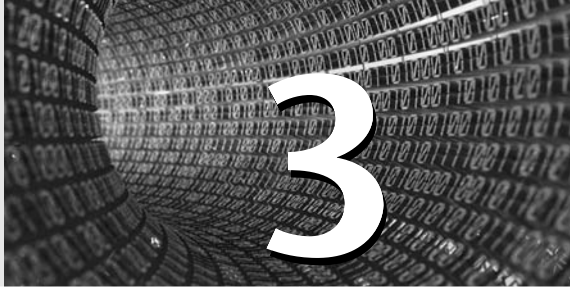

The Inverse

3.1 Introduction

Definition 1 An *inverse* of an $n \times n$ matrix $\mathbf{A}$ is a $n \times n$ matrix $\mathbf{B}$ having the property that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}. \tag{1}$$

Here, $\mathbf{B}$ is called an inverse of $\mathbf{A}$ and is usually denoted by $\mathbf{A}^{-1}$. If a square matrix $\mathbf{A}$ has an inverse, it is said to be *invertible* or *nonsingular*. If $\mathbf{A}$ does not possess an inverse, it is said to be *singular*. Note that inverses are only defined for square matrices. In particular, the identity matrix is invertible and is its own inverse because

$$\mathbf{II} = \mathbf{I}.$$

Example 1 Determine whether

$$\mathbf{B} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \quad \text{or} \quad \mathbf{C} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

are inverses for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution $\mathbf{B}$ is an inverse if and only if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$; $\mathbf{C}$ is an inverse if and only if $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$. Here,

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & 1 \\ \frac{13}{3} & \frac{5}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

while

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \mathbf{CA}.$$

Thus, **B** is not an inverse for **A**, but **C** is. We may write $\mathbf{A}^{-1} = \mathbf{C}$. ■

Definition 1 is a test for checking whether a given matrix is an inverse of another given matrix. In the Final Comments to this chapter we prove that if $\mathbf{AB} = \mathbf{I}$ for two square matrices of the same order, then **A** and **B** commute, and $\mathbf{BA} = \mathbf{I}$. Thus, we can reduce the checking procedure by half. A matrix **B** is an inverse for a square matrix **A** if either $\mathbf{AB} = \mathbf{I}$ or $\mathbf{BA} = \mathbf{I}$; each equality automatically guarantees the other for square matrices. We will show in Section 3.4 that an inverse is unique. If a square matrix has an inverse, it has only one.

Definition 1 does not provide a method for finding inverses. We develop such a procedure in the next section. Still, inverses for some matrices can be found directly.

The inverse for a diagonal matrix **D** having only nonzero elements on its main diagonal is also a diagonal matrix whose diagonal elements are the reciprocals of the corresponding diagonal elements of **D**. That is, if

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & & \mathbf{0} \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \lambda_n \end{bmatrix},$$

then

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & & & \mathbf{0} \\ & \frac{1}{\lambda_2} & & & \\ & & \frac{1}{\lambda_3} & & \\ & & & \ddots & \\ \mathbf{0} & & & & \frac{1}{\lambda_n} \end{bmatrix}.$$

It is easy to show that if any diagonal element in a diagonal matrix is zero, then that matrix is singular. (See Problem 57.)

An elementary matrix **E** is a square matrix that generates an elementary row operation on a matrix **A** (which need not be square) under the multiplication **EA**. Elementary, matrices are constructed by applying the desired elementary row operation to an identity matrix of appropriate order. The appropriate order for both **I** and **E** is a square matrix having as many columns as there are rows in **A**; then, the multiplication **EA** is defined. Because identity matrices contain many zeros, the process for constructing elementary matrices can be simplified still further. After all, nothing is accomplished by interchanging the positions of zeros, multiplying zeros by nonzero constants, or adding zeros to zeros.

(i) To construct an elementary matrix that interchanges the ith row with the jth row, begin with an identity matrix of the appropriate order. First, interchange the unity element in the $i-i$ position with the zero in the $j-i$ position, and then interchange the unity element in the $j-j$ position with the zero in the $i-j$ position.

(ii) To construct an elementary matrix that multiplies the ith row of a matrix by the nonzero scalar k, replace the unity element in the $i-i$ position of the identity matrix of appropriate order with the scalar k.

(iii) To construct an elementary matrix that adds to the jth row of a matrix k times the ith row, replace the zero element in the $j-i$ position of the identity matrix of appropriate order with the scalar k.

Example 2 Find elementary matrices that when multiplied on the right by any 4×3 matrix **A** will (a) interchange the second and fourth rows of **A**, (b) multiply the third row of **A** by 3, and (c) add to the fourth row of **A** -5 times its second row.

Solution

$$\text{(a)}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&3&0\\0&0&0&1\end{bmatrix},\quad \text{(c)}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-5&0&1\end{bmatrix}. \quad \blacksquare$$

Example 3 Find elementary matrices that when multiplied on the right by any 3×5 matrix **A** will (a) interchange the first and second rows of **A**, (b) multiply the third row of **A** by -0.5, and (c) add to the third row of **A** -1 times its second row.

Solution

$$\text{(a)}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-0.5\end{bmatrix},\quad \text{(c)}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}. \quad \blacksquare$$

The inverse of an elementary matrix that interchanges two rows is the matrix itself, it is its own inverse. The inverse of an elementary matrix that multiplies one row by a nonzero scalar k is gotten by replacing k by $1/k$. The inverse of an elementary matrix which adds to one row a constant k times another row is obtained by replacing the scalar k by $-k$.

Example 4 Compute the inverses of the elementary matrices found in Example 2.

Solution

$$\text{(a)}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{3}&0\\0&0&0&1\end{bmatrix},\quad \text{(c)}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&5&0&1\end{bmatrix}. \quad \blacksquare$$

Example 5 Compute the inverses of the elementary matrices found in Example 3.

