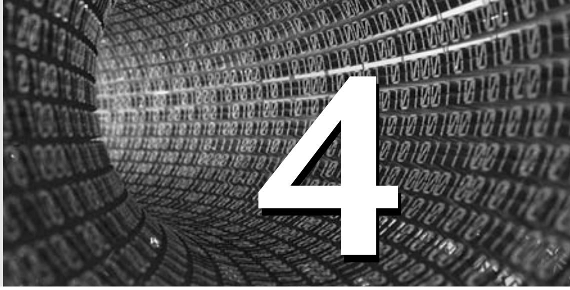

An Introduction to Optimization

4.1 Graphing Inequalities

Many times in real life, solving simple *equations* can give us solutions to everyday problems.

Example 1 Suppose we enter a supermarket and are informed that a certain brand of coffee is sold in 3-lb bags for \$6.81. If we wanted to determine the cost per unit pound, we could *model* this problem as follows:

Let x be the cost per unit pound of coffee; then the following equation represents the total cost of the coffee:

$$x + x + x = 3x = 6.81. \tag{1}$$

Dividing both sides of (1) by 3 gives the cost of \$2.27 per pound of coffee. ■

Example 2 Let's suppose that we are going to rent a car. If the daily fixed cost is \$100.00, with the added price of \$1.25 per mile driven, then

$$C = 100 + 1.25m \tag{2}$$

represents the total daily cost, C, where m is the number of miles traveled on a particular day.

What if we had a daily budget of \$1000.00? We would then use (2) to determine the number of miles we could travel given this budget. Using elementary algebra, we see that we would be able to drive 720 miles. ■

These two simple examples illustrate how equations can assist us in our daily lives. But sometimes things can be a bit more complicated.

Example 3 Suppose we are employed in a factory that produces two types of bicycles: a standard model (S) and a deluxe model (D). Let us assume that the revenue (R) on the former is \$250 per bicycle and the revenue on the latter is \$300 per bicycle. Then the total revenue can be expressed by the following equation:

$$R = 250S + 300D. \tag{3}$$

Now suppose manufacturing costs are \$10,000; so to make a profit, R has to be *greater than* \$10,000. Hence the following *inequality* is used to relate the bicycles and revenue with respect to showing a profit:

$$250S + 300D > 10{,}000. \tag{4}$$

Relationship (4) illustrates the occurrence of inequalities. However, before we can solve problems related to this example, it is important to "visualize" inequalities, because the graphing of such relationships will assist us in many ways. ■

For the rest of this section, we will sketch inequalities in two dimensions.

Example 4 Sketch the inequality $x + y \leq 2$. The *equation* $x + y = 2$ is a straight line passing through the points $(2, 0)$—the x-intercept—and $(0, 2)$—the y-intercept. The *inequality* $x + y \leq 2$ merely includes the region "under" the line.

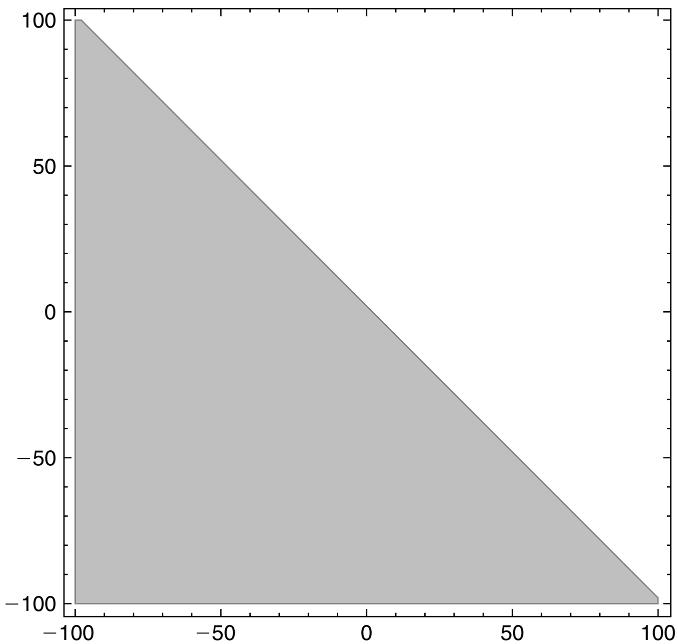

Figure 4.1 ■

Remark 1 Notice that the lower left-hand part of the graph is *shaded*. An easy way to check is to pick a point, say $(-50, -50)$; clearly $-50 + -50 \leq 2$, therefore the "half-region" containing this point must be the shaded portion.

