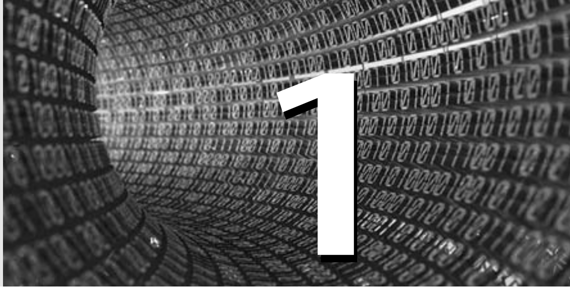

Matrices

1.1 Basic Concepts

Definition 1 A *matrix* is a rectangular array of elements arranged in horizontal rows and vertical columns. Thus,

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & -1 \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}, \tag{2}$$

and

$$\begin{bmatrix} \sqrt{2} \\ \pi \\ 19.5 \end{bmatrix} \tag{3}$$

are all examples of a matrix.

The matrix given in (1) has two rows and three columns; it is said to have *order* (or size) 2×3 (read two by three). By convention, the row index is always given first. The matrix in (2) has order 3×3, while that in (3) has order 3×1. The entries of a matrix are called *elements*.

In general, a matrix $\mathbf{A}$ (matrices will always be designated by uppercase boldface letters) of order $p \times n$ is given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} \end{bmatrix}, \tag{4}$$

which is often abbreviated to $[a_{ij}]_{p \times n}$ or just $[a_{ij}]$. In this notation, a_{ij} represents the general element of the matrix and appears in the ith row and the jth column. The subscript i, which represents the row, can have any value 1 through p, while the subscript j, which represents the column, runs 1 through n. Thus, if $i = 2$ and $j = 3$, a_{ij} becomes a_{23} and designates the element in the second row and third column. If $i = 1$ and $j = 5$, a_{ij} becomes a_{15} and signifies the element in the first row, fifth column. Note again that the row index is always given before the column index.

Any element having its row index equal to its column index is a *diagonal element*. Thus, the diagonal elements of a matrix are the elements in the 1–1 position, 2–2 position, 3–3 position, and so on, for as many elements of this type that exist. Matrix (1) has 1 and 0 as its diagonal elements, while matrix (2) has 4, 2, and 2 as its diagonal elements.

If the matrix has as many rows as columns, $p = n$, it is called a *square matrix*; in general it is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}. \tag{5}$$

In this case, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ lie on and form the *main* (or principal) *diagonal*.

It should be noted that the elements of a matrix need not be numbers; they can be, and quite often arise physically as, functions, operators or, as we shall see later, matrices themselves. Hence,

$$\begin{bmatrix} \int_0^1 (t^2 + 1)dt & t^2 & \sqrt{3t} & 2 \end{bmatrix}, \quad \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix},$$

and

$$\begin{bmatrix} x^2 & x \\ e^x & \frac{d}{dx}\ln x \\ 5 & x + 2 \end{bmatrix}$$

are good examples of matrices. Finally, it must be noted that a matrix is an entity unto itself; it is not a number. If the reader is familiar with determinants, he will undoubtedly recognize the similarity in form between the two. *Warning*: the similarity ends there. Whereas a determinant (see Chapter 5) can be evaluated to yield a number, a matrix cannot. A matrix is a rectangular array, period.

Problems 1.1

1. Determine the orders of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 & 4 & 7 \\ 2 & 5 & -6 & 5 & 7 \\ 0 & 3 & 1 & 2 & 0 \\ -3 & -5 & 2 & 2 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 3 & 2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & -7 & 8 \\ 10 & 11 & 12 & 12 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & t & t^2 & 0 \\ t-2 & t^4 & 6t & 5 \\ t+2 & 3t & 1 & 2 \\ 2t-3 & -5t^2 & 2t^5 & 3t^2 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{5} & -\frac{5}{6} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 \\ 5 \\ 10 \\ 0 \\ -4 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \sqrt{313} & -505 \\ 2\pi & 18 \\ 46.3 & 1.043 \\ 2\sqrt{5} & -\sqrt{5} \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{J} = [1 \quad 5 \quad -30].$$

2. Find, if they exist, the elements in the 1–3 and the 2–1 positions for each of the matrices defined in Problem 1.

3. Find, if they exist, $a_{23}, a_{32}, b_{31}, b_{32}, c_{11}, d_{22}, e_{13}, g_{22}, g_{23}$, and h_{32} for the matrices defined in Problem 1.

4. Construct the 2×2 matrix $\mathbf{A}$ having $a_{ij} = (-1)^{i+j}$.

5. Construct the 3×3 matrix $\mathbf{A}$ having $a_{ij} = i/j$.

6. Construct the $n \times n$ matrix $\mathbf{B}$ having $b_{ij} = n - i - j$. What will this matrix be when specialized to the 3×3 case?

7. Construct the 2×4 matrix $\mathbf{C}$ having

$$c_{ij} = \begin{cases} i & \text{when } i = 1, \\ j & \text{when } i = 2. \end{cases}$$

8. Construct the 3×4 matrix $\mathbf{D}$ having

$$d_{ij} = \begin{cases} i + j & \text{when } i > j, \\ 0 & \text{when } i = j, \\ i - j & \text{when } i < j. \end{cases}$$

9. Express the following times as matrices: (a) A quarter after nine in the morning. (b) Noon. (c) One thirty in the afternoon. (d) A quarter after nine in the evening.

10. Express the following dates as matrices:

(a) July 4, 1776 (b) December 7, 1941

(c) April 23, 1809 (d) October 31, 1688

11. A gasoline station currently has in inventory 950 gallons of regular unleaded gasoline, 1253 gallons of premium, and 98 gallons of super. Express this inventory as a matrix.

12. Store 1 of a three store chain has 3 refrigerators, 5 stoves, 3 washing machines, and 4 dryers in stock. Store 2 has in stock no refrigerators, 2 stoves, 9 washing machines, and 5 dryers, while store 3 has in stock 4 refrigerators, 2 stoves, and no washing machines or dryers. Present the inventory of the entire chain as a matrix.

13. The number of damaged items delivered by the SleepTight Mattress Company from its various plants during the past year is given by the matrix

$$\begin{bmatrix} 80 & 12 & 16 \\ 50 & 40 & 16 \\ 90 & 10 & 50 \end{bmatrix}.$$

The rows pertain to its three plants in Michigan, Texas, and Utah. The columns pertain to its regular model, its firm model, and its extra-firm model, respectively. The company's goal for next year to is to reduce by 10% the number of damaged regular mattresses shipped by each plant, to reduce by 20% the number of damaged firm mattresses shipped by its Texas plant, to reduce by 30% the number of damaged extra-firm mattresses shipped by its Utah plant, and to keep all other entries the same as last year. What will next year's matrix be if all goals are realized?

14. A person purchased 100 shares of AT&T at \$27 per share, 150 shares of Exxon at \$45 per share, 50 shares of IBM at \$116 per share, and 500 shares of PanAm at \$2 per share. The current price of each stock is \$29, \$41, \$116, and \$3, respectively. Represent in a matrix all the relevant information regarding this person's portfolio.

15. On January 1, a person buys three certificates of deposit from different institutions, all maturing in one year. The first is for \$1000 at 7%, the second is for \$2000 at 7.5%, and the third is for \$3000 at 7.25%. All interest rates are effective on an annual basis.

(a) Represent in a matrix all the relevant information regarding this person's holdings.

(b) What will the matrix be one year later if each certificate of deposit is renewed for the current face amount and accrued interest at rates one half a percent higher than the present?

16. **(Markov Chains, see Chapter 9)** A finite Markov chain is a set of objects, a set of consecutive time periods, and a finite set of different states such that

(i) during any given time period, each object is in only state (although different objects can be in different states), and

(ii) the probability that an object will move from one state to another state (or remain in the same state) over a time period depends only on the beginning and ending states.

A Markov chain can be represented by a matrix $\mathbf{P} = [p_{ij}]$ where p_{ij} represents the probability of an object moving from state i to state j in one time period. Such a matrix is called a *transition matrix*.

Construct a transition matrix for the following Markov chain: Census figures show a population shift away from a large mid-western metropolitan city to its suburbs. Each year, 5% of all families living in the city move to the suburbs while during the same time period only 1% of those living in the suburbs move into the city. *Hint*: Take state 1 to represent families living in the city, state 2 to represent families living in the suburbs, and one time period to equal a year.

17. Construct a transition matrix for the following Markov chain: Every four years, voters in a New England town elect a new mayor because a town ordinance prohibits mayors from succeeding themselves. Past data indicate that a Democratic mayor is succeeded by another Democrat 30% of the time and by a Republican 70% of the time. A Republican mayor, however, is succeeded by another Republican 60% of the time and by a Democrat 40% of the time. *Hint*: Take state 1 to represent a Republican mayor in office, state 2 to represent a Democratic mayor in office, and one time period to be four years.

18. Construct a transition matrix for the following Markov chain: The apple harvest in New York orchards is classified as poor, average, or good. Historical data indicates that if the harvest is poor one year then there is a 40% chance of having a good harvest the next year, a 50% chance of having an average harvest, and a 10% chance of having another poor harvest. If a harvest is average one year, the chance of a poor, average, or good harvest the next year is 20%, 60%, and 20%, respectively. If a harvest is good, then the chance of a poor, average, or good harvest the next year is 25%, 65%, and 10%, respectively. *Hint*: Take state 1 to be a poor harvest, state 2 to be an average harvest, state 3 to be a good harvest, and one time period to equal one year.