Solution

$$\text{(a)}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix},\quad \text{(c)}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}.$$

Finally, if $\mathbf{A}$ can be partitioned into the block diagonal form,

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & & & & \mathbf{0}\\ & \mathbf{A}_2 & & & \\ & & \mathbf{A}_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{A}_n\end{bmatrix},$$

then $\mathbf{A}$ is invertible if and only if each of the diagonal blocks $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ is invertible and

$$\mathbf{A}^{-1} = \begin{bmatrix}\mathbf{A}_1^{-1} & & & & \mathbf{0}\\ & \mathbf{A}_2^{-1} & & & \\ & & \mathbf{A}_3^{-1} & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{A}_n^{-1}\end{bmatrix}. \quad \blacksquare$$

Example 6 Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution Set

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

then, $\mathbf{A}$ is in the block diagonal form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & \mathbf{0} \\ & \mathbf{A}_2 & \\ \mathbf{0} & & \mathbf{A}_3 \end{bmatrix}.$$

Here $\mathbf{A}_1$ is a diagonal matrix with nonzero diagonal elements, $\mathbf{A}_2$ is an elementary matrix that adds to the second row four times the first row, and $\mathbf{A}_3$ is an elementary matrix that interchanges the second and third rows; thus

$$\mathbf{A}_1^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}, \quad \mathbf{A}_2^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad \blacksquare$$

Problems 3.1

1. Determine if any of the following matrices are inverses for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}:$$

$$\text{(a)} \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{9} \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} -1 & -3 \\ -2 & -9 \end{bmatrix},$$

$$\text{(c)} \begin{bmatrix} 3 & -1 \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \qquad \text{(d)} \begin{bmatrix} 9 & -3 \\ -2 & 1 \end{bmatrix}.$$

2. Determine if any of the following matrices are inverses for

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}:$$

$$\text{(a)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\text{(c)} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \text{(d)} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

3. Calculate directly the inverse of

$$\mathbf{A} = \begin{bmatrix} 8 & 2 \\ 5 & 3 \end{bmatrix}.$$

Hint: Define

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Calculate **AB**, set the product equal to **I**, and then solve for the elements of **B**.

4. Use the procedure described in Problem 3 to calculate the inverse of

$$\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

5. Use the procedure described in Problem 3 to calculate the inverse of

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

6. Show directly that the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

when $ad - bc \neq 0$ is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

7. Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}.$$

8. Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

9. Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

10. Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}.$$

In Problems 11 through 26, find elementary matrices that when multiplied on the right by a matrix **A** will generate the specified result.

11. Interchange the order of the first and second row of the 2×2 matrix **A**.

12. Multiply the first row of a 2×2 matrix **A** by three.

13. Multiply the second row of a 2×2 matrix **A** by -5.

14. Multiply the second row of a 3×3 matrix **A** by -5.

15. Add to the second row of a 2×2 matrix **A** three times its first row.

16. Add to the first row of a 2×2 matrix **A** three times its second row.

17. Add to the second row of a 3×3 matrix **A** three times its third row.

18. Add to the third row of a 3×4 matrix **A** five times its first row.

19. Add to the second row of a 4×4 matrix **A** eight times its fourth row.

20. Add to the fourth row of a 5×7 matrix **A** -2 times its first row.

21. Interchange the second and fourth rows of a 4×6 matrix **A**.

22. Interchange the second and fourth rows of a 4×4 matrix $\mathbf{A}$.

23. Interchange the second and fourth rows of a 6×6 matrix $\mathbf{A}$.

24. Multiply the second row of a 2×5 matrix $\mathbf{A}$ by seven.

25. Multiply the third row of a 5×2 matrix $\mathbf{A}$ by seven.

26. Multiply the second row of a 3×5 matrix $\mathbf{A}$ by -0.2.

In Problems 27 through 42, find the inverses of the given elementary matrices.

27. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, **28.** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, **29.** $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, **30.** $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

31. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **32.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **33.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$,

34. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **35.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, **36.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$,

37. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, **38.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, **39.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

40. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, **41.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, **42.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

In Problems 43 through 55, find the inverses, if they exist, of the given diagonal or block diagonal matrices.

43. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, **44.** $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, **45.** $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$, **46.** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$,

47. $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, **48.** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, **49.** $\begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{3}{5} \end{bmatrix}$,

50. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, **51.** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, **52.** $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

53. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, **54.** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$, **55.** $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

56. Prove that a square zero matrix does not have an inverse.

57. Prove that if a diagonal matrix has at least one zero on its main diagonal, then that matrix cannot have an inverse.

58. Prove that if $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{A}$.

3.2 Calculating Inverses

In Section 2.3, we developed a method for transforming any matrix into row-reduced form using elementary row operations. If we now restrict our attention to square matrices, we may say that the resulting row-reduced matrices are upper triangular matrices having either a unity or zero element in each entry on the main diagonal. This provides a simple test for determining which matrices have inverses.

Theorem 1 *A square matrix has an inverse if and only if reduction to row-reduced form by elementary row operations results in a matrix having all unity elements on the main diagonal.*

We shall prove this theorem in the Final Comments to this chapter as

Theorem 2 *An $n \times n$ matrix has an inverse if and only if it has rank n.*

Theorem 1 not only provides a test for determining when a matrix is invertible, but it also suggests a technique for obtaining the inverse when it exists. Once a matrix has been transformed to a row-reduced matrix with unity elements on the main diagonal, it is a simple matter to reduce it still further to the identity matrix. This is done by applying elementary row operation (E3)—adding to one row of a matrix a scalar times another row of the same matrix—to each column of the matrix, *beginning with the last column and moving sequentially toward the first column*, placing zeros in all positions above the diagonal elements.

Example 1 Use elementary row operations to transform the upper triangular matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

to the identity matrix.

Solution

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left\{ \begin{array}{l} \text{by adding to} \\ \text{the second row } (-3) \\ \text{times the third row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left\{ \begin{array}{l} \text{by adding to} \\ \text{the first row } (-1) \\ \text{times the third row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left\{ \begin{array}{l} \text{by adding to} \\ \text{the first row } (-2) \\ \text{times the second row} \end{array} \right. \quad \blacksquare$$

To summarize, we now know that a square matrix $\mathbf{A}$ has an inverse if and only if it can be transformed into the identity matrix by elementary row operations. Moreover, it follows from the previous section that each elementary row operation is represented by an elementary matrix $\mathbf{E}$ that generates the row operation under the multiplication $\mathbf{EA}$. Therefore, $\mathbf{A}$ has an inverse if and only if there exist a sequence of elementary matrices. $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ such that

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_3\,\mathbf{E}_2\,\mathbf{E}_1\,\mathbf{A} = \mathbf{I}.$$

