ 0 \end{pmatrix}$ is $12 \cdot 40 + 5 \cdot 130 + 15 \cdot 30 = 1780$

- The value of an LP is the value of the optimal sol-n (also called the optimal value).

- LP is called infeasible if there is no solution $\bar{x}$ that satisfies all constraints at the same time.

  E.g., Suppose demand for bookcases $= 300$.

  Then labor constraint is violated:

  $10 \cdot 300 > 800$.

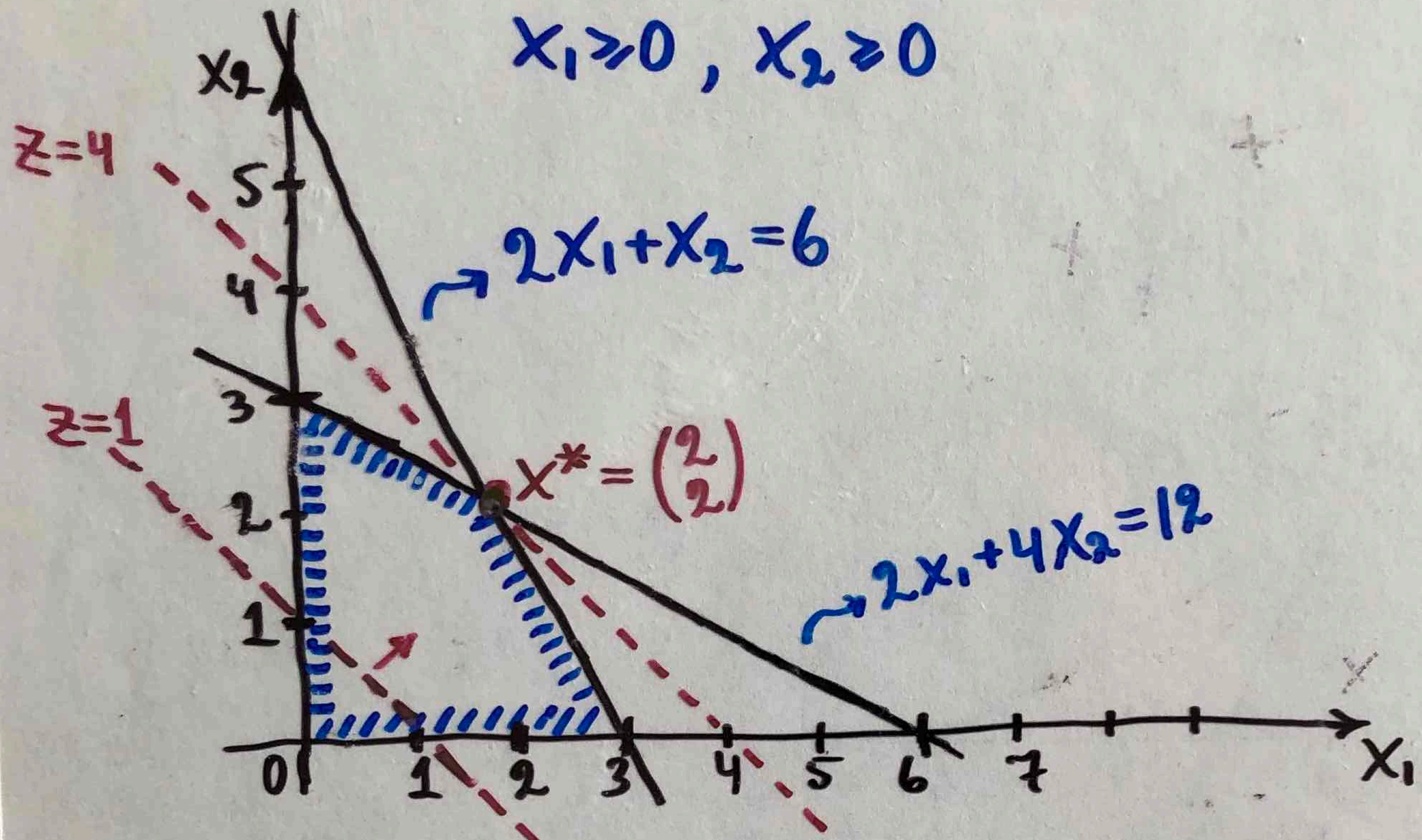
  $\Rightarrow$ this new problem is infeasible.

Problem is infeasible $\Leftrightarrow$ feasible region is empty

# Graphical presentation of LP

An abstract example:

$$\max\ Z = x_1 + x_2$$
$$\text{s.t.}\quad 2x_1 + x_2 \le 6$$
$$2x_1 + 4x_2 \le 12$$
$$x_1 \ge 0,\ x_2 \ge 0$$




- Each constraint defines a halfspace $\Rightarrow$ LP feasible region is intersection of halfspaces
- Draw "iso-profit lines" for objective function.