19. Construct a transition matrix for the following Markov chain. Brand X and brand Y control the majority of the soap powder market in a particular region, and each has promoted its own product extensively. As a result of past advertising campaigns, it is known that over a two year period of time 10% of brand Y customers change to brand X and 25% of all other customers change to brand X. Furthermore, 15% of brand X customers change to brand Y and 30% of all other customers change to brand Y. The major brands also lose customers to smaller competitors, with 5% of brand X customers switching to a minor brand during a two year time period and 2% of brand Y customers doing likewise. All other customers remain loyal to their past brand of soap powder. *Hint*: Take state 1 to be a brand X customer, state 2 a brand Y customer, state 3 another brand customer, and one time period to be two years.

1.2 Operations

The simplest relationship between two matrices is equality. Intuitively one feels that two matrices should be equal if their corresponding elements are equal. This is the case, providing the matrices are of the same order.

Definition 1 Two matrices $\mathbf{A} = [a_{ij}]_{p\times n}$ and $\mathbf{B} = [b_{ij}]_{p\times n}$ are equal if they have the same order and if $a_{ij} = b_{ij}$ $(i = 1, 2, 3, \ldots, p;\ j = 1, 2, 3, \ldots, n)$. Thus, the equality

$$\begin{bmatrix} 5x + 2y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

implies that $5x + 2y = 7$ and $x - 3y = 1$.

The intuitive definition for matrix addition is also the correct one.

Definition 2 If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are both of order $p \times n$, then $\mathbf{A} + \mathbf{B}$ is a $p \times n$ matrix $\mathbf{C} = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$ $(i = 1, 2, 3, \ldots, p;\ j = 1, 2, 3, \ldots, n)$. Thus,

$$\begin{bmatrix} 5 & 1 \\ 7 & 3 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 2 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 + (-6) & 1 + 3 \\ 7 + 2 & 3 + (-1) \\ (-2) + 4 & (-1) + 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 9 & 2 \\ 2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} t^2 & 5 \\ 3t & 0 \end{bmatrix} + \begin{bmatrix} 1 & -6 \\ t & -t \end{bmatrix} = \begin{bmatrix} t^2 + 1 & -1 \\ 4t & -t \end{bmatrix};$$

but the matrices

$$\begin{bmatrix} 5 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -6 & 2 \\ 1 & 1 \end{bmatrix}$$

cannot be added since they are not of the same order.

It is not difficult to show that the addition of matrices is both commutative and associative: that is, if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ represent matrices of the same order, then

(A1) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$,

(A2) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

We define a zero matrix $\mathbf{0}$ to be a matrix consisting of only zero elements. Zero matrices of every order exist, and when one has the same order as another matrix $\mathbf{A}$, we then have the additional property

(A3) $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

Subtraction of matrices is defined in a manner analogous to addition: the orders of the matrices involved must be identical and the operation is performed elementwise.

Thus,

$$\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 6 & -1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -7 & 3 \end{bmatrix}.$$

Another simple operation is that of multiplying a scalar times a matrix. Intuition guides one to perform the operation elementwise, and once again intuition is correct. Thus, for example,

$$7\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ -21 & 28 \end{bmatrix} \quad \text{and} \quad t\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 3t & 2t \end{bmatrix}.$$

Definition 3 If $\mathbf{A} = [a_{ij}]$ is a $p \times n$ matrix and if λ is a scalar, then $\lambda\mathbf{A}$ is a $p \times n$ matrix $\mathbf{B} = [b_{ij}]$ where $b_{ij} = \lambda a_{ij} (i = 1, 2, 3, \ldots, p;\ j = 1, 2, 3, \ldots, n)$.

Example 1 Find $5\mathbf{A} - \frac{1}{2}\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -20 \\ 18 & 8 \end{bmatrix}$$

Solution

$$\begin{aligned} 5\mathbf{A} - \tfrac{1}{2}\mathbf{B} &= 5\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 6 & -20 \\ 18 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 5 \\ 0 & 15 \end{bmatrix} - \begin{bmatrix} 3 & -10 \\ 9 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 15 \\ -9 & 11 \end{bmatrix}. \quad \blacksquare \end{aligned}$$

It is not difficult to show that if λ_1 and λ_2 are scalars, and if $\mathbf{A}$ and $\mathbf{B}$ are matrices of identical order, then

(S1) $\lambda_1\mathbf{A} = \mathbf{A}\lambda_1$,

(S2) $\lambda_1(\mathbf{A} + \mathbf{B}) = \lambda_1\mathbf{A} + \lambda_1\mathbf{B}$,

(S3) $(\lambda_1 + \lambda_2)\mathbf{A} = \lambda_1\mathbf{A} + \lambda_2\mathbf{A}$,

(S4) $\lambda_1(\lambda_2\mathbf{A}) = (\lambda_1\lambda_2)\mathbf{A}$.

The reader is cautioned that there is *no* such operation as matrix division. We will, however, define a somewhat analogous operation, namely matrix inversion, in Chapter 3.

Problems 1.2

In Problems 1 through 26, let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

1. Find $2\mathbf{A}$.

2. Find $-5\mathbf{A}$.

3. Find $3\mathbf{D}$.

4. Find $10\mathbf{E}$.

5. Find $-\mathbf{F}$.

6. Find $\mathbf{A} + \mathbf{B}$.

7. Find $\mathbf{C} + \mathbf{A}$.

8. Find $\mathbf{D} + \mathbf{E}$.

9. Find $\mathbf{D} + \mathbf{F}$.

10. Find $\mathbf{A} + \mathbf{D}$.

11. Find $\mathbf{A} - \mathbf{B}$.

12. Find $\mathbf{C} - \mathbf{A}$.

13. Find $\mathbf{D} - \mathbf{E}$.

14. Find $\mathbf{D} - \mathbf{F}$.

15. Find $2\mathbf{A} + 3\mathbf{B}$.

16. Find $3\mathbf{A} - 2\mathbf{C}$.

17. Find $0.1\mathbf{A} + 0.2\mathbf{C}$.

18. Find $-2\mathbf{E} + \mathbf{F}$.

19. Find $\mathbf{X}$ if $\mathbf{A} + \mathbf{X} = \mathbf{B}$.

20. Find $\mathbf{Y}$ if $2\mathbf{B} + \mathbf{Y} = \mathbf{C}$.

21. Find $\mathbf{X}$ if $3\mathbf{D} - \mathbf{X} = \mathbf{E}$.

22. Find $\mathbf{Y}$ if $\mathbf{E} - 2\mathbf{Y} = \mathbf{F}$.

23. Find $\mathbf{R}$ if $4\mathbf{A} + 5\mathbf{R} = 10\mathbf{C}$.

24. Find $\mathbf{S}$ if $3\mathbf{F} - 2\mathbf{S} = \mathbf{D}$.

25. Verify directly that $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

26. Verify directly that $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$.

27. Find $6\mathbf{A} - \theta\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} \theta^2 & 2\theta - 1 \\ 4 & 1/\theta \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \theta^2 - 1 & 6 \\ 3/\theta & \theta^3 + 2\theta + 1 \end{bmatrix}.$$

28. Prove Property (A1).

29. Prove Property (A3).

30. Prove Property (S2).

31. Prove Property (S3).

32. (a) Mr. Jones owns 200 shares of IBM and 150 shares of AT&T. Determine a portfolio matrix that reflects Mr. Jones' holdings.

(b) Over the next year, Mr. Jones triples his holdings in each company. What is his new portfolio matrix?

(c) The following year Mr. Jones lists changes in his portfolio as $\begin{bmatrix} -50 & 100 \end{bmatrix}$. What is his new portfolio matrix?

33. The inventory of an appliance store can be given by a 1×4 matrix in which the first entry represents the number of television sets, the second entry the number of air conditioners, the third entry the number of refrigerators, and the fourth entry the number of dishwashers.

(a) Determine the inventory given on January 1 by $\begin{bmatrix} 15 & 2 & 8 & 6 \end{bmatrix}$.

(b) January sales are given by $\begin{bmatrix} 4 & 0 & 2 & 3 \end{bmatrix}$. What is the inventory matrix on February 1?

(c) February sales are given by $\begin{bmatrix} 5 & 0 & 3 & 3 \end{bmatrix}$, and new stock added in February is given by $\begin{bmatrix} 3 & 2 & 7 & 8 \end{bmatrix}$. What is the inventory matrix on March 1?

34. The daily gasoline supply of a local service station is given by a 1×3 matrix in which the first entry represents gallons of regular, the second entry gallons of premium, and the third entry gallons of super.

(a) Determine the supply of gasoline at the close of business on Monday given by $\begin{bmatrix} 14{,}000 & 8{,}000 & 6{,}000 \end{bmatrix}$.

(b) Tuesday's sales are given by $\begin{bmatrix} 3{,}500 & 2{,}000 & 1{,}500 \end{bmatrix}$. What is the inventory matrix at day's end?

(c) Wednesday's sales are given by $\begin{bmatrix} 5{,}000 & 1{,}500 & 1{,}200 \end{bmatrix}$. In addition, the station received a delivery of 30,000 gallons of regular, 10,000 gallons of premium, but no super. What is the inventory at day's end?

35. On a recent shopping trip Mary purchased 6 oranges, a dozen grapefruits, 8 apples, and 3 lemons. John purchased 9 oranges, 2 grapefruits, and 6 apples. Express each of their purchases as 1×4 matrices. What is the physical significance of the sum of these matrices?

1.3 Matrix Multiplication

Matrix multiplication is the first operation we encounter where our intuition fails. First, two matrices are *not* multiplied together elementwise. Secondly, it is not always possible to multiply matrices of the same order while it is possible to multiply certain matrices of different orders. Thirdly, if **A** and **B** are two matrices for which multiplication is defined, it is generally not the case that $\mathbf{AB} = \mathbf{BA}$; that is, *matrix multiplication is not a commutative operation.* There are other properties of matrix multiplication, besides the three mentioned that defy our intuition, and we shall illustrate them shortly. We begin by determining which matrices can be multiplied.