But, if we denote the product of these elementary matrices as $\mathbf{B}$, we then have $\mathbf{BA} = \mathbf{I}$, which implies that $\mathbf{B} = \mathbf{A}^{-1}$. That is, the inverse of a square matrix $\mathbf{A}$ of full rank is the product of those elementary matrices that reduce $\mathbf{A}$ to the identity matrix! Thus, to calculate the inverse of $\mathbf{A}$, we need only keep a record of the elementary row operations, or equivalently the elementary matrices, that were used to reduce $\mathbf{A}$ to $\mathbf{I}$. This is accomplished by simultaneously applying the same elementary row operations to both $\mathbf{A}$ and an identity matrix of the same order, because if

$$\mathbf{E}_k\,\mathbf{E}_{k-1}\cdots\mathbf{E}_3\,\mathbf{E}_2\,\mathbf{E}_1\,\mathbf{A} = \mathbf{I},$$

then

$$(\mathbf{E}_k\,\mathbf{E}_{k-1}\cdots\mathbf{E}_3\,\mathbf{E}_2\,\mathbf{E}_1)\mathbf{I} = \mathbf{E}_k\,\mathbf{E}_{k-1}\cdots\mathbf{E}_3\,\mathbf{E}_2\,\mathbf{E}_1 = \mathbf{A}^{-1}.$$

We have, therefore, the following procedure for calculating inverses when they exist. Let $\mathbf{A}$ be the $n \times n$ matrix we wish to invert. Place next to it another $n \times n$ matrix $\mathbf{B}$ which is initially the identity. Using elementary row operations on $\mathbf{A}$, transform it into the identity. Each time an operation is performed on $\mathbf{A}$, repeat the exact same operation on $\mathbf{B}$. After $\mathbf{A}$ is transformed into the identity, the matrix obtained from transforming $\mathbf{B}$ will be $\mathbf{A}^{-1}$.

If $\mathbf{A}$ cannot be transformed into an indentity matrix, which is equivalent to saying that its row-reduced from contains at least one zero row, then $\mathbf{A}$ does not have an inverse.

Example 2 Invert

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to} \\ \text{the second row } (-3) \\ \text{times the first row}\end{array}\right.$$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]. \quad \left\{\begin{array}{l}\text{by multiplying} \\ \text{the second row by } (-\frac{1}{2})\end{array}\right.$$

$\mathbf{A}$ has been transformed into row-reduced form with a main diagonal of only unity elements; it has an inverse. Continuing with transformation process, we get

$$\to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]. \quad \left\{\begin{array}{l}\text{by adding to} \\ \text{the first row } (-2) \\ \text{times the second row}\end{array}\right.$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}. \quad \blacksquare$$

Example 3 Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}.$$

Solution

$$\left[\begin{array}{ccc|ccc} 5 & 8 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{first row by } (0.2)\end{array}\right.$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3.4 & -1.8 & -0.8 & 0 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-4) \\ \text{times the first row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.5 & 0 \\ 0 & -3.4 & -1.8 & -0.8 & 0 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by multiplying the}\\ \text{second row by (0.5)}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & -0.1 & -0.8 & 1.7 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{third row (3.4)}\\ \text{times the second row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right]. \quad \left\{\begin{array}{l}\text{by multiplying the}\\ \text{third row by (−0.1)}\end{array}\right.$$

A has been transformed into row-reduced form with a main diagonal of only unity elements; it has an inverse. Continuing with the transformation process, we get

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & -4 & 9 & 5 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{second row (−0.5)}\\ \text{times the third row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0 & -1.4 & 3.4 & 2 \\ 0 & 1 & 0 & -4 & 9 & 5 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{first row (−0.2)}\\ \text{times the third row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -11 & -6 \\ 0 & 1 & 0 & -4 & 9 & 5 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right]. \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{first row (−1.6)}\\ \text{times the second row}\end{array}\right.$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}. \quad \blacksquare$$

Example 4 Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Solution

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by interchanging the}\\ \text{first and second rows}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{the third row } (-1)\\ \text{times the first row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \left\{\begin{array}{l}\text{by multiplying the}\\ \text{third row by } (\frac{1}{2})\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{second row } (-1)\\ \text{times the third row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{first row } (-1)\\ \text{times the third row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]. \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{first row } (-1)\\ \text{times the second row}\end{array}\right.$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \quad \blacksquare$$

Example 5 Invert

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]. \quad \left\{\begin{array}{l}\text{by adding to}\\ \text{the second row } (-2)\\ \text{times the first row}\end{array}\right.$$

A has been transformed into row-reduced form. Since the main diagonal contains a zero element, here in the 2–2 position, the matrix **A** does not have an inverse. It is singular. ■

Problems 3.2

In Problems 1–20, find the inverses of the given matrices, if they exist.

1. $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, **2.** $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, **3.** $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$,

4. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, **5.** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$, **6.** $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$,

7. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, **8.** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, **9.** $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$,

10. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, **11.** $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$, **12.** $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$,

13. $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 3 & 9 & 2 \end{bmatrix}$, **14.** $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, **15.** $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -2 & -4 \\ 2 & 3 & -1 \end{bmatrix}$,

16. $\begin{bmatrix} 2 & 4 & 3 \\ 3 & -4 & -4 \\ 5 & 0 & -1 \end{bmatrix}$, **17.** $\begin{bmatrix} 5 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$, **18.** $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix}$,

19. $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$, **20.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 6 & 2 & 0 \\ 3 & 2 & 4 & -1 \end{bmatrix}$.

21. Use the results of Problems 11 and 20 to deduce a theorem involving inverses of lower triangular matrices.

22. Use the results of Problems 12 and 19 to deduce a theorem involving the inverses of upper triangular matrices.

23. Matrix inversion can be used to encode and decode sensitive messages for transmission. Initially, each letter in the alphabet is assigned a unique positive integer, with the simplest correspondence being

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓.
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

Zeros are used to separate words. Thus, the message

SHE IS A SEER

is encoded

19 8 5 0 9 19 0 1 0 19 5 5 18 0.

This scheme is too easy to decipher, however, so a scrambling effect is added prior to transmission. One scheme is to package the coded string as a set of 2-tuples, multiply each 2-tuple by a 2×2 invertible matrix, and then transmit the new string. For example, using the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

the coded message above would be scrambled into

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 19 \\ 8 \end{bmatrix} = \begin{bmatrix} 35 \\ 62 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} 47 \\ 75 \end{bmatrix}, \qquad \text{etc.,}$$

and the scrambled message becomes

35 62 5 10 47 75 . . . .

Note an immediate benefit from the scrambling: the letter S, which was originally always coded as 19 in each of its three occurrences, is now coded as a 35 the first time and as 75 the second time. Continue with the scrambling, and determine the final code for transmitting the above message.