Remark 2 The graph of the *strict* inequality $x + y < 2$ yields the same picture with the line *dashed* (instead of solid) to indicate that points on the line $x + y = 2$ are *not* included.

Example 5 Sketch $2x + 3y \geq 450$.

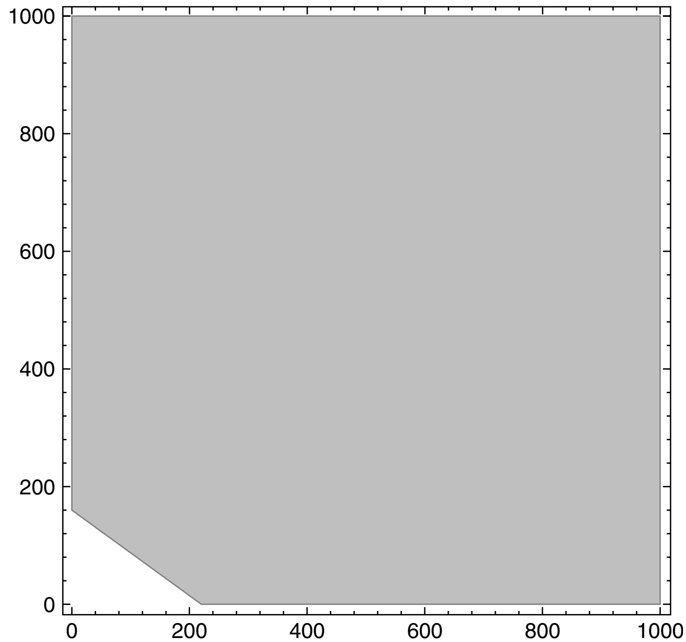

Figure 4.2 ■

Remark 3 Notice that we have restricted this graph to the first quadrant. Many times the variables involved will have non-negative values, such as volume, area, etc. Notice, too, that the region is *infinite*, as is the region in Example 4.

Example 6 Sketch $4x + y \leq 12$ and $2x + 5y \leq 24$, where $x \geq 0$ and $y \geq 0$.

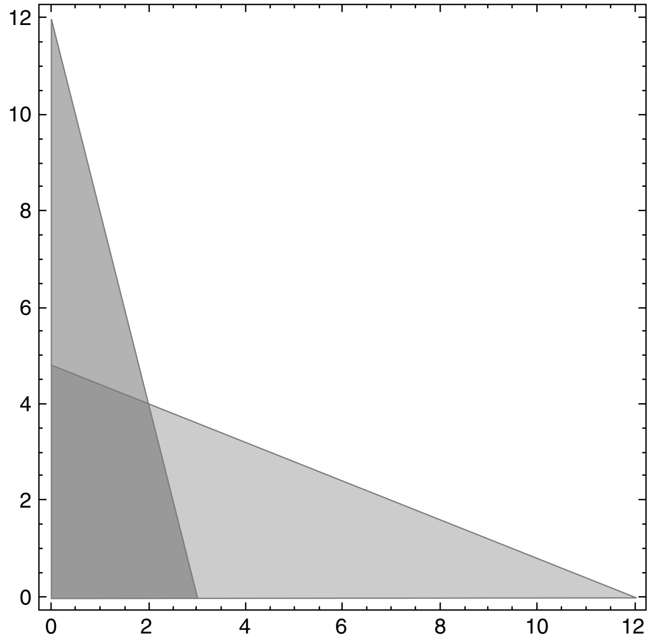

Figure 4.3 ■

Remark 4 Note that the "upper-right" corner point is $(2, 4)$. This point is the *intersection* of the straight lines given by the equations $4x + y = 12$ and $2x + 5y = 24$; in Chapter 2 we covered techniques used in solving simultaneous equations. Here the added constraints of $x \geq 0$ and $y \geq 0$ render a *bounded* or *finite* region.