  E.g., $x_1+x_2=1$, $x_1+x_2=4$, ...
- Find "highest" iso-profit line that touches the feasible region; its intersection with the feasible region gives optimal solution(s)

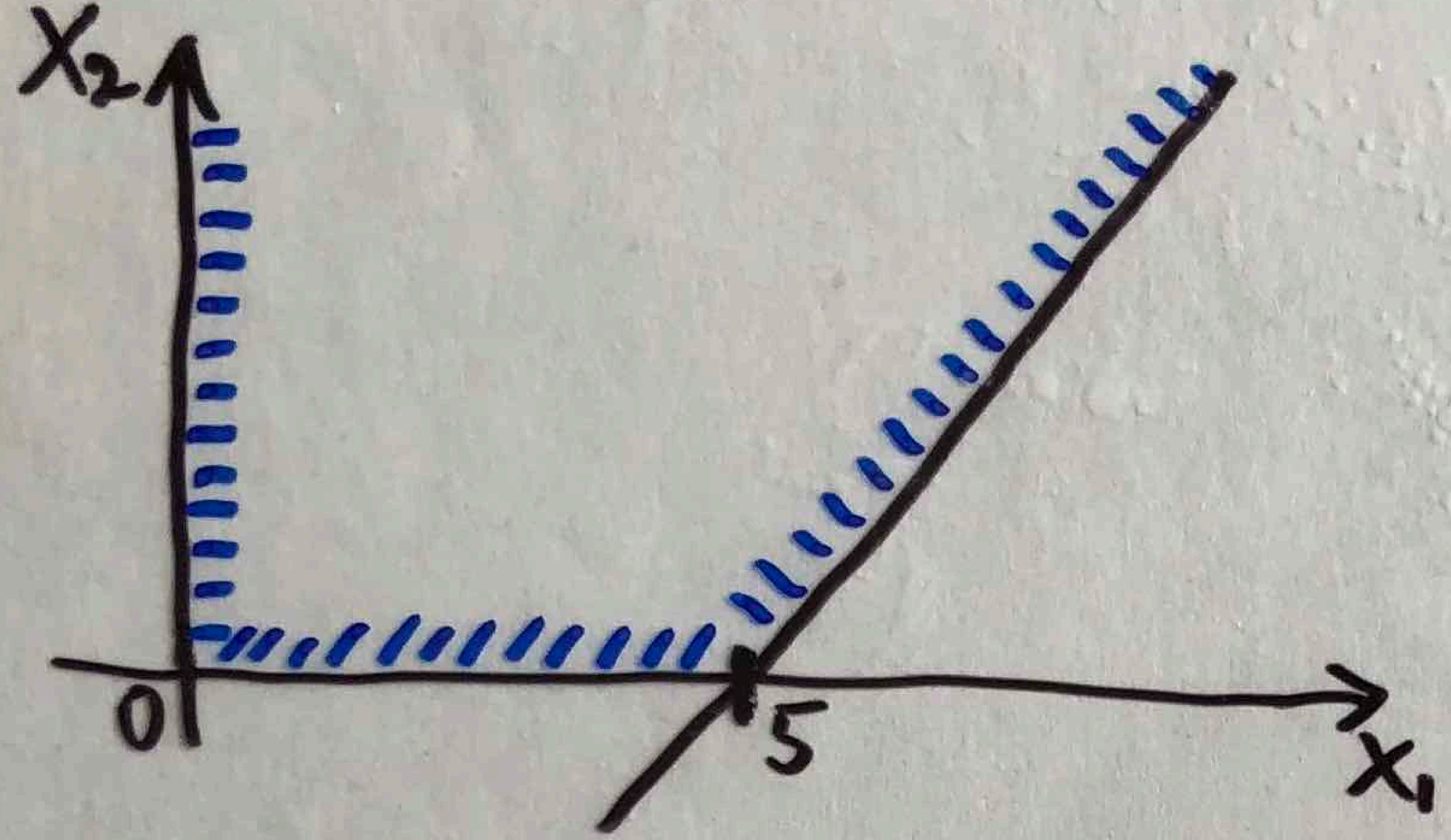
E.g., in our example $x^*=\binom{2}{2}$ is optimal with value $Z^*=2+2=4$

- This method is useful for 2-dimensional problems. What about higher dimensions?

# Classifications of LP problems 

Other example:

$$\max \quad x_1$$
$$\text{s.t.} \quad x_1 - x_2 \leq 5$$
$$x_1 \geq 0, \; x_2 \geq 0$$