24. Scramble the message SHE IS A SEER using, matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}.$$

25. Scramble the message AARON IS A NAME using the matrix and steps described in Problem 23.

26. Transmitted messages are unscrambled by again packaging the received message into 2-tuples and multiplying each vector by the inverse of **A**. To decode the scrambled message

$$18 \quad 31 \quad 44 \quad 72$$

using the encoding scheme described in Problem 23, we first calculate

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix},$$

and then

$$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 18 \\ 31 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 44 \\ 72 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \end{bmatrix}.$$

The unscrambled message is

$$8 \quad 5 \quad 12 \quad 16$$

which, according to the letter-integer correspondence given in Problem 23, translates to HELP. Using the same procedure, decode the scrambled message

$$26 \quad 43 \quad 40 \quad 60 \quad 18 \quad 31 \quad 28 \quad 51.$$

27. Use the decoding procedure described in Problem 26, but with the matrix **A** given in Problem 24, to decipher the transmitted message

$$16 \quad 120 \quad -39 \quad 131 \quad -27 \quad 45 \quad 38 \quad 76 \quad -51 \quad 129 \quad 28 \quad 56.$$

28. Scramble the message SHE IS A SEER by packaging the coded letters into 3-tuples and then multiplying by the 3×3 invertible matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Add as many zeros as necessary to the end of the message to generate complete 3-tuples.

3.3 Simultaneous Equations

One use of the inverse is in the solution of systems of simultaneous linear equations. Recall, from Section 1.3 that any such system may be written in the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{2}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{b}$ is a known vector, and $\mathbf{x}$ is the unknown vector we wish to find. If $\mathbf{A}$ is invertible, then we can premultiply (2) by $\mathbf{A}^{-1}$ and obtain

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

But $\mathbf{A}^{-1}\mathbf{A} = \mathbf{1}$, therefore

$$\mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

or

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3}$$

Hence, (3) shows that if $\mathbf{A}$ is invertible, then $\mathbf{x}$ can be obtained by premultiplying $\mathbf{b}$ by the inverse of $\mathbf{A}$.

Example 1 Solve the following system for x and y:

$$\begin{aligned} x - 2y &= -9, \\ -3x + \;\; y &= \;\;\, 2. \end{aligned}$$

Solution Define

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -9 \\ 2 \end{bmatrix};$$

then the system can be written as $\mathbf{A}\mathbf{x} = \mathbf{b}$, hence $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Using the method given in Section 3.2 we find that

$$\mathbf{A}^{-1} = \left(-\tfrac{1}{5}\right)\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \left(-\tfrac{1}{5}\right)\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -9 \\ 2 \end{bmatrix} = \left(-\tfrac{1}{5}\right)\begin{bmatrix} -5 \\ -25 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Using the definition of matrix equality (two matrices are equal if and only if their corresponding elements are equal), we have that $x = 1$ and $y = 5$. ■

Example 2 Solve the following system for x, y, and z:

$$\begin{aligned} 5x + 8y + z &= 2, \\ 2y + z &= -1, \\ 4x + 3y - z &= 3. \end{aligned}$$

Solution

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

$\mathbf{A}^{-1}$ is found to be (see Example 3 of Section 3.2)

$$\begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix},$$

hence $x = 3$, $y = -2$, and $z = 3$. ■

Not only does the invertibility of $\mathbf{A}$ provide us with a solution of the system $\mathbf{Ax} = \mathbf{b}$, it also provides us with a means of showing that this solution is unique (that is, there is no other solution to the system).

Theorem 1 *If* $\mathbf{A}$ *is invertible, then the system of simultaneous linear equations given by* $\mathbf{Ax} = \mathbf{b}$ *has one and only one solution.*

Proof. Define $\mathbf{w} = \mathbf{A}^{-1}\mathbf{b}$. Since we have already shown that $\mathbf{w}$ is a solution to $\mathbf{Ax} = \mathbf{b}$, it follows that

$$\mathbf{Aw} = \mathbf{b}. \tag{4}$$

Assume that there exists another solution $\mathbf{y}$. Since $\mathbf{y}$ is a solution, we have that

$$\mathbf{Ay} = \mathbf{b}. \tag{5}$$

Equations (4) and (5) imply that

$$\mathbf{Aw} = \mathbf{Ay}. \tag{6}$$

Premultiply both sides of (6) by $\mathbf{A}^{-1}$. Then

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{w} = \mathbf{A}^{-1}\mathbf{A}\mathbf{y},$$

$$\mathbf{I}\mathbf{w} = \mathbf{I}\mathbf{y},$$

or

$$\mathbf{w} = \mathbf{y}.$$

Thus, we see that if $\mathbf{y}$ is assumed to be a solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$, it must, in fact, equal $\mathbf{w}$. Therefore, $\mathbf{w} = \mathbf{A}^{-1}\mathbf{b}$ is the only solution to the problem. □

If $\mathbf{A}$ is singular, so that $\mathbf{A}^{-1}$ does not exist, then (3) is not valid and other methods, such as Gaussian elimination, must be used to solve the given system of simultaneous equations.

Problems 3.3

In Problems 1 through 12, use matrix inversion, if possible, to solve the given systems of equations:

1. $\begin{aligned} x + 2y &= -3, \\ 3x + y &= 1. \end{aligned}$

2. $\begin{aligned} a + 2b &= 5, \\ -3a + b &= 13. \end{aligned}$

3. $\begin{aligned} 4x + 2y &= 6, \\ 2x - 3y &= 7. \end{aligned}$

4. $\begin{aligned} 4l - p &= 1, \\ 5l - 2p &= -1. \end{aligned}$

5. $\begin{aligned} 2x + 3y &= 8, \\ 6x + 9y &= 24. \end{aligned}$

6. $\begin{aligned} x + 2y - z &= -1, \\ 2x + 3y + 2z &= 5, \\ y - z &= 2. \end{aligned}$

7. $\begin{aligned} 2x + 3y - z &= 4, \\ -x - 2y + z &= -2, \\ 3x - y &= 2. \end{aligned}$

8. $\begin{aligned} 60l + 30m + 20n &= 0, \\ 30l + 20m + 15n &= -10, \\ 20l + 15m + 12n &= -10. \end{aligned}$

9. $\begin{aligned} 2r + 4s &= 2, \\ 3r + 2s + t &= 8, \\ 5r - 3s + 7t &= 15. \end{aligned}$

10. $\begin{aligned} 2r + 4s &= 3, \\ 3r + 2s + t &= 8, \\ 5r - 3s + 7t &= 15. \end{aligned}$

11. $\begin{aligned} 2r + 3s - 4t &= 12, \\ 3r - 2s &= -1, \\ 8r - s - 4t &= 10. \end{aligned}$

12. $\begin{aligned} x + 2y - 2z &= -1, \\ 2x + y + z &= 5, \\ -x + y - z &= -2. \end{aligned}$

13. Use matrix inversion to determine a production schedule that satisfies the requirements of the manufacturer described in Problem 12 of Section 2.1.