We will see regions like Figure 4.3 again both in Section 4.2 (with regard to modeling) and Section 4.3 (using the technique of *linear programming*).

Problems 4.1

Sketch the following inequalities:

1. $y \leq 0$

2. $x \geq 0$

3. $y \geq \pi$

4. $x + 4y \leq 12$

5. $x + 4y < 12$

6. $x + 4y \geq 12$

7. $x + 4y > 12$

Sketch the inequalities on the same set of axes:

8. $x + 4y \leq 12, x \geq 0, y \geq 0$

9. $x + 4y \leq 12, 5x + 2y \leq 24$

10. $x + 4y \geq 12, 5x + 2y \geq 24$

11. $x + 2y \leq 12, 2x + y \leq 16, x + 2y \leq 20$

12. $x - y \geq 100$

13. $x + y \geq 100, 3x + 3y \leq 60$

14. $x + y \leq 10, -x + y \leq 10, x - y \leq 10, -x - y \leq 10$

4.2 Modeling with Inequalities

Consider the following situation. Suppose a toy company makes two types of wagons, X and Y. Let us further assume that during any work period, each X takes 3 hours to construct and 2 hours to paint, while each Y takes 1 hour to construct and 2 hours to paint. Finally, the maximum number of hours allotted for construction is 1500 and the limit on hours available for painting is 1200 hours. If the profit on each X is \$50 and the profit on each Y is \$60, how many of each type of wagon should be produced to maximize the profit?

We can model the above with a number of inequalities. First, we must define our variables. Let X represent the number of X wagons produced and Y represent the number of Y wagons produced. This leads to the following four relationships:

$$3X + Y \leq 1500 \tag{5}$$

$$2X + 2Y \leq 1200 \tag{6}$$

$$X \geq 0 \tag{7}$$

$$Y \geq 0. \tag{8}$$

Note that (5) represents the constraint due to *construction* (in hours) while (6) represents the constraint due to *painting* (also in hours). The inequalities (7) and (8) merely state that the number of each type of wagon cannot be negative.

These four inequalities can be graphed as follows in Figure 4.4:

Let us make a few observations. We will call the shaded region that satisfies all four inequalities the *region of feasibility*. Next, the shaded region has four "corner points" called *vertices*. The coordinates of these points are given by (0, 0), (0, 600), (450, 150) and (500, 0). Lastly, this region has the property that, given any two points in the interior of the region, the straight line segment connecting these two points lies entirely within the region. We call regions with this property *convex*.

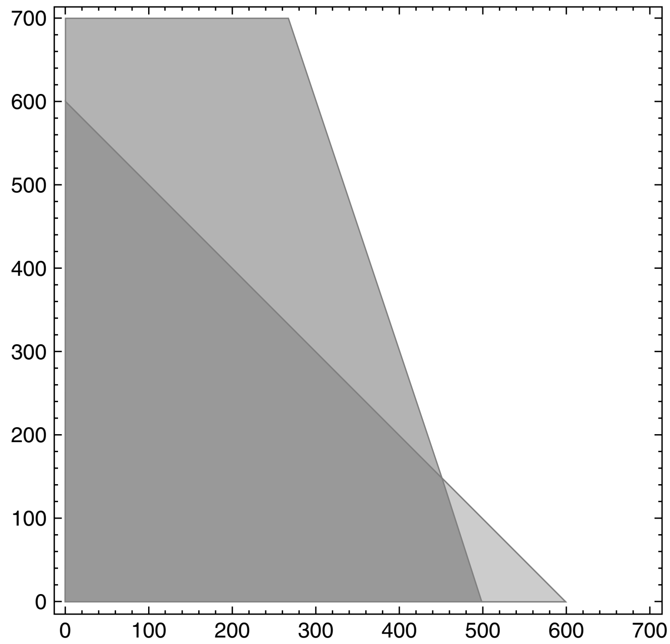

Figure 4.4

The following equation gives the profit (in dollars):