Rule 1 The product of two matrices **AB** is defined if the number of columns of **A** equals the number of rows of **B**.

Thus, if **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 3 & 2 & -2 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix}, \tag{6}$$

then the product **AB** is defined since **A** has three columns and **B** has three rows. The product **BA**, however, is not defined since **B** has four columns while **A** has only two rows.

When the product is written **AB**, **A** is said to *premultiply* **B** while **B** is said to *postmultiply* **A**.

Rule 2 If the product **AB** is defined, then the resultant matrix will have the same number of rows as **A** and the same number of columns as **B**.

Thus, the product **AB**, where **A** and **B** are given in (6), will have two rows and four columns since **A** has two rows and **B** has four columns.

An easy method of remembering these two rules is the following: write the orders of the matrices on paper in the sequence in which the multiplication is to be carried out; that is, if **AB** is to be found where **A** has order 2×3 and **B** has order 3×4, write

$$(2 \times \underset{\smile}{3)(3} \times 4) \tag{7}$$

If the two adjacent numbers (indicated in (7) by the curved arrow) are both equal (in the case they are both three), the multiplication is defined. The order of the product matrix is obtained by canceling the adjacent numbers and using the two remaining numbers. Thus in (7), we cancel the adjacent 3's and are left with 2×4, which in this case, is the order of **AB**.

As a further example, consider the case where **A** is 4×3 matrix while **B** is a 3×5 matrix. The product **AB** is defined since, in the notation $(4 \times \underset{\smile}{3)(3} \times 5)$, the adjacent numbers denoted by the curved arrow are equal. The product will be a 4×5 matrix. The product **BA** however is not defined since in the notation $(3 \times \underset{\smile}{5)(4} \times 3)$ the adjacent numbers are not equal. In general, one may schematically state the method as

$$(k \times \underset{\smile}{n)(n} \times p) = (k \times p).$$

Rule 3 If the product $\mathbf{AB} = \mathbf{C}$ is defined, where **C** is denoted by $[c_{ij}]$, then the element c_{ij} is obtained by multiplying the elements in ith row of **A** by the corresponding elements in the jth column of **B** and adding. Thus, if **A** has order $k \times n$, and B has order $n \times p$, and

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kp} \end{bmatrix},$$

then c_{11} is obtained by multiplying the elements in the first row of **A** by the corresponding elements in the first column of **B** and adding; hence,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}.$$

The element c_{12} is found by multiplying the elements in the first row of **A** by the corresponding elements in the second column of **B** and adding; hence.

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}.$$

The element c_{kp} is obtained by multiplying the elements in the kth row of **A** by the corresponding elements in the pth column of **B** and adding; hence,

$$c_{kp} = a_{k1}b_{1p} + a_{k2}b_{2p} + \cdots + a_{kn}b_{np}.$$

Example 1 Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -7 & -8 \\ 9 & 10 \\ 0 & -11 \end{bmatrix}.$$

Solution

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -7 & -8 \\ 9 & 10 \\ 0 & -11 \end{bmatrix} \\
&= \begin{bmatrix} 1(-7) + 2(9) + 3(0) & 1(-8) + 2(10) + 3(-11) \\ 4(-7) + 5(9) + 6(0) & 4(-8) + 5(10) + 6(-11) \end{bmatrix} \\
&= \begin{bmatrix} -7 + 18 + 0 & -8 + 20 - 33 \\ -28 + 45 + 0 & -32 + 50 - 66 \end{bmatrix} = \begin{bmatrix} 11 & -21 \\ 17 & -48 \end{bmatrix}, \\
\mathbf{BA} &= \begin{bmatrix} -7 & -8 \\ 9 & 10 \\ 0 & -11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\
&= \begin{bmatrix} (-7)1 + (-8)4 & (-7)2 + (-8)5 & (-7)3 + (-8)6 \\ 9(1) + 10(4) & 9(2) + 10(5) & 9(3) + 10(6) \\ 0(1) + (-11)4 & 0(2) + (-11)5 & 0(3) + (-11)6 \end{bmatrix} \\
&= \begin{bmatrix} -7 - 32 & -14 - 40 & -21 - 48 \\ 9 + 40 & 18 + 50 & 27 + 60 \\ 0 - 44 & 0 - 55 & 0 - 66 \end{bmatrix} = \begin{bmatrix} -39 & -54 & -69 \\ 49 & 68 & 87 \\ -44 & -55 & -66 \end{bmatrix}. \quad \blacksquare
\end{aligned}$$

The preceding three rules can be incorporated into the following formal definition:

Definition 1 If $\mathbf{A} = [a_{ij}]$ is a $k \times n$ matrix and $\mathbf{B} = [b_{ij}]$ is an $n \times p$ matrix, then the product $\mathbf{AB}$ is defined to be a $k \times p$ matrix $\mathbf{C} = [c_{ij}]$ where $c_{ij} = \sum_{l=1}^{n} a_{il}b_{lj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$ $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, p)$.

Example 2 Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 1 & 5 & -1 \\ 4 & -2 & 1 & 0 \end{bmatrix}.$$

Solution

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 & -1 \\ 4 & -2 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2(3)+1(4) & 2(1)+1(-2) & 2(5)+1(1) & 2(-1)+1(0) \\ -1(3)+0(4) & -1(1)+0(-2) & -1(5)+0(1) & -1(-1)+0(0) \\ 3(3)+1(4) & 3(1)+1(-2) & 3(5)+1(1) & 3(-1)+1(0) \end{bmatrix} \\ &= \begin{bmatrix} 10 & 0 & 11 & -2 \\ -3 & -1 & -5 & 1 \\ 13 & 1 & 16 & -3 \end{bmatrix}. \end{aligned}$$

Note that in this example the product **BA** is not defined. ■

Example 3 Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Solution

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(4)+1(1) & 2(0)+1(2) \\ -1(4)+3(1) & -1(0)+3(2) \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ -1 & 6 \end{bmatrix};$$

$$\mathbf{BA} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4(2)+0(-1) & 4(1)+0(3) \\ 1(2)+2(-1) & 1(1)+2(3) \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 0 & 7 \end{bmatrix}.$$

This, therefore, is an example where both products **AB** and **BA** are defined but unequal. ■

Example 4 Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution

$$\mathbf{AB} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 8 \end{bmatrix},$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 8 \end{bmatrix}.$$

This, therefore, is an example where both products **AB** and **BA** are defined and equal. ■

In general, it can be shown that matrix multiplication has the following properties:

(M1) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (Associative Law)

(M2) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (Left Distributive Law)

(M3) $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ (Right Distributive Law)

providing that the matrices **A**, **B**, **C** have the correct order so that the above multiplications and additions are defined. The one basic property that matrix multiplication does not possess is commutativity; that is, in general, **AB** does not equal **BA** (see Example 3). We hasten to add, however, that while matrices in general do not commute, it may very well be the case that, given two particular matrices, they do commute as can be seen from Example 4.

Commutativity is not the only property that matrix multiplication lacks. We know from our experiences with real numbers that if the product $xy = 0$, then either $x = 0$ or $y = 0$ or both are zero. Matrices do not possess this property as the following example shows:

Example 5 Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix}.$$

Solution

$$\mathbf{AB} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 4(3) + 2(-6) & 4(-4) + 2(8) \\ 2(3) + 1(-6) & 2(-4) + 1(8) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, even though neither **A** nor **B** is zero, their product is zero. ■

One final "unfortunate" property of matrix multiplication is that the equation $\mathbf{AB} = \mathbf{AC}$ does not imply $\mathbf{B} = \mathbf{C}$.

Example 6 Find **AB** and **AC** if

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}.$$

Solution

$$\mathbf{AB} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4(1)+2(2) & 4(1)+2(1) \\ 2(1)+1(2) & 2(1)+1(1) \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix};$$

$$\mathbf{AC} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4(2)+2(0) & 4(2)+2(-1) \\ 2(2)+1(0) & 2(2)+1(-1) \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix}.$$

Thus, cancellation is not a valid operation in the matrix algebra. ■

The reader has no doubt wondered why this seemingly complicated procedure for matrix multiplication has been introduced when the more obvious methods of multiplying matrices termwise could be used. The answer lies in systems of simultaneous linear equations. Consider the set of simultaneous linear equations given by

$$\begin{aligned} 5x - 3y + 2z &= 14, \\ x + y - 4z &= -7, \\ 7x \qquad - 3z &= 1. \end{aligned} \tag{8}$$

This system can easily be solved by the method of substitution. Matrix algebra, however, will give us an entirely new method for obtaining the solution.

Consider the matrix equation

$$\mathbf{Ax} = \mathbf{b} \tag{9}$$

where

$$\mathbf{A} = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 1 & -4 \\ 7 & 0 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 14 \\ -7 \\ 1 \end{bmatrix}.$$

Here $\mathbf{A}$, called the *coefficient matrix*, is simply the matrix whose elements are the coefficients of the unknowns x, y, z in (8). (Note that we have been very careful to put all the x coefficients in the first column, all the y coefficients in the second column, and all the z coefficients in the third column. The zero in the (3, 2) entry appears because the y coefficient in the third equation of system (8) is zero.) **x** and **b** are obtained in the obvious manner. One note of warning: there is a basic difference between the unknown matrix **x** in (9) and the unknown variable x. The reader should be especially careful not to confuse their respective identities.