14. Use matrix inversion to determine a production schedule that satisfies the requirements of the manufacturer described in Problem 13 of Section 2.1.

15. Use matrix inversion to determine a production schedule that satisfies the requirements of the manufacturer described in Problem 14 of Section 2.1.

16. Use matrix inversion to determine the bonus for the company described in Problem 16 of Section 2.1.

17. Use matrix inversion to determine the number of barrels of gasoline that the producer described in Problem 17 of Section 2.1 must manufacture to break even.

18. Use matrix inversion to solve the Leontief input–output model described in Problem 22 of Section 2.1.

19. Use matrix inversion to solve the Leontief input–output model described in Problem 23 of Section 2.1.

3.4 Properties of the Inverse

Theorem 1 *If* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are square matrices of the same order with* $\mathbf{AB} = \mathbf{I}$ *and* $\mathbf{CA} = \mathbf{I}$, *then* $\mathbf{B} = \mathbf{C}$.

Proof. $\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}$. □

Theorem 2 *The inverse of a matrix is unique.*

Proof. Suppose that $\mathbf{B}$ and $\mathbf{C}$ are inverse of $\mathbf{A}$. Then, by (1), we have that

$$\mathbf{AB} = \mathbf{I}, \quad \mathbf{BA} = \mathbf{I}, \quad \mathbf{AC} = \mathbf{I}, \quad \text{and} \quad \mathbf{CA} = \mathbf{I}. \qquad \square$$

It follows from Theorem 1 that $\mathbf{B} = \mathbf{C}$. Thus, if $\mathbf{B}$ and $\mathbf{C}$ are both inverses of $\mathbf{A}$, they must in fact be equal. Hence, the inverse is unique.

Using Theorem 2, we can prove some useful properties of the inverse of a matrix $\mathbf{A}$ when $\mathbf{A}$ is nonsingular.

Property 1 $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$.

Proof. See Problem 1. □

Property 2 $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof. $(\mathbf{AB})^{-1}$ denotes the inverse of $\mathbf{AB}$. However, $\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) = \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$. Thus, $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is also an inverse for $\mathbf{AB}$, and, by uniqueness of the inverse, $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$. □

Property 3 $(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1}\,\mathbf{A}_{n-1}^{-1} \cdots \mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}$.

Proof. This is an extension of Property 2 and, as such, is proved in a similar manner. □

CAUTION. Note that Property 3 states that the inverse of a product is *not* the product of the inverses but rather the product of the inverses commuted.

Property 4 $\left(\mathbf{A}^{\mathsf{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathsf{T}}$

Proof. $\left(\mathbf{A}^{\mathsf{T}}\right)^{-1}$ denotes the inverse of $\mathbf{A}^{\mathsf{T}}$. However, using the property of the transpose that $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$, we have that

$$\left(\mathbf{A}^{\mathsf{T}}\right)\left(\mathbf{A}^{-1}\right)^{\mathsf{T}} = \left(\mathbf{A}^{-1}\mathbf{A}\right)^{\mathsf{T}} = \mathbf{I}^{\mathsf{T}} = \mathbf{I}.$$

Thus, $\left(\mathbf{A}^{-1}\right)^{\mathsf{T}}$ is an inverse of $\mathbf{A}^{\mathsf{T}}$, and by uniqueness of the inverse, $\left(\mathbf{A}^{-1}\right)^{\mathsf{T}} = \left(\mathbf{A}^{\mathsf{T}}\right)^{-1}$. □

Property 5 $(\lambda\mathbf{A})^{-1} = (1/\lambda)\,(\mathbf{A})^{-1}$ *if* λ *is a nonzero scalar.*

Proof. $(\lambda\mathbf{A})^{-1}$ denotes the inverse of $\lambda\mathbf{A}$. However,

$$(\lambda\mathbf{A})(1/\lambda)\mathbf{A}^{-1} = \lambda(1/\lambda)\mathbf{A}\mathbf{A}^{-1} = 1 \cdot \mathbf{I} = \mathbf{I}.$$

Thus, $(1/\lambda)\mathbf{A}^{-1}$ is an inverse of $\lambda\mathbf{A}$, and by uniqueness of the inverse $(1/\lambda)\mathbf{A}^{-1} = (\lambda\mathbf{A})^{-1}$. □

Property 6 *The inverse of a nonsingular symmetric matrix is symmetric.*

Proof. See Problem 18. □

Property 7 *The inverse of a nonsingular upper or lower triangular matrix is again an upper or lower triangular matrix respectively.*

Proof. This is immediate from Theorem 2 and the constructive procedure described in Section 3.2 for calculating inverses.

Finally, the inverse provides us with a straightforward way of defining square matrices raised to negative integral powers. If $\mathbf{A}$ is nonsingular then we define $\mathbf{A}^{-n} = \left(\mathbf{A}^{-1}\right)^{n}$. □

Example 1 Find $\mathbf{A}^{-2}$ if

$$\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Solution

$$\mathbf{A}^{-2} = \left(\mathbf{A}^{-1}\right)^2$$
$$= \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}^2 = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 180 & -96 \\ -96 & 52 \end{bmatrix}. \quad \blacksquare$$

Problems 3.4

1. Prove Property 1.

2. Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}.$$

3. Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}.$$

4. Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

6. Verify the result of Problem 5 if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}.$$

7. Verify Property 4 for the matrix **A** defined in Problem 2.

8. Verify Property 4 for the matrix **A** defined in Problem 3.

9. Verify Property 4 for the matrix **A** defined in Problem 4.

10. Verify Property 5 for $\lambda = 2$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \\ -1 & 0 & 3 \end{bmatrix}.$$

11. Find $\mathbf{A}^{-2}$ and $\mathbf{B}^{-2}$ for the matrices defined in Problem 2.

12. Find $\mathbf{A}^{-3}$ and $\mathbf{B}^{-3}$ for the matrices defined in Problem 2.

13. Find $\mathbf{A}^{-2}$ and $\mathbf{B}^{-4}$ for the matrices defined in Problem 3.

14. Find $\mathbf{A}^{-2}$ and $\mathbf{B}^{-2}$ for the matrices defined in Problem 4.

15. Find $\mathbf{A}^{-3}$ and $\mathbf{B}^{-3}$ for the matrices defined in Problem 4.

16. Find $\mathbf{A}^{-3}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

17. If $\mathbf{A}$ is symmetric, prove the identity

$$\left(\mathbf{B}\mathbf{A}^{-1}\right)^{\mathsf{T}}\left(\mathbf{A}^{-1}\mathbf{B}^{\mathsf{T}}\right)^{-1} = \mathbf{I}.$$