$$P(X, Y) = 50X + 60Y. \tag{9}$$

Note that Equation (9) is called the *objective function*. The notation $P(X, Y)$ is read "P of X and Y" and is evaluated by simply substituting the respective values into the expression. For example, $P(0{,}600) = 50(0) + 60(600) = 0 + 36{,}000 = 36{,}000$ dollars, while $P(450{,}150) = 50(450) + 60(150) = 22{,}500 + 9000 = 31{,}500$ dollars.

Equation (9), the inequalities (5)–(8), and Figure 4.4 model the situation above, which is an example of an *optimization problem*. In this particular example, our goal was to *maximize* a quantity (profit). Our next example deals with *minimization*.

Suppose a specific diet calls for the following minimum daily requirements: 186 units of Vitamin A and 120 units of Vitamin B. Pill X contains 6 units of Vitamin A and 3 units of Vitamin B, while pill Y contains 2 units of Vitamin A and 2 units of Vitamin B. What is the least number of pills needed to satisfy both vitamin requirements?

Let us allow X to represent the number of X pills ingested and let Y represent the number of Y pills taken. Then the following inequalities hold:

$$6X + 2Y \geq 186 \tag{10}$$

$$3X + 2Y \geq 120 \tag{11}$$

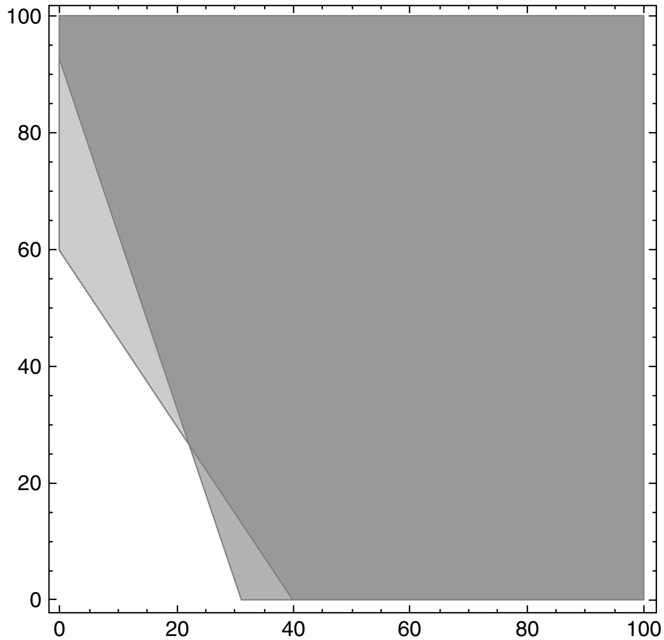

Figure 4.5

$$X \geq 0 \tag{12}$$

$$Y \geq 0. \tag{13}$$

Note that (10) models the minimum daily requirement of units of Vitamin A, while (11) refers to the minimum daily requirement of units of Vitamin B. The quantity to be minimized, the total number of pills, is given by the objective function:

$$N(X, Y) = X + Y. \tag{14}$$

We note that while this region of feasibility is convex, it is also *unbounded*. Our vertices are $(40, 0)$, $(0, 93)$, and $(22, 27)$.

In the next section we will solve problems such as these by applying a very simple, yet extremely powerful, theorem of *linear programming*.

Problems 4.2

Model the following situations by defining all variables and giving all inequalities, the objective function and the region of feasibility.

1. Farmer John gets \$5000 for every truck of wheat sold and \$6000 for every truck of corn sold. He has two fields: field A has 23 acres and field B has 17 acres.

For every 2 acres of field A, Farmer John produces a truck of wheat, while 3 acres are required of field B for the same amount of wheat. Regarding the corn, 3 acres of field A are required for a truck, while only 1 acre of field B is needed. How many trucks of each commodity should be produced to maximize Farmer John's profit?