Now using our definition of matrix multiplication, we have that

$$\mathbf{Ax} = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 1 & -4 \\ 7 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (5)(x) + (-3)(y) + (2)(z) \\ (1)(x) + (1)(y) + (-4)(z) \\ (7)(x) + (0)(y) + (-3)(z) \end{bmatrix}$$

$$= \begin{bmatrix} 5x - 3y + 2z \\ x + \ y - 4z \\ 7x \qquad - 3z \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \\ 1 \end{bmatrix}. \tag{10}$$

Using the definition of matrix equality, we see that (10) is precisely system (8). Thus (9) is an alternate way of representing the original system. It should come as no surprise, therefore, that by redefining the matrices **A**, **x**, **b**, appropriately, we can represent any system of simultaneous linear equations by the matrix equation $\mathbf{Ax} = \mathbf{b}$.

Example 7 Put the following system into matrix form:

$$\begin{aligned} x - \ y + \ z + \ w &= 5, \\ 2x + \ y - \ z \qquad &= 4, \\ 3x + 2y \qquad + 2w &= 0, \\ x - 2y + 3z + 4w &= -1. \end{aligned}$$

Solution Define

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 0 & 2 \\ 1 & -2 & 3 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ -1 \end{bmatrix}.$$

The original system is then equivalent to the matrix system $\mathbf{Ax} = \mathbf{b}$. ■

Unfortunately, we are not yet in a position to solve systems that are in matrix form $\mathbf{Ax} = \mathbf{b}$. One method of solution depends upon the operation of inversion, and we must postpone a discussion of it until the inverse has been defined. For the present, however, we hope that the reader will be content with the knowledge that matrix multiplication, as we have defined it, does serve some useful purpose.

Problems 1.3

1. The order of **A** is 2×4, the order of **B** is 4×2, the order of **C** is 4×1, the order of **D** is 1×2, and the order of **E** is 4×4. Find the orders of

(a) **AB**, (b) **BA**, (c) **AC**, (d) **CA**,
(e) **CD**, (f) **AE**, (g) **EB**, (h) **EA**,
(i) **ABC**, (j) **DAE**, (k) **EBA**, (l) **EECD**.

In Problems 2 through 19, let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix},$$

$$\mathbf{X} = [1 \quad -2], \qquad \mathbf{Y} = [1 \quad 2 \quad 1].$$

2. Find **AB**. **3.** Find **BA**. **4.** Find **AC**.

5. Find **BC**. **6.** Find **CB**. **7.** Find **XA**.

8. Find **XB**. **9.** Find **XC**. **10.** Find **AX**.

11. Find **CD**. **12.** Find **DC**. **13.** Find **YD**.

14. Find **YC**. **15.** Find **DX**. **16.** Find **XD**.

17. Find **EF**. **18.** Find **FE**. **19.** Find **YF**.

20. Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}.$$

Note that $\mathbf{AB} = \mathbf{0}$ but neither **A** nor **B** equals the zero matrix.

21. Find **AB** and **CB** if

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 6 \\ 3 & -4 \end{bmatrix}.$$

Thus show that $\mathbf{AB} = \mathbf{CB}$ but $\mathbf{A} \neq \mathbf{C}$

22. Compute the product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

23. Compute the product

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

24. Compute the product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

25. Compute the product

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

26. Evaluate the expression $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I}$ for the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

27. Evaluate the expression $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 2\mathbf{I})$ for the matrix*

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}.$$

28. Evaluate the expression $(\mathbf{I} - \mathbf{A})(\mathbf{A}^2 - \mathbf{I})$ for the matrix*

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

29. Verify property (M1) for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}.$$

30. Prove Property (M2).

31. Prove Property (M3).

32. Put the following system of equations into matrix form:

$$\begin{aligned} 2x + 3y &= 10, \\ 4x - 5y &= 11. \end{aligned}$$

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*$\mathbf{I}$ is defined in Section 1.4

33. Put the following system of equations into matrix form:

$$\begin{aligned} x + \quad z + y &= 2, \\ 3z + 2x + y &= 4, \\ y + x &= 0. \end{aligned}$$

34. Put the following system of equations into matrix form:

$$\begin{aligned} 5x + 3y + 2z + 4w &= 5, \\ x + y + w &= 0, \\ 3x + 2y + 2z &= -3, \\ x + y + 2z + 3w &= 4. \end{aligned}$$

35. The price schedule for a Chicago to Los Angeles flight is given by $\mathbf{P} = [200 \quad 350 \quad 500]$, where the matrix elements pertain, respectively, to coach tickets, business-class tickets, and first-class tickets. The number of tickets purchased in each category for a particular flight is given by

$$\mathbf{N} = \begin{bmatrix} 130 \\ 20 \\ 10 \end{bmatrix}.$$

Compute the products (a) **PN**, and (b) **NP**, and determine their significance.

36. The closing prices of a person's portfolio during the past week are given by the matrix

$$\mathbf{P} = \begin{bmatrix} 40 & 40\frac{1}{2} & 40\frac{7}{8} & 41 & 41 \\ 3\frac{1}{4} & 3\frac{5}{8} & 3\frac{1}{2} & 4 & 3\frac{7}{8} \\ 10 & 9\frac{3}{4} & 10\frac{1}{8} & 10 & 9\frac{5}{8} \end{bmatrix},$$

where the columns pertain to the days of the week, Monday through Friday, and the rows pertain to the prices of Orchard Fruits, Lion Airways, and Arrow Oil. The person's holdings in each of these companies are given by the matrix $\mathbf{H} = [100 \quad 500 \quad 400]$. Compute the products (a) **HP**, and (b) **PH**, and determine their significance.

37. The time requirements for a company to produce three products is given by the matrix

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 1.2 & 2.3 & 0.7 \\ 0.8 & 3.1 & 1.2 \end{bmatrix},$$

where the rows pertain to lamp bases, cabinets, and tables, respectively. The columns pertain to the hours of labor required for cutting the wood, assembling, and painting, respectively. The hourly wages of a carpenter to cut wood, of a craftsperson to assemble a product, and of a decorator to paint is given, respectively, by the elements of the matrix

$$\mathbf{W} = \begin{bmatrix} 10.50 \\ 14.00 \\ 12.25 \end{bmatrix}.$$

Compute the product **TW** and determine its significance.

38. Continuing with the data given in the previous problem, assume further that the number of items on order for lamp bases, cabinets, and tables, respectively, is given by the matrix $\mathbf{O} = [1000 \quad 100 \quad 200]$. Compute the product **OTW**, and determine its significance.

39. The results of a flu epidemic at a college campus are collected in the matrix

$$\mathbf{F} = \begin{bmatrix} 0.20 & 0.20 & 0.15 & 0.15 \\ 0.10 & 0.30 & 0.30 & 0.40 \\ 0.70 & 0.50 & 0.55 & 0.45 \end{bmatrix}.$$

The elements denote percents converted to decimals. The columns pertain to freshmen, sophomores, juniors, and seniors, respectively, while the rows represent bedridden students, infected but ambulatory students, and well students, respectively. The male–female composition of each class is given by the matrix

$$\mathbf{C} = \begin{bmatrix} 1050 & 950 \\ 1100 & 1050 \\ 360 & 500 \\ 860 & 1000 \end{bmatrix}.$$

Compute the product **FC**, and determine its significance.

1.4 Special Matrices

There are certain types of matrices that occur so frequently that it becomes advisable to discuss them separately. One such type is the *transpose*. Given a matrix **A**, the transpose of **A**, denoted by $\mathbf{A}^{\mathsf{T}}$ and read **A**-transpose, is obtained by changing all the rows of **A** into columns of $\mathbf{A}^{\mathsf{T}}$ while preserving the order; hence, the first row of **A** becomes the first column of $\mathbf{A}^{\mathsf{T}}$, while the second row of **A** becomes the second column of $\mathbf{A}^{\mathsf{T}}$, and the last row of **A** becomes the last column of $\mathbf{A}^{\mathsf{T}}$. Thus if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then } \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

and if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad \text{then } \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}.$$

Definition 1 If $\mathbf{A}$, denoted by $[a_{ij}]$ is an $n \times p$ matrix, then the transpose of $\mathbf{A}$, denoted by $\mathbf{A}^{\mathsf{T}} = [a_{ij}^{\mathsf{T}}]$ is a $p \times n$ matrix where $a_{ij}^{\mathsf{T}} = a_{ji}$.

It can be shown that the transpose possesses the following properties:

(1) $(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$,
(2) $(\lambda\mathbf{A})^{\mathsf{T}} = \lambda\mathbf{A}^{\mathsf{T}}$ where λ represents a scalar,
(3) $(\mathbf{A} + \mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$,
(4) $(\mathbf{A} + \mathbf{B} + \mathbf{C})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}} + \mathbf{C}^{\mathsf{T}}$,
(5) $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$,
(6) $(\mathbf{ABC})^{\mathsf{T}} = \mathbf{C}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

Transposes of sums and products of more than three matrices are defined in the obvious manner. We caution the reader to be alert to the ordering of properties (5) and (6). In particular, one should be aware that the transpose of a product is not the product of the transposes but rather the *commuted* product of the transposes.

Example 1 Find $(\mathbf{AB})^{\mathsf{T}}$ and $\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Solution

$$\mathbf{AB} = \begin{bmatrix} -3 & 6 & 3 \\ -1 & 7 & 4 \end{bmatrix}, \quad (\mathbf{AB})^{\mathsf{T}} = \begin{bmatrix} -3 & -1 \\ 6 & 7 \\ 3 & 4 \end{bmatrix};$$

$$\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 6 & 7 \\ 3 & 4 \end{bmatrix}.$$

Note that $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ but $\mathbf{A}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}$ is not defined. ■

A zero row in a matrix is a row containing only zeros, while a nonzero row is one that contains at least one nonzero element. A matrix is in *row-reduced form* if it satisfies four conditions:

(R1) All zero rows appear below nonzero rows when both types are present in the matrix.

(R2) The first nonzero element in any nonzero row is unity.

(R3) All elements directly below (that is, in the same column but in succeeding rows from) the first nonzero element of a nonzero row are zero.

