Last time

- CPF solutions
- Structure of Simplex Method
- Basic solutions

Today

- Continue with basic solutions
- Details of Simplex Method

- $\bar{X}_B$ – basic variables
- The set of basic variables is called basis
- $\bar{X}_N$ – nonbasic variables

Ex.:
$$
\begin{aligned}
2X_1 + X_2 + X_3 \quad\quad &= 6 \\
2X_1 + 4X_2 \quad\quad + X_4 &= 12 \\
X_i \geq 0 \ \forall i &
\end{aligned}
$$

basic sol-ns:

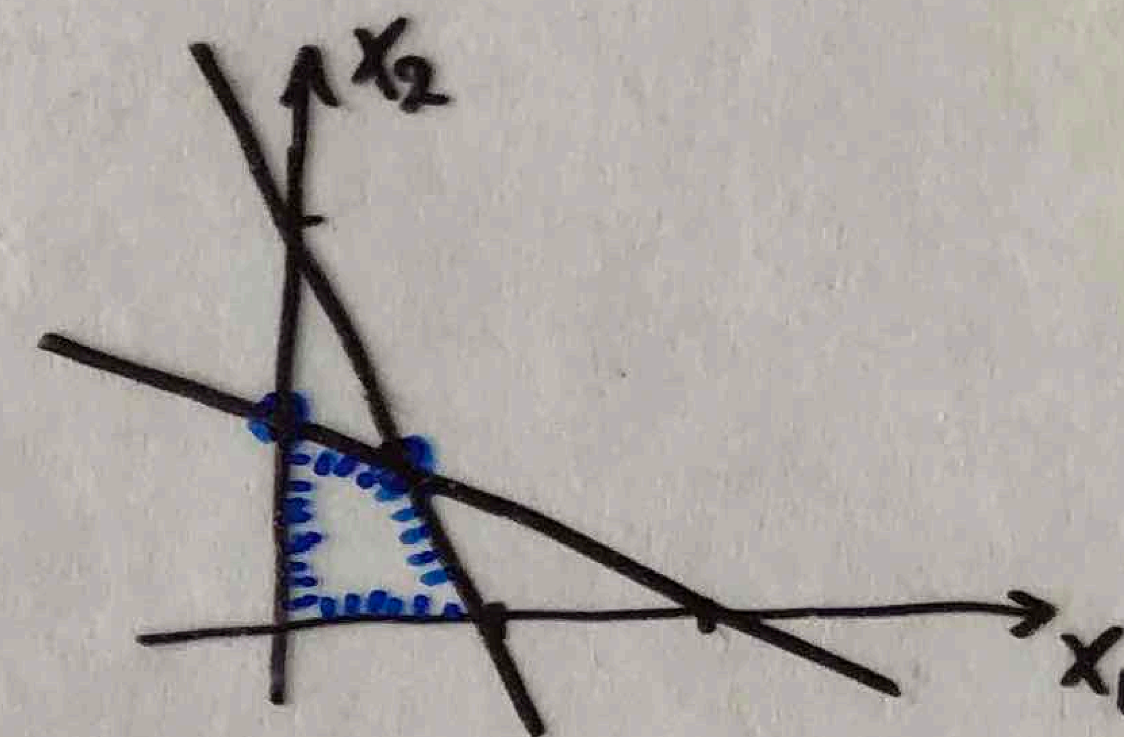
$$
x \in \left\{ \underset{\substack{B=\{3,4\} \\ N=\{1,2\}}}{\begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \end{pmatrix}}, \underset{B=\{1,4\}}{\begin{pmatrix} 3 \\ 0 \\ 0 \\ 6 \end{pmatrix}}, \underset{B=\{2,3\}}{\begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}}, \underset{B=\{1,2\}}{\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}}, \underset{B=\{2,4\}}{\begin{pmatrix} 0 \\ 6 \\ 0 \\ -12 \end{pmatrix}}, \underset{B=\{1,3\}}{\begin{pmatrix} 6 \\ 0 \\ -6 \\ 0 \end{pmatrix}} \right\}
$$

Correspond exactly to intersection of constraints!

Def-n: If all components ≥ 0, then basic sol-n is called **feasible** (BF) for LP; otherwise it is called **infeasible**.

- CPFs and BFs are the same in LP.
- If two CPFs are adjacent then corresponding BFs are also called adjacent.
- Two BFs are adjacent $\Leftrightarrow$ all but one of their nonbasic variables are the same.

Ex.: $\begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$



Rough algorithm statement (Simplex Method)

- Start with a BF sol-n (Q: how to find one?)
- REPEAT
  - look for an adjacent BF sol-n with a better obj. f-n value (how?)