2. Redo Problem (1) if Farmer John gets \$8000 for every truck of wheat and \$5000 for every truck of corn.

3. Dr. Lori Pesciotta, a research scientist, is experimenting with two forms of a special compound, *H-Turebab*. She needs at least 180 units of one form of the compound (α) and at least 240 units of the second form of the compound (β). Two mixtures are used: X and Y. Every unit of X contains two units of α and three units of β, while each unit of Y has the opposite concentration. What combination of X and Y will minimize Dr. Pesciotta's costs, if each unit of X costs \$500 and each unit of Y costs \$750?

4. Redo Problem (3) if X costs \$750 per unit and Y costs \$500 per unit.

5. Redo Problem (3) if, in addition, Dr. Pesciotta needs at least 210 units of a third form (γ) of *H-Turebab*, and it is known that every unit of both X and Y contains 10 units of γ.

6. Cereal X costs \$.05 per ounce while Cereal Y costs \$.04 per ounce. Every ounce of X contains 2 milligrams (mg) of Zinc and 1 mg of Calcium, while every ounce of Y contains 1 mg of Zinc and 4 mg of Calcium. The minimum daily requirement (MDR) is 10 mg of Zinc and 15 mg of Calcium. Find the least expensive combination of the cereals which would satisfy the MDR.

7. Redo Problem (6) with the added constraint of at least 12 mg of Sodium if each ounce of X contains 3 mg of Sodium and every ounce of Y has 2 mg of Sodium.

8. Redo Problem (7) if Cereal X costs \$.07 an ounce and Cereal Y costs \$.08 an ounce.

9. Consider the following group of inequalities along with a corresponding objective function. For each one, sketch the region of feasibility (except for 9 g) and construct a scenario that might model each set of inequalities:

(a) $x \geq 0, y \geq 0, 2x + 5y \leq 10, 3x + 4y \leq 12, F(x, y) = 100x + 55y$

(b) $x \geq 0, y \leq 0, x + y \leq 40, x + 2y \leq 60, G(x, y) = 7x + 6y$

(c) $x \geq 2, y \geq 3, x + y \leq 40, x + 2y \leq 60, H(x, y) = x + 3y$

(d) $x \geq 0, y \geq 0, x + y \leq 600, 3x + y \leq 900, x + 2y \leq 1000, J(x, y) = 10x + 4y$

(e) $2x + 9y \geq 1800, 3x + y \geq 750, K(x, y) = 4x + 11y$

(f) $x + y \geq 100, x + 3y \geq 270, 3x + y \geq 240, L(x, y) = 600x + 375y$

(g) $x \geq 0, y \geq 0, z \geq 0, x + y + 2z \leq 12, 2x + y + z \leq 14, x + 3y + z \leq 15,$

$M(x, y, z) = 2x + 3y + 4z$ (Do *not* sketch the region of feasibility for this problem.)

4.3 Solving Problems Using Linear Programming

We are now ready to solve a fairly large class of optimization problems using a special form of the *Fundamental Theorem of Linear Programming*. We will not prove this theorem, but many references to the proof of a more general result are available (for example, see Luenberger, D. G., *Linear and Nonlinear Programming*, 2nd Ed., Springer 2003).

The Fundamental Theorem of Linear Programming Let Γ be a convex region of feasibility in the xy-plane. Then the objective function $F(x, y) = ax + by$, where a and b are real numbers, takes on both maximum and minimum values—if they exist—on one or more vertices of Γ.

Remark 1 The theorem holds only *if* maximum and/or minimum values exist.

Remark 2 It is possible to have infinitely many values where an optimal (maximum or minimum) value exists. In this case, they would lie on one of the line segments that form the boundary of the region of feasibility. See Example 3 below.

Remark 3 The word *programming* has nothing to do with computer programming, but rather the systematic order followed by the procedure, which can also be termed an *algorithm*.

Some examples are in order.