(R4) The first nonzero element of any nonzero row appears in a later column (further to the right) than the first nonzero element in any preceding row.

Such matrices are invaluable for solving sets of simultaneous linear equations and developing efficient algorithms for performing important matrix operations. We shall have much more to say on these matters in later chapters. Here we are simply interested in recognizing when a given matrix is or is not in row-reduced form.

Example 2 Determine which of the following matrices are in row-reduced form:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 & 4 & 7 \\ 0 & 0 & -6 & 5 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 5 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & -2 & 3 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution Matrix **A** is not in row-reduced form because the first nonzero element of the second row is not unity. This violates (R2). If a_{23} had been unity instead of -6, then the matrix would be in row-reduced form. Matrix **B** is not in row-reduced form because the second row is a zero row and it appears before the third row which is a nonzero row. This violates (R1). If the second and third rows had been interchanged, then the matrix would be in row-reduced form. Matrix **C** is not in row-reduced form because the first nonzero element in row two appears in a later column, column 3, than the first nonzero element of row three. This violates (R4). If the second and third rows had been interchanged, then the matrix would be in row-reduced form. Matrix **D** is not in row-reduced form because the first nonzero element in row two appears in the third column, and everything below d_{23} is not zero. This violates (R3). Had the 3–3 element been zero instead of unity, then the matrix would be in row-reduced form. ■

For the remainder of this section, we concern ourselves with square matrices; that is, matrices having the same number of rows as columns. A *diagonal matrix* is a square matrix all of whose elements are zero except possibly those on the main diagonal. (Recall that the main diagonal consists of all the diagonal elements a_{11}, a_{22}, a_{33}, and so on.) Thus,

$$\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are both diagonal matrices of order 2×2 and 3×3 respectively. The zero matrix is the special diagonal matrix having all the elements on the main diagonal equal to zero.

An *identity* matrix is a diagonal matrix worthy of special consideration. Designated by $\mathbf{I}$, an identity is defined to be a diagonal matrix having all diagonal elements equal to one. Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are the 2×2 and 4×4 identities respectively. The identity is perhaps the most important matrix of all. If the identity is of the appropriate order so that the following multiplication can be carried out, then for any arbitrary matrix $\mathbf{A}$,

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IA} = \mathbf{A}.$$

A *symmetric* matrix is a matrix that is equal to its transpose while a *skew symmetric* matrix is a matrix that is equal to the negative of its transpose. Thus, a matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$ while it is skew symmetric if $\mathbf{A} = -\mathbf{A}^{\mathsf{T}}$. Examples of each are respectively

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

A matrix $\mathbf{A} = [a_{ij}]$ is called *lower triangular* if $a_{ij} = 0$ for $j > i$ (that is, if all the elements above the main diagonal are zero) and *upper triangular* if $a_{ij} = 0$ for $i > j$ (that is, if all the elements below the main diagonal are zero).

Examples of lower and upper triangular matrices are, respectively,

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 & 4 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Theorem 1 *The product of two lower (upper) triangular matrices is also lower (upper) triangular.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ both be $n \times n$ lower triangular matrices. Set $\mathbf{C} = \mathbf{AB}$. We need to show that $\mathbf{C}$ is lower triangular, or equivalently, that $c_{ij} = 0$ when $i < j$. Now,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{j-1} a_{ik} b_{kj} + \sum_{k=j}^{n} a_{ik} b_{kj}.$$

We are given that $a_{ik} = 0$ when $i < k$, and $b_{kj} = 0$ when $k < j$, because both $\mathbf{A}$ and $\mathbf{B}$ are lower triangular. Thus,

$$\sum_{k=1}^{j-1} a_{ik}b_{kj} = \sum_{k=1}^{j-1} a_{ik}(0) = 0$$

because k is always less than j. Furthermore, if we restrict $i < j$, then

$$\sum_{k=j}^{n} a_{ik}b_{kj} = \sum_{k=j}^{n} (0)b_{kj} = 0$$

because $k \geq j > i$. Therefore, $c_{ij} = 0$ when $i < j$. □

Finally, we define positive integral powers of a matrix in the obvious manner: $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA}$ and, in general, if n is a positive integer,

$$\mathbf{A}^n = \underbrace{\mathbf{AA}\ldots\mathbf{A}.}_{n \text{ times}}$$

Thus, if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}, \quad \text{then } \mathbf{A}^2 = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 4 & 7 \end{bmatrix}.$$

It follows directly from Property 5 that

$$(\mathbf{A}^2)^{\mathsf{T}} = (\mathbf{AA})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})^2.$$

We can generalize this result to the following property for any integral positive power n:

$$(7)\quad (\mathbf{A}^n)^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})^n.$$

Problems 1.4

1. Verify that $(\mathbf{A} + \mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix}.$$

2. Verify that $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$, where

$$\mathbf{A} = \begin{bmatrix} t & t^2 \\ 1 & 2t \\ 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3 & t & t+1 & 0 \\ t & 2t & t^2 & t^3 \end{bmatrix}.$$

3. Simplify the following expressions:

(a) $(\mathbf{AB}^{\mathsf{T}})^{\mathsf{T}}$, (b) $\mathbf{A}^{\mathsf{T}} + (\mathbf{A} + \mathbf{B}^{\mathsf{T}})^{\mathsf{T}}$,
(c) $(\mathbf{A}^{\mathsf{T}}(\mathbf{B} + \mathbf{C}^{\mathsf{T}}))^{\mathsf{T}}$, (d) $((\mathbf{AB})^{\mathsf{T}} + \mathbf{C})^{\mathsf{T}}$,
(e) $((\mathbf{A} + \mathbf{A}^{\mathsf{T}})(\mathbf{A} - \mathbf{A}^{\mathsf{T}}))^{\mathsf{T}}$.

4. Find $\mathbf{X}^{\mathsf{T}}\mathbf{X}$ and $\mathbf{XX}^{\mathsf{T}}$ when

$$\mathbf{X} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

5. Find $\mathbf{X}^{\mathsf{T}}\mathbf{X}$ and $\mathbf{XX}^{\mathsf{T}}$ when $\mathbf{X} = [1 \quad -2 \quad 3 \quad -4]$.

6. Find $\mathbf{X}^{\mathsf{T}}\mathbf{AX}$ when

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

7. Determine which, if any, of the following matrices are in row-reduced form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 4 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 4 & -7 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 4 & -7 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 4 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix}.$$

8. Determine which, if any, of the matrices in Problem 7 are upper triangular.

9. Must a square matrix in row-reduced form necessarily be upper triangular?

10. Must an upper triangular matrix necessarily be in row-reduced form?

11. Can a matrix be both upper and lower triangular simultaneously?

12. Show that $\mathbf{AB} = \mathbf{BA}$, where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

13. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices of the same order, then $\mathbf{AB} = \mathbf{BA}$.

14. Does a 2×2 diagonal matrix commute with every other 2×2 matrix?

15. Compute the products $\mathbf{AD}$ and $\mathbf{BD}$ for the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

What conclusions can you make about postmultiplying a square matrix by a diagonal matrix?

16. Compute the products $\mathbf{DA}$ and $\mathbf{DB}$ for the matrices defined in Problem 15. What conclusions can you make about premultiplying a square matrix by a diagonal matrix?

17. Prove that if a 2×2 matrix $\mathbf{A}$ commutes with every 2×2 diagonal matrix, then $\mathbf{A}$ must also be diagonal. *Hint*: Consider, in particular, the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

18. Prove that if an $n \times n$ matrix $\mathbf{A}$ commutes with every $n \times n$ diagonal matrix, then $\mathbf{A}$ must also be diagonal.

19. Compute $\mathbf{D}^2$ and $\mathbf{D}^3$ for the matrix $\mathbf{D}$ defined in Problem 15.

20. Find $\mathbf{A}^3$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

21. Using the results of Problems 19 and 20 as a guide, what can be said about $\mathbf{D}^n$ if $\mathbf{D}$ is a diagonal matrix and n is a positive integer?

22. Prove that if $\mathbf{D} = [d_{ij}]$ is a diagonal matrix, then $\mathbf{D}^2 = [d_{ij}^2]$.

23. Calculate $\mathbf{D}^{50} - 5\mathbf{D}^{35} + 4\mathbf{I}$, where

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

24. A square matrix $\mathbf{A}$ is *nilpotent* if $\mathbf{A}^n = \mathbf{0}$ for some positive integer n. If n is the smallest positive integer for which $\mathbf{A}^n = \mathbf{0}$ then $\mathbf{A}$ is nilpotent of *index* n. Show that

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & -3 \\ -5 & -2 & -6 \\ 2 & 1 & 3 \end{bmatrix}$$

is nilpotent of index 3.

25. Show that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent. What is its index?

26. Prove that if $\mathbf{A}$ is a square matrix, then $\mathbf{B} = (\mathbf{A} + \mathbf{A}^\mathsf{T})/2$ is a symmetric matrix.

27. Prove that if $\mathbf{A}$ is a square matrix, then $\mathbf{C} = (\mathbf{A} - \mathbf{A}^\mathsf{T})/2$ is a skew symmetric matrix.

28. Using the results of the preceding two problems, prove that any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

29. Write the matrix $\mathbf{A}$ in Problem 1 as the sum of a symmetric matrix and skew-symmetric matrix.

30. Write the matrix $\mathbf{B}$ in Problem 1 as the sum of a symmetric matrix and a skew-symmetric matrix.

31. Prove that if $\mathbf{A}$ is any matrix, then $\mathbf{A}\mathbf{A}^\mathsf{T}$ is symmetric.

32. Prove that the diagonal elements of a skew-symmetric matrix must be zero.

33. Prove that the transpose of an upper triangular matrix is lower triangular, and vice versa.

34. If $\mathbf{P} = [p_{ij}]$ is a transition matrix for a Markov chain (see Problem 16 of Section 1.1), then it can be shown with elementary probability theory that the $i - j$ element of $\mathbf{P}^2$ denotes the probability of an object moving from state i to stage j over two time periods. More generally, the $i - j$ element of $\mathbf{P}^n$ for any positive integer n denotes the probability of an object moving from state i to state j over n time periods.