18. Prove Property 6.

3.5 LU Decomposition

Matrix inversion of elementary matrices (see Section 3.1) can be combined with the third elementary row operation (see Section 2.3) to generate a good numerical technique for solving simultaneous equations. It rests on being able to decompose a *nonsingular* square matrix $\mathbf{A}$ into the product of lower triangular matrix $\mathbf{L}$ with an upper triangular matrix $\mathbf{U}$. Generally, there are many such factorizations. If, however, we add the additional condition that all diagonal elements of $\mathbf{L}$ be unity, then the decomposition, when it exists, is unique, and we may write

$$\mathbf{A} = \mathbf{L}\mathbf{U} \tag{7}$$

with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

and

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

To decompose $\mathbf{A}$ into from (7), we first reduce $\mathbf{A}$ to upper triangular from using just the third elementary row operation: namely, add to one row of a matrix a scalar times another row of that same matrix. This is completely analogous to transforming a matrix to row-reduced form, except that we no longer use the first two elementary row operations. We do not interchange rows, and we do not multiply a row by a nonzero constant. Consequently, we no longer require the first nonzero element of each nonzero row to be unity, and if any of the pivots are zero—which in the row-reduction scheme would require a row interchange operation—then the decomposition scheme we seek cannot be done.

Example 1 Use the third elementary row operation to transform the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix}$$

into upper triangular form.

Solution

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ -6 & -1 & 2 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{second row } (-2) \text{ times} \\ \text{the first row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & -4 & 11 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{third row } (3) \text{ times} \\ \text{the first row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}. \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{third row } (1) \text{ times} \\ \text{the second row} \end{array} \right. \quad \blacksquare$$

If a square matrix **A** can be reduced to upper triangular form **U** by a sequence of elementary row operations of the third type, then there exists a sequence of elementary matrices $\mathbf{E}_{21}$, $\mathbf{E}_{31}$, $\mathbf{E}_{41}, \ldots, \mathbf{E}_{n,n-1}$ such that

$$\left(\mathbf{E}_{n-1,n} \cdots \mathbf{E}_{41}\mathbf{E}_{31}\mathbf{E}_{21}\right)\mathbf{A} = \mathbf{U}, \tag{8}$$

where $\mathbf{E}_{21}$ denotes the elementary matrix that places a zero in the 2–1 position, $\mathbf{E}_{31}$ denotes the elementary matrix that places a zero in the 3–1 position, $\mathbf{E}_{41}$ denotes the elementary matrix that places a zero in the 4–1 position, and so on. Since elementary matrices have inverses, we can write (8) as

$$\mathbf{A} = \left(\mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{41}^{-1} \cdots \mathbf{E}_{n,n-1}^{-1}\right)\mathbf{U}. \tag{9}$$

Each elementary matrix in (8) is lower triangular. If follows from Property 7 of Section 3.4 that each of the inverses in (9) are lower triangular, and then from

Theorem 1 of Section 1.4 that the product of these lower triangular matrices is itself lower triangular. Setting

$$\mathbf{L} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{41}^{-1}\cdots\mathbf{E}_{n,n-1}^{-1},$$

we see that (9) is identical to (7), and we have the decomposition we seek.

Example 2 Construct an **LU** decomposition for the matrix given in Example 1.

Solution The elementary matrices associated with the elementary row operations described in Example 1 are

$$\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{E}_{42} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

with inverses given respectively by

$$\mathbf{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{E}_{42}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}$$

or, upon multiplying together the inverses of the elementary matrices,

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}. \quad \blacksquare$$

Example 2 suggests an important simplification of the decomposition process. Note the elements in **L** below the main diagonal are *the negatives of the scalars* used in the elementary row operations to reduce the original matrix to upper triangular form! This is no coincidence. In general,

Observation 1 If an elementary row operation is used to put a zero in the $i-j$ position of $\mathbf{A}(i > j)$ by adding to row i a scalar k times row j, then the $i-j$ element of **L** in the **LU** decomposition of **A** is $-k$.

We summarize the decomposition process as follows: Use only the third elementary row operation to transform a given square matrix **A** to upper triangular from. If this is not possible, because of a zero pivot, then stop; otherwise, the **LU** decomposition is found by defining the resulting upper triangular matrix as **U** and constructing the lower triangular matrix **L** utilizing Observation 1.

Example 3 Construct an **LU** decomposition for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 6 & 2 & 4 & 8 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix}.$$

Solution Transforming **A** to upper triangular form, we get

$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 6 & 2 & 4 & 8 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{second row } (-3) \text{ times} \\ \text{the first row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & -\frac{3}{2} & -1 & \frac{5}{2} \\ 0 & 1 & -3 & -4 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{third row } \left(-\frac{1}{2}\right) \text{ times} \\ \text{the first row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{third row } \left(-\frac{3}{2}\right) \text{ times} \\ \text{the second row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -5 & -5 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{fourth row } (1) \text{ times} \\ \text{the second row} \end{array} \right.$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{by adding to the} \\ \text{fourth row } \left(\frac{5}{2}\right) \text{ times} \\ \text{the third row} \end{array} \right.$$

We now have an upper triangular matrix **U**. To get the lower triangular matrix **L** in the decomposition, we note that we used the scalar -3 to place a zero in the 2–1 position, so its negative $-(-3) = 3$ goes into the 2–1 position of **L**. We used the scalar $-\frac{1}{2}$ to place a zero in the 3–1 position in the second step of the above triangularization process, so its negative, $\frac{1}{2}$, becomes the 3–1 element in **L**; we used the scalar $\frac{5}{2}$ to place a zero in the 4–3 position during the last step of the triangularization process, so its negative, $-\frac{5}{2}$, becomes the 4–3 element in **L**. Continuing in this manner, we generate the decomposition

$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 6 & 2 & 4 & 8 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{2} & 1 & 0 \\ 0 & -1 & -\frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}. \quad \blacksquare$$

LU decompositions, when they exist, can be used to solve systems of simultaneous linear equations. If a square matrix **A** can be factored into $\mathbf{A} = \mathbf{LU}$, then the system of equations $\mathbf{Ax} = \mathbf{b}$ can be written as $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. To find **x**, we first solve the system

$$\mathbf{Ly} = \mathbf{b} \tag{10}$$

for **y**, and then, once **y** is determined, we solve the system

$$\mathbf{Ux} = \mathbf{y} \tag{11}$$

for **x**. Both systems (10) and (11) are easy to solve, the first by forward substitution and the second by backward substitution.