Example 1 (Wagons): Consider the inequalities (5) through (8), along with Equation (9), from Section 4.2. We again give the region of feasibility below in Figure 4.6 (same as Figure 4.4):

Evaluation our objective function,

$$P(X, Y) = 50X + 60Y, \tag{15}$$

at each of the four vertices yields the following results:

$$\begin{cases} P(0,0) = 0 \\ P(0,600) = 36{,}000 \\ P(450,150) = 31{,}500 \\ P(500,0) = 25{,}000. \end{cases}$$

By the Fundamental Theorem of Linear Programming, we see that the maximum profit of \$36,000 occurs if no X wagons are produced and 600 Y wagons are made. ■

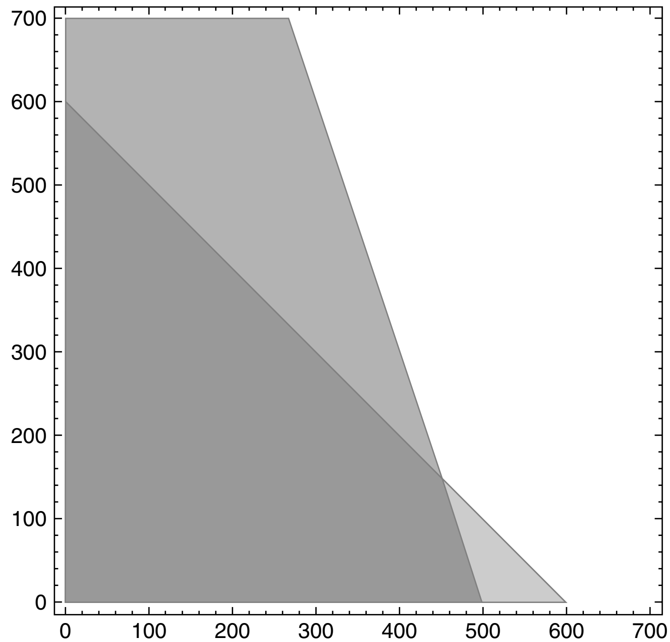

Figure 4.6

Example 2 (Wagons): Suppose the profit function in the previous example is given by

$$R(X, Y) = 80X + 50Y. \tag{16}$$

Then

$$\begin{cases} R(0,0) = 0 \\ R(0,600) = 30{,}000 \\ R(450,150) = 43{,}500 \\ R(500,0) = 40{,}000. \end{cases}$$

We see, in this situation, that the maximum profit of \$43,500 occurs if 450 X wagons are produced, along with 150 Y wagons. ■

Example 3 (Wagons): Consider the examples above with the profit function given by

$$L(X, Y) = 75X + 75Y. \tag{17}$$

Then

$$\begin{cases} L(0,0) = 0 \\ L(0,600) = 45{,}000 \\ L(450,150) = 45{,}000 \\ L(500,0) = 37{,}500. \end{cases}$$

Note that we have *two* situations in which the profit is maximized at \$45,000; in fact, there are *many* points where this occurs. For example,

$$L(300, 300) = 45{,}000. \tag{18}$$

This occurs *at any point* along the constraint given by inequality (2). The reason lies in the fact that *coefficients* of X and Y in (2) and in Equation (7) have the same ratio. ■

Example 4 (Vitamins): Consider constraints (10) through (13) above in Section 4.2; minimize the objective function given by Equation (14).

$$N(X, Y) = X + Y. \tag{19}$$

■

The region of feasibility (same as Figure 4.5) is given below in Figure 4.7: Evaluating our objective function (19) at the three vertices, we find that

$$\begin{cases} N(40, 0) = 40 \\ N(0, 93) = 93 \\ N(22, 27) = 49, \end{cases}$$

so the minimum number of pills needed to satisfy the minimum daily requirement is 40.

Sometimes a constraint is *redundant*; that is, the other constraints "include" the redundant constraint.