  UNTIL

  no better adjacent BF sol-n can be found.

Simplex applied to our example

$$\max Z$$

$$\text{s.t. } Z - x_1 - x_2 = 0 \quad (0)$$

$$2x_1 + x_2 + x_3 = 6 \quad (1)$$

$$2x_1 + 4x_2 + x_4 = 12 \quad (2)$$

$$x_i \geq 0 \quad \forall i = 1, \dots, 4$$

- Start with "obvious" BF sol-n:

  $N = \{1,2\}$, $B = \{3,4\}$ and so $x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \end{pmatrix}$, $Z = 0$

ITERATION 1:

- Is there a better adjacent BF sol-n?

  Increasing x_1 increases the obj. f-n $\rightarrow$
  - Increase x_1 (enters the basis)
  - while keeping x_2 at 0

- What should we do with x_3 and x_4 to stay feasible?

  - $\left.\begin{array}{r} x_3 \geq 0 \\ x_3 = 6 - 2x_1 \end{array}\right\} \Rightarrow 6 - 2x_1 \geq 0 \Rightarrow x_1 \leq \frac{6}{2} = 3 \Rightarrow$

    $\Rightarrow$ if x_1 gets greater than 3, we will lose feasibility.

  - $\left.\begin{array}{r} x_4 \geq 0 \\ x_4 = 12 - 2x_1 \end{array}\right\} \Rightarrow 12 - 2x_1 \geq 0 \Rightarrow x_1 \leq \frac{12}{2} = 6 \Rightarrow$

    $\Rightarrow$ if x_1 gets greater than 6, we will lose feasibility

Thus, x_1 can be increased up to 3, and at that point x_3 drops to 0 (leaves the basis)

<u>New basis</u>:

$B = \{1,4\}$, $N = \{2,3\}$.

- Do elementary algebraic operations to have coefficients of basic variables
  - 0's in equation (0);
  - identity matrix in the functional constraints.

To realize that, the elementary operations are
- divide equation (1) by 2
- subtract 2 times new eq-n (1) from eq-n (2)
- add new eq-n (1) to eq-n (0)

The resulting system is:

$$
\begin{array}{lrcl}
\max Z & & & \\
\text{s.t. } Z & -\frac{1}{2}x_2 + \frac{1}{2}x_3 & = 3 & (0') \\
& x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 & = 3 & (1') \\
& 3x_2 - x_3 + x_4 & = 6 & (2') \\
& x_i \geq 0 \quad \forall i = 1,..,4 & &
\end{array}
$$

This new LP is <u>equivalent</u> to the original one.

From the new system, $x = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 6 \end{pmatrix}$, $Z = 3$

<u>END OF ITERATION 1</u>

ITERATION 2:

- Increasing x_2 improves the obj. f-n value → x_2 enters the basis
- To stay feasible, perform minimum ratio test:

$$\min\left(\frac{3}{1/2}, \frac{6}{3}\right) = \min(6, 2) = 2 \rightarrow$$

can increase x_2 by at most 2, x_4 leaves the basis

New basis: $B = \{1, 2\}$, $N = \{3, 4\}$

- After elementary operations, we get

$$\begin{aligned}
\max\ & z \\
\text{s.t. } & z \qquad\quad + \tfrac{1}{3}x_3 + \tfrac{1}{6}x_4 = 4 \qquad (0'') \\
& x_1 \qquad + \tfrac{2}{3}x_3 - \tfrac{1}{6}x_4 = 2 \qquad (1'') \\
& \quad x_2 - \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 = 2 \qquad (2'') \\
& x_i \geq 0 \quad \forall i = 1, .., 4
\end{aligned}$$

From this system, $x = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $z = 4$

END OF ITERATION 2

ITERATION 3:

- Increasing x_3 or x_4 decreases z → we are at an optimal BF sol-n → return $x^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $z^* = 4$

Restatement of Simplex Method

① Start with a BF sol-n

② Determine entering nonbasic variable X_j by examining objective f-n coefficients.
If all coefficients ≥ 0 then stop (we are at an optimal sol-n) otherwise choose most negative coefficient.

③ Determine leaving basic variable by performing "min-ratio test": find basic variable ($\in$ row i) such that

$$\left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0 \right\} \text{ is minimal.}$$

④ Do elementary algebraic operations to get an equivalent system conforming the new basis. Go to step ②.

Still need more details to make Simplex work in all situations.

Important Questions

- How to find initial BF sol-n?
- How to deal with infeasible problems?
- How to deal with unboundedness?
- Multiple optimal sol-ns?
- How to break ties?

Answers next time.