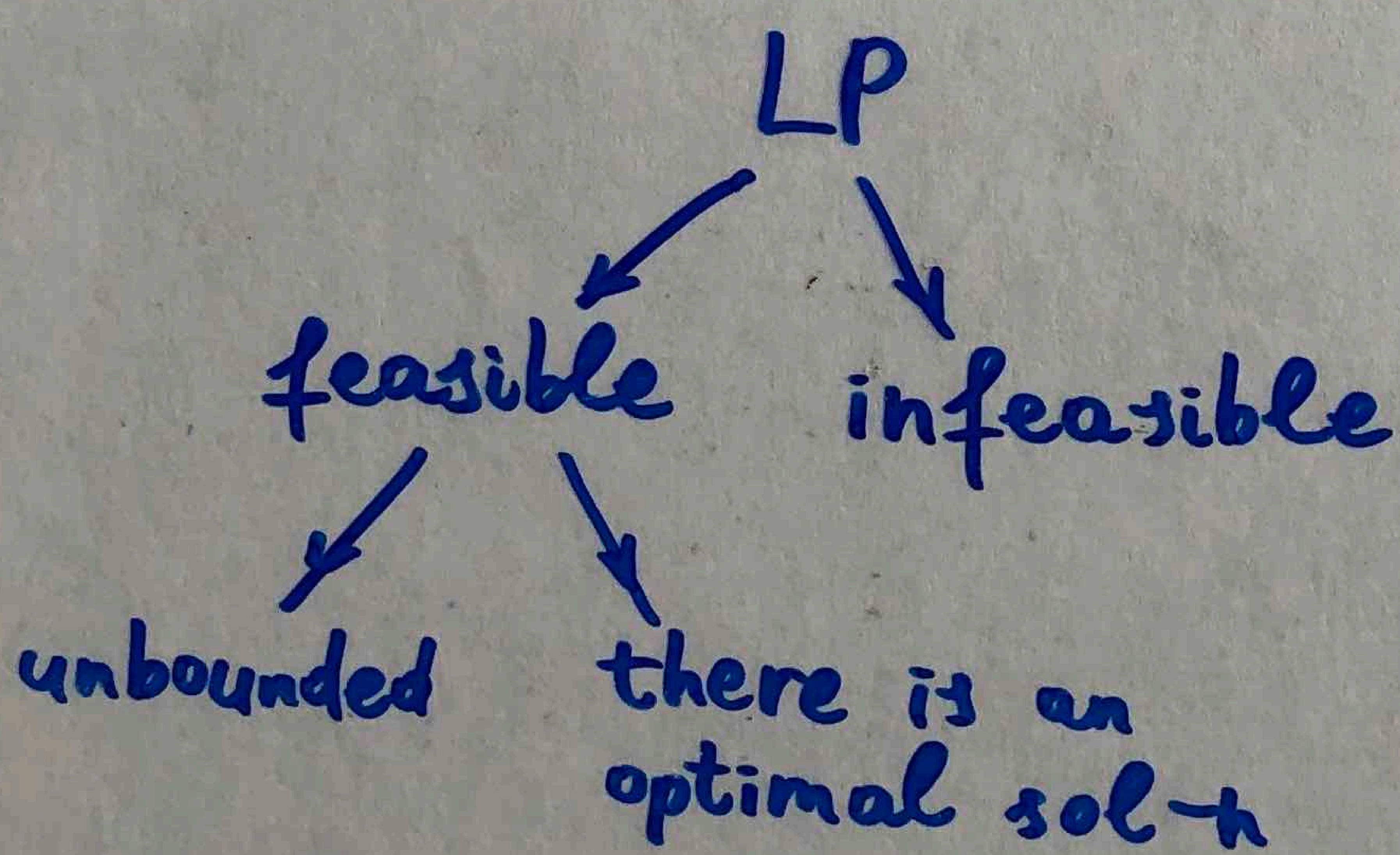



This problem is "unbounded": for every feasible point $\bar{x}$ there is another feasible point $\bar{x}'$ s.t.

$$z(\bar{x}') > z(\bar{x}).$$

$\Rightarrow$ no optimal sol-n.

- We learned 3 cases for LP:



# Converting LP's to Standard form

Other example:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1 - x_2 \geq 5 \\ & x_1 \geq 0,\ x_2 \text{ free (unrestricted)} \end{aligned}$$

- Recall the Standard form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

- Can make transformations to get the Standard form:
  - $\min Z \iff \max -Z$

    In our ex., $\min x_1 + x_2 \iff \max -x_1 - x_2$
  - "=" constraint $\iff$ "$\geq$" and "$\leq$" constraints

    In our ex., $x_1 + x_2 = 1 \iff \left\{ \begin{array}{l} x_1 + x_2 \geq 1 \\ x_1 + x_2 \leq 1 \end{array} \right\}$
  - $g(x) \geq b \iff -g(x) \leq -b$

    In our ex., $x_1 - x_2 \geq 5 \iff -x_1 + x_2 \leq -5$
  - $x_2$ free. Introduce new variables $x_3 \geq 0, x_4 \geq 0$ and write $x_2$ as $x_3 - x_4$
- The standard form of our example is:

$$\begin{aligned} \max \quad & -x_1 - x_3 + x_4 \\ \text{s.t.} \quad & x_1 + x_3 - x_4 \leq 1 \\ & -x_1 - x_3 + x_4 \leq -1 \\ & -x_1 + x_3 - x_4 \leq -5 \\ & x_1, x_3, x_4 \geq 0 \end{aligned}$$