Example 4 Solve the system of equations:

$$\begin{aligned} 2x - y + 3z &= 9, \\ 4x + 2y + z &= 9, \\ -6x - y + 2z &= 12. \end{aligned}$$

Solution This system has the matrix form

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 12 \end{bmatrix}.$$

The **LU** decomposition for the coefficient matrix **A** is given in Example 2. If we define the components of **y** by α, β, and γ, respectively, the matrix system $\mathbf{Ly} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 12 \end{bmatrix},$$

which is equivalent to the system of equations

$$\begin{aligned} \alpha \qquad\qquad &= 9, \\ 2\alpha + \beta \qquad &= 9, \\ -3\alpha - \beta + \gamma &= 12. \end{aligned}$$

Solving this system from top to bottom, we get $\alpha = 9$, $\beta = -9$, and $\gamma = 30$. Consequently, the matrix system $\mathbf{U}\mathbf{x} = \mathbf{y}$ is

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 30 \end{bmatrix}.$$

which is equivalent to the system of equations

$$\begin{aligned} 2x - y + 3z &= 9, \\ 4y - 5z &= -9, \\ 6z &= 30. \end{aligned}$$

Solving this system from bottom to top, we obtain the final solution $x = -1$, $y = 4$, and $z = 5$. ■

Example 5 Solve the system:

$$\begin{aligned} 2a + b + 2c + 3d &= 5, \\ 6a + 2b + 4c + 8d &= 8, \\ a - b \qquad + 4d &= -4, \\ b - 3c - 4d &= -3. \end{aligned}$$

Solution The matrix representation for this system has as its coefficient matrix the matrix $\mathbf{A}$ of Example 3. Define.

$$\mathbf{y} = [\alpha,\ \beta,\ \gamma,\ \delta]^{\mathsf{T}}.$$

Then, using the decomposition determined in Example 3, we can write the matrix system $\mathbf{L}\mathbf{y} = \mathbf{b}$ as the system of equations

$$\begin{aligned} \alpha \qquad\qquad\qquad &= 5, \\ 3\alpha + \beta \qquad\qquad &= 8, \\ \tfrac{1}{2}\alpha + \tfrac{3}{2}\beta + \gamma \qquad &= -4, \\ -\beta - \tfrac{5}{2}\gamma + \delta &= -3, \end{aligned}$$

which has as its solution $\alpha = 5$, $\beta = -7$, $\gamma = 4$, and $\delta = 0$. Thus, the matrix system $\mathbf{U}\mathbf{x} = \mathbf{y}$ is equivalent to the system of equations

$$\begin{aligned} 2a + b + 2c + 3d &= 5, \\ -b - 2c - d &= -7, \\ 2c + 4d &= 4, \\ 5d &= 0. \end{aligned}$$

Solving this set from bottom to top, we calculate the final solution $a = -1$, $b = 3$, $c = 2$, and $d = 0$. ■

LU decomposition and Gaussian elimination are equally efficient for solving $\mathbf{Ax} = \mathbf{b}$, when the decomposition exists. **LU** decomposition is superior when $\mathbf{Ax} = \mathbf{b}$ must be solved repeatedly for different values of **b** but the same **A**, because once the factorization of **A** is determined it can be used with all **b**. (See Problems 17 and 18.) A disadvantage of **LU** decomposition is that it does not exist for all nonsingular matrices, in particular whenever a pivot is zero. Fortunately, this occurs rarely, and when it does the difficulty usually is overcome by simply rearranging the order of the equations. (See Problems 19 and 20.)

Problems 3.5

In Problems 1 through 14, **A** and **b** are given. Construct an **LU** decomposition for the matrix **A** and then use it to solve the system $\mathbf{Ax} = \mathbf{b}$ for **x**.

1. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$

2. $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}.$

3. $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 625 \\ 550 \end{bmatrix}.$

4. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.$

5. $\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 1 \\ 2 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}.$

6. $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ -2 & -1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ -40 \\ 0 \end{bmatrix}.$

7. $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 3 & 9 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 50 \\ 80 \\ 20 \end{bmatrix}.$

8. $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 80 \\ 159 \\ -75 \end{bmatrix}.$

9. $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}.$

10. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}.$

11. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 4 \\ -3 \\ -2 \\ -2 \end{bmatrix}$.

12. $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 4 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 1000 \\ 200 \\ 100 \\ 100 \end{bmatrix}$.

13. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 30 \\ 30 \\ 10 \\ 10 \end{bmatrix}$.

14. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 6 \\ -4 & 3 & 1 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 9 \\ 4 \end{bmatrix}$.

15. (a) Use **LU** decomposition to solve the system

$$\begin{aligned} -x + 2y &= 9, \\ 2x + 3y &= 4. \end{aligned}$$

(b) Resolve when the right sides of each equation are replaced by 1 and -1, respectively.

16. (a) Use **LU** decomposition to solve the system

$$\begin{aligned} x + 3y - z &= -1, \\ 2x + 5y + z &= 4, \\ 2x + 7y - 4z &= -6. \end{aligned}$$

(b) Resolve when the right sides of each equation are replaced by 10, 10, and 10, respectively.

17. Solve the system $\mathbf{Ax} = \mathbf{b}$ for the following vectors $\mathbf{b}$ when $\mathbf{A}$ is given as in Problem 4:

(a) $\begin{bmatrix} 5 \\ 7 \\ -4 \end{bmatrix}$, $\quad$ (b) $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\quad$ (c) $\begin{bmatrix} 40 \\ 50 \\ 20 \end{bmatrix}$, $\quad$ (d) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

18. Solve the system $\mathbf{Ax} = \mathbf{b}$ for the following vectors $\mathbf{b}$ when $\mathbf{A}$ is given as in Problem 13:

$$\text{(a)} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 190 \\ 130 \\ 160 \\ 60 \end{bmatrix}, \quad \text{(d)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

19. Show that **LU** decomposition cannot be used to solve the system

$$\begin{aligned} 2y + z &= -1, \\ x + y + 3z &= 8, \\ 2x - y - z &= 1, \end{aligned}$$

but that the decomposition can be used if the first two equations are interchanged.

20. Show that **LU** decomposition cannot be used to solve the system

$$\begin{aligned} x + 2y + z &= 2, \\ 2x + 4y - z &= 7, \\ x + y + 2z &= 2, \end{aligned}$$

but that the decomposition can be used if the first and third equations are interchanged.