For example, suppose we want to maximize the objective function

$$Z(X, Y) = 4X + 3Y, \tag{20}$$

given the constraints

$$4X + 2Y \leq 40 \tag{21}$$

$$3X + 4Y \leq 60 \tag{22}$$

$$X \geq 0 \tag{23}$$

$$Y \geq 0. \tag{24}$$

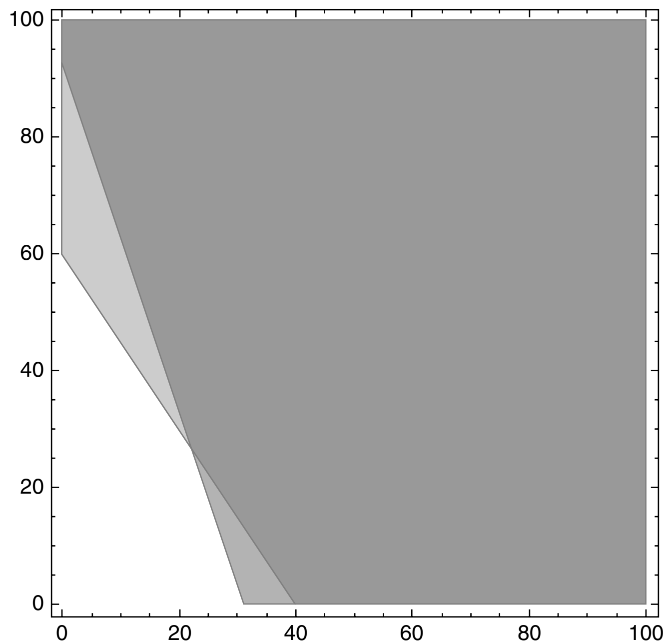

Figure 4.7

The vertices of the region of feasibility are *(0, 0), (0, 15), (4, 12),* and *(10, 0),* as seen below in Figure 4.8.

Note that (11) is maximized at $Z(4, 12) = 52$.

Suppose we now add a third constraint:

$$X + Y \leq 30. \tag{25}$$

Figure 4.9 below reflects this added condition. Note, however, that the region of feasibility is *not* changed and the four vertices are unaffected by this *redundant* constraint. It follows, therefore, that our objective function $Z(X, Y) = 4X + 3Y$ is still maximized at $Z(4, 12) = 52$.

Remark 4 Sometimes a vertex does not have whole number coordinates (see problem (15) below). If the physical model does not make sense to have a fractional or decimal answer—for example 2.5 bicycles or 1/3 cars—then we should check the closest points with whole number coordinates, *provided these points lie in the region of feasibility*. For example, if (2.3, 7.8) is the vertex which gives the optimal value for an objective function, then the following points should be checked: (2, 7), (2, 8), (3, 7) and (3, 8).