(a) Calculate $\mathbf{P}^2$ and $\mathbf{P}^3$ for the two-state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{bmatrix}.$$

(b) Determine the probability of an object beginning in state 1 and ending in state 1 after two time periods.

(c) Determine the probability of an object beginning in state 1 and ending in state 2 after two time periods.

(d) Determine the probability of an object beginning in state 1 and ending in state 2 after three time periods.

(e) Determine the probability of an object beginning in state 2 and ending in state 2 after three time periods.

35. Consider a two-state Markov chain. List the number of ways an object in state 1 can end in state 1 after three time periods.

36. Consider the Markov chain described in Problem 16 of Section 1.1. Determine (a) the probability that a family living in the city will find themselves in the suburbs after two years, and (b) the probability that a family living in the suburbs will find themselves living in the city after two years.

37. Consider the Markov chain described in Problem 17 of Section 1.1. Determine (a) the probability that there will be a Republican mayor eight years after a Republican mayor serves, and (b) the probability that there will be a Republican mayor 12 years after a Republican mayor serves.

38. Consider the Markov chain described in Problem 18 of Section 1.1. It is known that this year the apple harvest was poor. Determine (a) the probability that next year's harvest will be poor, and (b) the probability that the harvest in two years will be poor.

39. Consider the Markov chain described in Problem 19 of Section 1.1. Determine (a) the probability that a brand X customer will be a brand X customer after 4 years, (b) after 6 years, and (c) the probability that a brand X customer will be a brand Y customer after 4 years.

40. A graph consists of a set of nodes, which we shall designate by positive integers, and a set of arcs that connect various pairs of nodes. An *adjacency matrix* **M** associated with a particular graph is defined by

$$m_{ij} = \text{number of distinct arcs connecting node } i \text{ to node } j$$

(a) Construct an adjacency matrix for the graph shown in Figure 1.1.

(b) Calculate $\mathbf{M}^2$, and note that the $i - j$ element of $\mathbf{M}^2$ is the number of paths consisting of two arcs that connect node i to node j.

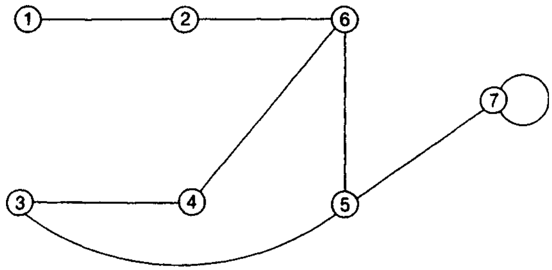

Figure 1.1

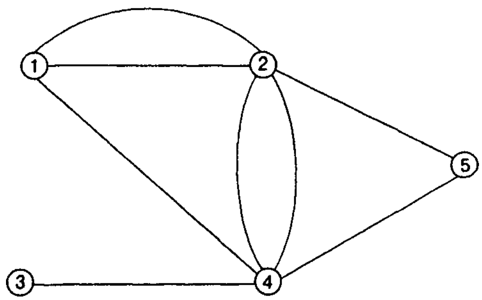

Figure 1.2

41. (a) Construct an adjacency matrix **M** for the graph shown in Figure 1.2.

(b) Calculate $\mathbf{M}^2$, and use that matrix to determine the number of paths consisting of two arcs that connect node 1 to node 5.

(c) Calculate $\mathbf{M}^3$, and use that matrix to determine the number of paths consisting of three arcs that connect node 2 to node 4.

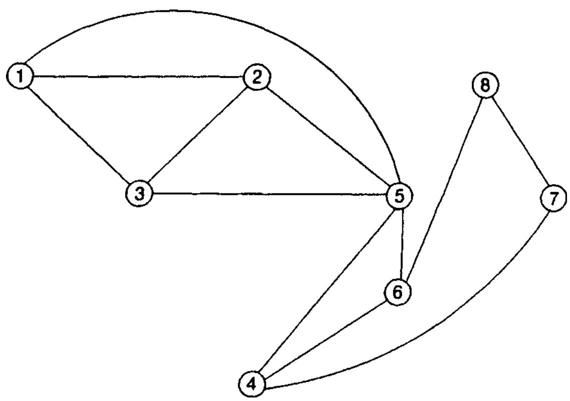

Figure 1.3

42. Figure 1.3 depicts a road network linking various cities. A traveler in city 1 needs to drive to city 7 and would like to do so by passing through the least number of intermediate cities. Construct an adjacency matrix for this road network. Consider powers of this matrix to solve the traveler's problem.

1.5 Submatrices and Partitioning

Given any matrix **A**, a *submatrix* of **A** is a matrix obtained from **A** by the removal of any number of rows or columns. Thus, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = [2 \quad 3 \quad 4], \tag{11}$$

then **B** and **C** are both submatrices of **A**. Here **B** was obtained by removing from **A** the first and second rows together with the first and third columns, while **C** was obtained by removing from **A** the second, third, and fourth rows together with the first column. By removing no rows and no columns from **A**, it follows that **A** is a submatrix of itself.

A matrix is said to be partitioned if it is divided into submatrices by horizontal and vertical lines between the rows and columns. By varying the choices of where to put the horizontal and vertical lines, one can partition a matrix in many different ways. Thus,

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right]$$

are examples of two different partitions of the matrix **A** given in (11).

If partitioning is carried out in a particularly judicious manner, it can be a great help in matrix multiplication. Consider the case where the two matrices **A** and **B** are to be multiplied together. If we partition both **A** and **B** into four submatrices, respectively, so that

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{D} \\ \hline \mathbf{E} & \mathbf{F} \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{c|c} \mathbf{G} & \mathbf{H} \\ \hline \mathbf{J} & \mathbf{K} \end{array}\right]$$

where **C** through **K** represent submatrices, then the product **AB** may be obtained by simply carrying out the multiplication as if the submatrices were themselves elements. Thus,

$$\mathbf{AB} = \left[\begin{array}{c|c} \mathbf{CG} + \mathbf{DJ} & \mathbf{CH} + \mathbf{DK} \\ \hline \mathbf{EG} + \mathbf{FJ} & \mathbf{EH} + \mathbf{FK} \end{array}\right], \tag{12}$$

providing the partitioning was such that the indicated multiplications are defined.

It is not unusual to need products of matrices having thousands of rows and thousands of columns. Problem 42 of Section 1.4 dealt with a road network connecting seven cities. A similar network for a state with connections between all cities in the state would have a very large adjacency matrix associated with it, and its square is then the product of two such matrices. If we expand the network to include the entire United States, the associated matrix is huge, with one row and one column for each city and town in the country. Thus, it is not difficult to visualize large matrices that are too big to be stored in the internal memory of any modern day computer. And yet the product of such matrices must be computed.

The solution procedure is partitioning. Large matrices are stored in external memory on peripheral devices, such as disks, and then partitioned. Appropriate submatrices are fetched from the peripheral devices as needed, computed, and the results again stored on the peripheral devices. An example is the product given in (12). If **A** and **B** are too large for the internal memory of a particular computer, but **C** through **K** are not, then the partitioned product can be computed. First, **C** and **G** are fetched from external memory and multiplied; the product is then stored in external memory. Next, **D** and **J** are fetched and multiplied. Then, the product **CG** is fetched and added to the product **DJ**. The result, which is the first partition of **AB**, is then stored in external memory, and the process continues.

Example 1 Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & -1 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution We first partition **A** and **B** in the following manner

$$\mathbf{A} = \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 4 & -1 \\ \hline 3 & 1 & 2 \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 0 & 1 \\ \hline 0 & 1 & 1 \end{array}\right];$$

then,

$$\mathbf{AB} = \left[\begin{array}{c|c} \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} \\ \hline \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 2 & 9 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 2 \end{bmatrix} & \begin{bmatrix} 7 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 2 & 11 & 9 \\ -3 & 2 & 5 \\ \hline 2 & 11 & 9 \end{array}\right] = \begin{bmatrix} 2 & 11 & 9 \\ -3 & 2 & 5 \\ 2 & 11 & 9 \end{bmatrix}. \quad \blacksquare$$

Example 2 Find **AB** if

$$\mathbf{A} = \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & 0 & 0 \\ \hline 0 & 0 & 3 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Solution From the indicated partitions, we find that

$$\mathbf{AB} = \left[\begin{array}{c|c} \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{array}\right]$$

$$\mathbf{AB} = \left[\begin{array}{c|c} \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} 5 & 4 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \\ \hline 0 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that we partitioned in order to make maximum of the zero submatrices of both **A** and **B**. ■

A matrix **A** that can be partitioned into the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & & \\ & \mathbf{A}_2 & & & \\ & & \mathbf{A}_3 & & \mathbf{0} \\ & & & \ddots & \\ & \mathbf{0} & & & \mathbf{A}_n \end{bmatrix}$$

is called *block diagonal*. Such matrices are particularly easy to multiply because in partitioned form they act as diagonal matrices.

Problems 1.5

1. Which of the following are submatrices of the given $\mathbf{A}$ and why?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$.