21. (a) Show that the **LU** decomposition procedure given in this chapter cannot be applied to

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 9 \end{bmatrix}.$$

(b) Verify that $\mathbf{A} = \mathbf{LU}$, when

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 0 & 2 \\ 0 & 7 \end{bmatrix}.$$

(c) Verify that $\mathbf{A} = \mathbf{LU}$, when

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}.$$

(d) Why do you think the **LU** decomposition procedure fails for this **A**? What might explain the fact that **A** has more than one **LU** decomposition?

3.6 Final Comments on Chapter 3

We now prove the answers to two questions raised earlier. First, what matrices have inverses? Second, if $\mathbf{AB} = \mathbf{I}$, is it necessarily true that $\mathbf{AB} = \mathbf{I}$ too?

Lemma 1 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. If* $\mathbf{AB} = \mathbf{I}$*, then the system of equations* $\mathbf{Ax} = \mathbf{y}$ *has a solution for every choice of the vector* $\mathbf{y}$.

Proof. Once $\mathbf{y}$ is specified, set $\mathbf{x} = \mathbf{By}$. Then

$$\mathbf{Ax} = \mathbf{A}(\mathbf{By}) = (\mathbf{AB})\mathbf{y} = \mathbf{Iy} = \mathbf{y},$$

so $\mathbf{x} = \mathbf{By}$ is a solution of $\mathbf{Ax} = \mathbf{y}$. □

Lemma 2 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *matrices with* $\mathbf{AB} = \mathbf{I}$*, then* $\mathbf{A}$ *has rank* n.

Proof. Designate the rows of $\mathbf{A}$ by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$. We want to show that these n rows constitute a linearly independent set of vectors, in which case the rank of $\mathbf{A}$ is n. Designate the columns of $\mathbf{I}$ as the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, respectively. It follows from Lemma 1 that the set of equations $\mathbf{Ax} = \mathbf{e}_j$ $(j = 1, 2, \ldots, n)$ has a solution for each j. Denote these solutions by $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$, respectively. Therefore,

$$\mathbf{Ax}_j = \mathbf{e}_j.$$

Since $\mathbf{e}_j$ $(j = 1, 2, \ldots, n)$ is an n-dimensional column vector having a unity element in row j and zeros everywhere else, it follows form the last equation that

$$\mathbf{A}_i\mathbf{x}_j = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

This equation can be notationally simplified if we make use of the *Kronecker delta* δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j. \\ 0 & \text{when } i \neq j. \end{cases}$$

Then,

$$\mathbf{A}_i\mathbf{x}_j = \delta_{ij}.$$

Now consider the equation

$$\sum_{i=0}^{n} c_i\mathbf{A}_i = \mathbf{0}.$$

We wish to show that each constant c_i must be zero. Multiplying both sides of this last equation on the right by the vector $\mathbf{x}_j$, we have

$$\left(\sum_{i=0}^{n} c_i \mathbf{A}_i\right) \mathbf{x}_j = \mathbf{0}\mathbf{x}_j,$$

$$\sum_{i=0}^{n} (c_i \mathbf{A}_i)\, \mathbf{x}_j = 0,$$

$$\sum_{i=0}^{n} c_i \big(\mathbf{A}_i \mathbf{x}_j\big) = 0,$$

$$\sum_{i=0}^{n} c_i \delta_{ij} = 0,$$

$$c_j = 0.$$

Thus for each $\mathbf{x}_j$ $(j = 1, 2, \ldots, n)$ we have $c_j = 0$, which implies that $c_1 = c_2 = \cdots = c_n = 0$ and that the rows $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are linearly independent. □

It follows directly from Lemma 2 and the definition of an inverse that if an $n \times n$ matrix $\mathbf{A}$ has an inverse, then $\mathbf{A}$ must have rank n. This in turn implies directly that if $\mathbf{A}$ does not have rank n, then it does not have an inverse. We now want to show the converse: that is, if $\mathbf{A}$ has rank n, then $\mathbf{A}$ has an inverse.

We already have part of the result. If an $n \times n$ matrix $\mathbf{A}$ has rank n, then the procedure described in Section 3.2 is a constructive method for obtaining a matrix $\mathbf{C}$ having the property that $\mathbf{CA} = \mathbf{I}$. The procedure transforms $\mathbf{A}$ to an identity matrix by a sequence of elementary row operations $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_{k-1}, \mathbf{E}_k$. That is,

$$\mathbf{E}_k \mathbf{E}_{k-1} \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}.$$

Setting

$$\mathbf{C} = \mathbf{E}_k \mathbf{E}_{k-1} \ldots \mathbf{E}_2 \mathbf{E}_1, \tag{12}$$

we have

$$\mathbf{CA} = \mathbf{I}. \tag{13}$$

We need only show that $\mathbf{AC} = \mathbf{I}$, too.

Theorem 1 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *matrices such that* $\mathbf{AB} = \mathbf{I}$, *then* $\mathbf{BA} = \mathbf{I}$.

Proof. If $\mathbf{AB} = \mathbf{I}$, then from Lemma 1 $\mathbf{A}$ has rank n, and from (12) and (13) there exists a matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$. It follows from Theorem 1 of Section 3.4 that $\mathbf{B} = \mathbf{C}$. □

The major implication of Theorem 1 is that if $\mathbf{B}$ is a right inverse of $\mathbf{A}$, then $\mathbf{B}$ is also a left inverse of $\mathbf{A}$; and also if $\mathbf{A}$ is a left inverse of $\mathbf{B}$, then $\mathbf{A}$ is also a right inverse of $\mathbf{B}$. Thus, one needs only check whether a matrix is a right or left inverse; once one is verified for square matrices, the other is guaranteed. In particular, if an $n \times n$ matrix $\mathbf{A}$ has rank n, then (13) is valid. Thus, $\mathbf{C}$ is a left inverse of $\mathbf{A}$. As a result of Theorem 1, however, $\mathbf{C}$ is also a right inverse of $\mathbf{A}$—just replace $\mathbf{A}$ with $\mathbf{C}$ and $\mathbf{B}$ with $\mathbf{A}$ in Theorem 1—so $\mathbf{C}$ is both a left and right inverse of $\mathbf{A}$, which means that $\mathbf{C}$ is the inverse of $\mathbf{A}$. We have now proven:

Theorem 2 *An $n \times n$ matrix $\mathbf{A}$ has an inverse if and only if $\mathbf{A}$ has rank n.*