2. Determine all possible submatrices of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

3. Given the matrices $\mathbf{A}$ and $\mathbf{B}$ (as shown), find $\mathbf{AB}$ using the partitionings indicated:

$$\mathbf{A} = \left[\begin{array}{cc|c} 1 & -1 & 2 \\ \hline 3 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right], \quad \mathbf{B} = \left[\begin{array}{ccc|c} 5 & 2 & 0 & 2 \\ 1 & -1 & 3 & 1 \\ \hline 0 & 1 & 1 & 4 \end{array}\right].$$

4. Partition the given matrices $\mathbf{A}$ and $\mathbf{B}$ and, using the results, find $\mathbf{AB}$.

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

5. Compute $\mathbf{A}^2$ for the matrix $\mathbf{A}$ given in Problem 4 by partitioning $\mathbf{A}$ into block diagonal form.

6. Compute $\mathbf{B}^2$ for the matrix $\mathbf{B}$ given in Problem 4 by partitioning $\mathbf{B}$ into block diagonal form.

7. Use partitioning to compute $\mathbf{A}^2$ and $\mathbf{A}^3$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is $\mathbf{A}^n$ for any positive integral power of $n > 3$?

8. Use partitioning to compute $\mathbf{A}^2$ and $\mathbf{A}^3$ for

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -4 & 0 & 0 \\ 0 & 0 & -1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

What is $\mathbf{A}^n$ for any positive integral power of n?

1.6 Vectors

Definition 1 A *vector* is a $1 \times n$ or $n \times 1$ matrix.

A $1 \times n$ matrix is called a *row vector* while an $n \times 1$ matrix is a *column vector*. The elements are called the *components* of the vector while the number of components in the vector, in this case n, is its *dimension*. Thus,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is an example of a 3-dimensional column vector, while

$$\begin{bmatrix} t & 2t & -t & 0 \end{bmatrix}$$

is an example of a 4-dimensional row vector.

The reader who is already familiar with vectors will notice that we have not defined vectors as directed line segments. We have done this intentionally, first because in more than three dimensions this geometric interpretation loses its significance, and second, because in the general mathematical framework, vectors are not directed line segments. However, the idea of representing a finite dimensional vector by its components and hence as a matrix is one that is acceptable to the scientist, engineer, and mathematician. Also, as a bonus, since a vector is nothing more than a special matrix, we have already defined scalar multiplication, vector addition, and vector equality.

A vector $\mathbf{y}$ (vectors will be designated by boldface lowercase letters) has associated with it a nonnegative number called its *magnitude* or length designated by $\|\mathbf{y}\|$.

Definition 2 If $\mathbf{y} = [y_1\ y_2 \dots y_n]$ then $\|\mathbf{y}\| = \sqrt{(y_1)^2 + (y_2)^2 + \cdots + (y_n)^2}$.

Example 1 Find $\|\mathbf{y}\|$ if $\mathbf{y} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$.

Solution $\|\mathbf{y}\| = \sqrt{(1)^2 + (2)^2 + (3)^2 + (4)^2} = \sqrt{30}$. ■

If $\mathbf{z}$ is a column vector, $\|\mathbf{z}\|$ is defined in a completely analogous manner.

Example 2 Find $\|\mathbf{z}\|$ if

$$\mathbf{z} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}.$$

Solution $\|\mathbf{z}\| = \sqrt{(-1)^2 + (2)^2 + (-3)^2} = \sqrt{14}$. ■

A vector is called a *unit vector* if its magnitude is equal to one. A nonzero vector is said to be *normalized* if it is divided by its magnitude. Thus, a normalized vector is also a unit vector.

Example 3 Normalize the vector $[1 \quad 0 \quad -3 \quad 2 \quad -1]$.

Solution The magnitude of this vector is

$$\sqrt{(1)^2 + (0)^2 + (-3)^2 + (2)^2 + (-1)^2} = \sqrt{15}.$$

Hence, the normalized vector is

$$\begin{bmatrix} \frac{1}{\sqrt{15}} & 0 & \frac{-3}{\sqrt{15}} & \frac{2}{\sqrt{15}} & \frac{-1}{\sqrt{15}} \end{bmatrix}.$$ ■

In passing, we note that when a general vector is written $\mathbf{y} = [y_1 y_2 \ldots y_n]$ one of the subscripts of each element of the matrix is deleted. This is done solely for the sake of convenience. Since a row vector has only one row (a column vector has only one column), it is redundant and unnecessary to exhibit the row subscript (the column subscript).

Problems 1.6

1. Find p if $5\mathbf{x} - 2\mathbf{y} = \mathbf{b}$, where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ p \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 13 \\ -2 \end{bmatrix}.$$

2. Find $\mathbf{x}$ if $3\mathbf{x} + 2\mathbf{y} = \mathbf{b}$, where

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}.$$

3. Find $\mathbf{y}$ if $2\mathbf{x} - 5\mathbf{y} = -\mathbf{b}$, where

$$\mathbf{x} = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}.$$

4. Using the vectors defined in Problem 2, calculate, if possible,
(a) $\mathbf{yb}$, (b) $\mathbf{yb}^{\mathsf{T}}$,
(c) $\mathbf{y}^{\mathsf{T}}\mathbf{b}$, (d) $\mathbf{b}^{\mathsf{T}}\mathbf{y}$.

5. Using the vectors defined in Problem 3, calculate, if possible,
(a) $\mathbf{x} + 2\mathbf{b}$, (b) $\mathbf{xb}^{\mathsf{T}}$,
(c) $\mathbf{x}^{\mathsf{T}}\mathbf{b}$, (d) $\mathbf{b}^{\mathsf{T}}\mathbf{b}$.

6. Determine which of the following are unit vectors:
(a) $\begin{bmatrix} 1 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$, (c) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}$, (f) $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$,

(g) $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (h) $\frac{1}{6}\begin{bmatrix} 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}$, (i) $\frac{1}{\sqrt{3}}\begin{bmatrix} -1 & 0 & 1 & -1 \end{bmatrix}$.

7. Find $\|\mathbf{y}\|$ if
(a) $\mathbf{y} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, (b) $\mathbf{y} = \begin{bmatrix} 3 & 4 \end{bmatrix}$,
(c) $\mathbf{y} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$, (d) $\mathbf{y} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$,
(e) $\mathbf{y} = \begin{bmatrix} 2 & 1 & -1 & 3 \end{bmatrix}$, (f) $\mathbf{y} = \begin{bmatrix} 0 & -1 & 5 & 3 & 2 \end{bmatrix}$.

8. Find $\|\mathbf{x}\|$ if

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (b) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, (c) $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

(d) $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, (e) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, (f) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

9. Find $\|\mathbf{y}\|$ if

(a) $\mathbf{y} = [2 \quad 1 \quad -1 \quad 3]$, (b) $\mathbf{y} = [0 \quad -1 \quad 5 \quad 3 \quad 2]$.

10. Prove that a normalized vector must be a unit vector.

11. Show that the matrix equation

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 5 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \\ 5 \end{bmatrix}$$

is equivalent to the vector equation

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} + z \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \\ 5 \end{bmatrix}.$$

12. Convert the following system of equations into a vector equation:

$$\begin{aligned} 2x + 3y &= 10, \\ 4x + 5y &= 11. \end{aligned}$$

13. Convert the following system of equations into a vector equation:

$$\begin{aligned} 3x + 4y + 5z + 6w &= 1, \\ y - 2z + 8w &= 0, \\ -x + y + 2z - w &= 0. \end{aligned}$$

14. Using the definition of matrix multiplication, show that the jth column of $(\mathbf{AB}) = \mathbf{A} \times (j\text{th column of } \mathbf{B})$.

15. Verify the result of Problem 14 by showing that the first column of the product $\mathbf{AB}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & -3 \end{bmatrix}$$

is

$$\mathbf{A} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

while the second column of the product is

$$\mathbf{A}\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$

16. A *distribution row vector* $\mathbf{d}$ for an N-state Markov chain (see Problem 16 of Section 1.1 and Problem 34 of Section 1.4) is an N-dimensional row vector having as its components, one for each state, the probabilities that an object in the system is in each of the respective states. Determine a distribution vector for a three-state Markov chain if 50% of the objects are in state 1, 30% are in state 2, and 20% are in state 3.

17. Let $\mathbf{d}^{(k)}$ denote the distribution vector for a Markov chain after k time periods. Thus, $\mathbf{d}^{(0)}$ represents the initial distribution. It follows that

$$\mathbf{d}^{(k)} = \mathbf{d}^{(0)}\mathbf{P}^k = \mathbf{P}^{(k-1)}\mathbf{P},$$

where $\mathbf{P}$ is the transition matrix and $\mathbf{P}^k$ is its kth power.

Consider the Markov chain described in Problem 16 of Section 1.1.

(a) Explain the physical significance of saying $\mathbf{d}^{(0)} = [0.6 \quad 0.4]$.

(b) Find the distribution vectors $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$.

18. Consider the Markov chain described in Problem 19 of Section 1.1.

(a) Explain the physical significance of saying $\mathbf{d}^{(0)} = [0.4 \quad 0.5 \quad 0.1]$.

(b) Find the distribution vectors $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$.

19. Consider the Markov chain described in Problem 17 of Section 1.1. (a) Determine an initial distribution vector if the town currently has a Democratic mayor, and (b) show that the components of $\mathbf{d}^{(1)}$ are the probabilities that the next mayor will be a Republican and a Democrat, respectively.

20. Consider the Markov chain described in Problem 18 of Section 1.1. (a) Determine an initial distribution vector if this year's crop is known to be poor. (b) Calculate $\mathbf{d}^{(2)}$ and use it to determine the probability that the harvest will be good in two years.

1.7 The Geometry of Vectors

Vector arithmetic can be described geometrically for two- and three-dimensional vectors. For simplicity, we consider two dimensions here; the extension to three-dimensional vectors is straightforward. For convenience, we restrict our examples to row vectors, but note that all constructions are equally valid for column vectors.

A two dimensional vector $\mathbf{v} = [a \quad b]$ is identified with the point (a, b) on the plane, measured from the origin a units along the horizontal axis and then b

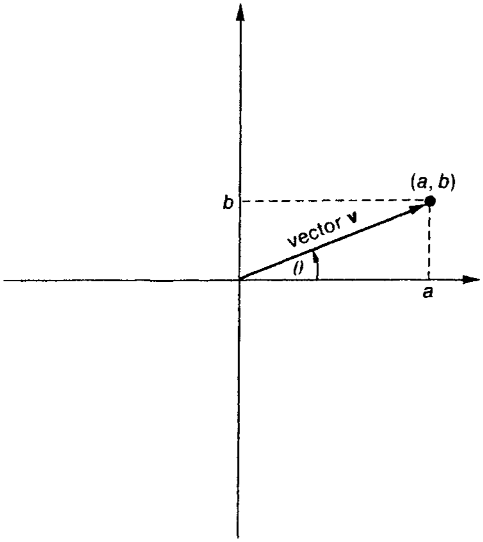

Figure 1.4

units parallel to the vertical axis. We can then draw an arrow beginning at the origin and ending at the point (a, b). This arrow or directed line segment, as shown in Figure 1.4, represents the vector geometrically. It follows immediately from Pythagoras's theorem and Definition 2 of Section 1.6 that the length of the directed line segment is the magnitude of the vector. The angle associated with a vector, denoted by θ in Figure 1.4, is the angle from the positive horizontal axis to the directed line segment measured in the counterclockwise direction.

Example 1 Graph the vectors $\mathbf{v} = [2 \quad 4]$ and $\mathbf{u} = [-1 \quad 1]$ and determine the magnitude and angle of each.

Solution The vectors are drawn in Figure 1.5. Using Pythagoras's theorem and elementary trigonometry, we have, for $\mathbf{v}$,

$$\|\mathbf{v}\| = \sqrt{(2)^2 + (4)^2} = 4.47, \quad \tan\theta = \frac{4}{2} = 2, \quad \text{and} \quad \theta = 63.4^\circ.$$

For $\mathbf{u}$, similar computations yield

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + (1)^2} = 1.14, \quad \tan\theta = \frac{1}{-1} = -1, \text{ and } \quad \theta = 135^\circ. \quad \blacksquare$$

To construct the sum of two vectors $\mathbf{u} + \mathbf{v}$ geometrically, graph $\mathbf{u}$ normally, translate $\mathbf{v}$ so that its initial point coincides with the terminal point of $\mathbf{u}$, *being careful to preserve both the magnitude and direction of* $\mathbf{v}$, and then draw an arrow from the origin to the terminal point of $\mathbf{v}$ after translation. This arrow geometrically

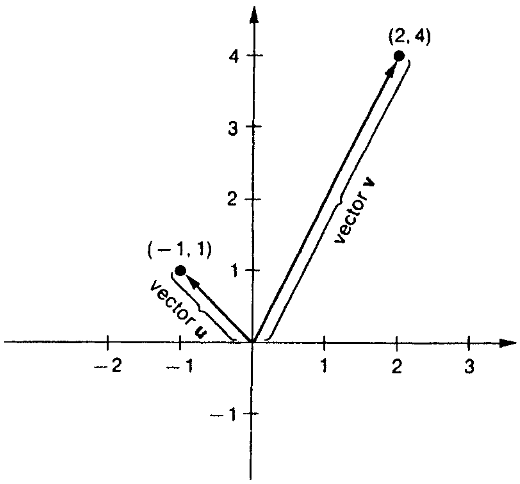

Figure 1.5

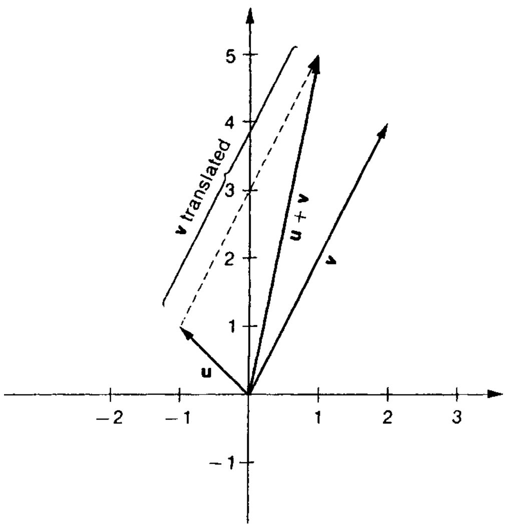

Figure 1.6

represents the sum $\mathbf{u} + \mathbf{v}$. The process is depicted in Figure 1.6 for the two vectors defined in Example 1.

To construct the difference of two vectors $\mathbf{u} - \mathbf{v}$ geometrically, graph both $\mathbf{u}$ and $\mathbf{v}$ normally and construct an arrow from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$. This arrow geometrically represents the difference $\mathbf{u} - \mathbf{v}$. The process is depicted in Figure 1.7 for the two vectors defined in Example 1. To measure the magnitude and direction of $\mathbf{u} - \mathbf{v}$, translate it so that its initial point is at the origin,

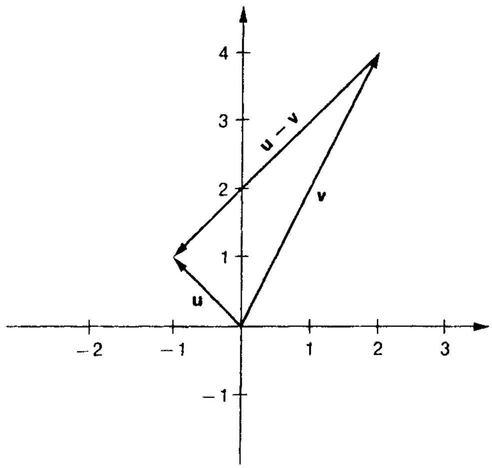

Figure 1.7

being careful to preserve both its magnitude and direction, and then measure the translated vector.

Both geometrical sums and differences involve translations of vectors. This suggests that a vector is not altered by translating it to another position in the plane providing both its magnitude and direction are preserved.

Many physical phenomena such as velocity and force are completely described by their magnitudes and directions. For example, a velocity of 60 miles per hour in the northwest direction is a complete description of that velocity, and *it is independent of where that velocity occurs*. This independence is the rationale behind translating vectors geometrically. Geometrically, vectors having the same magnitude and direction are called *equivalent*, and they are regarded as being equal even though they may be located at different positions in the plane.

A scalar multiplication $k\mathbf{u}$ is defined geometrically to be a vector having length $\|k\|$ times the length of $\mathbf{u}$ with direction equal to $\mathbf{u}$ when k is positive, and opposite to $\mathbf{u}$ when k is negative. Effectively, $k\mathbf{u}$ is an elongation of $\mathbf{u}$ by a factor of $\|k\|$ when $\|k\|$ is greater than unity, or a contraction of $\mathbf{u}$ by a factor of $\|k\|$ when $\|k\|$ is less than unity, followed by no rotation when k is positive, or a rotation of 180 degrees when k is negative.

Example 2 Find $-2\mathbf{u}$ and $\frac{1}{2}\mathbf{v}$ geometrically for the vectors defined in Example 1.

Solution To construct $-2\mathbf{u}$, we double the length of $\mathbf{u}$ and then rotate the resulting vector by 180°. To construct $\frac{1}{2}\mathbf{v}$ we halve the length of $\mathbf{v}$ and effect no rotation. These constructions are illustrated in Figure 1.8. ■

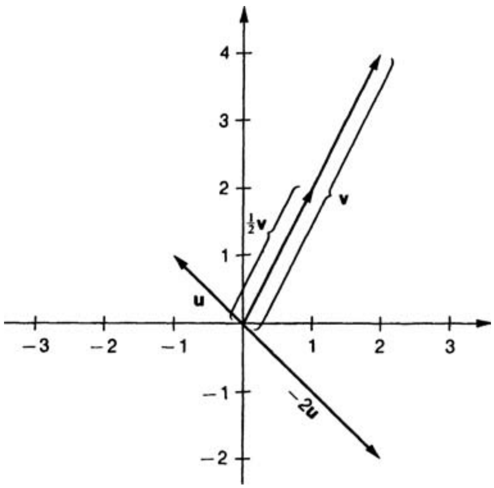

Figure 1.8

Problems 1.7

In Problems 1 through 16, geometrically construct the indicated vector operations for

$$\mathbf{u} = [3 \quad -1], \quad \mathbf{v} = [-2 \quad 5], \quad \mathbf{w} = [-4 \quad -4],$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

1. $\mathbf{u} + \mathbf{v}$. **2.** $\mathbf{u} + \mathbf{w}$. **3.** $\mathbf{v} + \mathbf{w}$. **4.** $\mathbf{x} + \mathbf{y}$.

5. $\mathbf{x} - \mathbf{y}$. **6.** $\mathbf{y} - \mathbf{x}$. **7.** $\mathbf{u} - \mathbf{v}$. **8.** $\mathbf{w} - \mathbf{u}$.

9. $\mathbf{u} - \mathbf{w}$. **10.** $2\mathbf{x}$. **11.** $3\mathbf{x}$. **12.** $-2\mathbf{x}$.

13. $\frac{1}{2}\mathbf{u}$. **14.** $-\frac{1}{2}\mathbf{u}$. **15.** $\frac{1}{3}\mathbf{v}$. **16.** $-\frac{1}{4}\mathbf{w}$.

17. Determine the angle of **u**.

18. Determine the angle of **v**.

19. Determine the angle of **w**.

20. Determine the angle of **x**.

21. Determine the angle of **y**.

22. For arbitrary two-dimensional row vectors construct on the same graph $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$.

(a) Show that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(b) Show that the sum is a diagonal of a parallelogram having **u** and **v** as two of its sides.

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