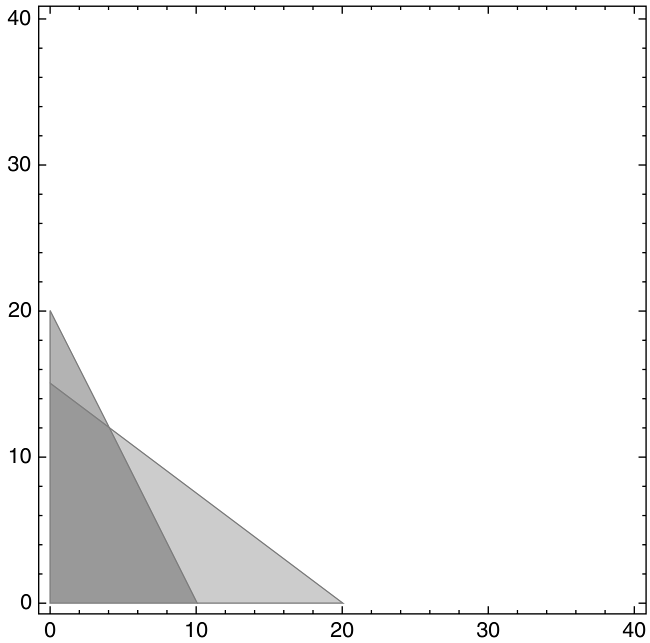

Figure 4.8

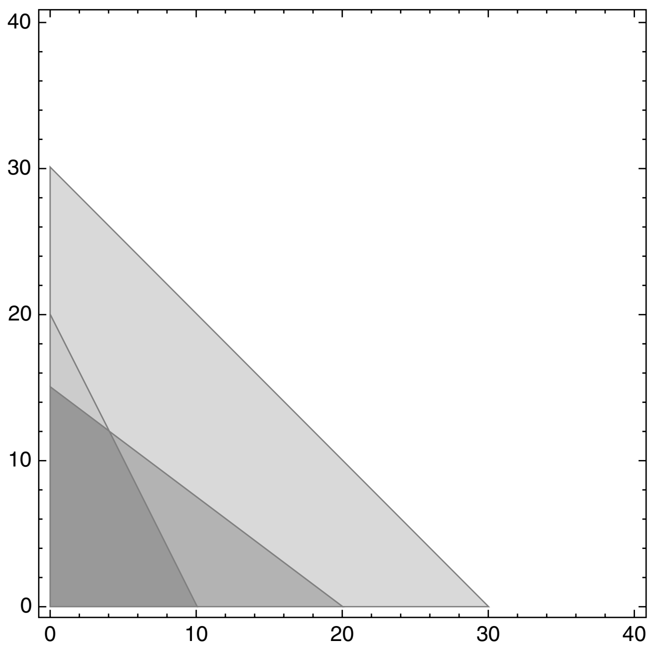

Figure 4.9

Problems 4.3

Using linear programming techniques, solve the following problems.

1. Section 4.2, Problem (1).
2. Section 4.2, Problem (2).
3. Section 4.2, Problem (3).
4. Section 4.2, Problem (4).
5. Section 4.2, Problem (5).
6. Section 4.2, Problem (6).
7. Section 4.2, Problem (7).
8. Section 4.2, Problem (8).
9. Section 4.2, Problem (9a); maximize $F(x, y)$.
10. Section 4.2, Problem (9b); maximize $G(x, y)$.
11. Section 4.2, Problem (9c); maximize $H(x, y)$.
12. Section 4.2, Problem (9d); maximize $J(x, y)$.
13. Section 4.2, Problem (9e); minimize $K(x, y)$.
14. Section 4.2, Problem (9f); minimize $L(x, y)$.
15. Maximize $P(x, y) = 7x + 6y$ subject to the constraints $x \geq 0$, $y \geq 0$, $2x + 3y \leq 1200$ and $6x + y \leq 1500$.

4.4 An Introduction to the Simplex Method

In most of the problems considered in the previous section, we had but two variables (usually X and Y) and two constraints, not counting the usual conditions of the non-negativity of X and Y. Once a third constraint is imposed, the region of feasibility becomes more complicated; and, with a fourth constraint, even more so.

Also, if a third variable, say Z, is brought into the discussion, then the region of feasibility becomes *three-dimensional*! This certainly makes the technique employed in the previous section much more difficult to apply, although theoretically it can be used.

We are fortunate that an alternate method exists which is valid for any number of variables and any number of constraints. It is known as the *Simplex Method*. This is a *classic* method that has been in use for *many* years. The reader may wish to consult G. Hadley's *Linear Programming* published by Addison-Wesley in 1963 for the theoretical underpinnings of this algorithm.

Before we illustrate this technique with a number of examples, describing and defining terms as we go along, we point out that this section will deal exclusively with *maximization* problems. We will address minimization in the next, and final, section of this chapter.

Example 1 Suppose we want to maximize the following function of two variables:

$$z = 7x_1 + 22x_2. \tag{26}$$

Note that we are using x_i instead of the usual x and y, due to the fact that, in later examples, we will have more than two independent variables.

Let us assume that the following constraints are imposed:

$$3x_1 + 10x_2 \leq 33{,}000 \tag{27}$$

$$5x_1 + 8x_2 \leq 42{,}000 \tag{28}$$

$$x_1 \geq 0 \tag{29}$$

$$x_2 \geq 0. \tag{30}$$

We now introduce the concept of *slack variables,* which we denote by s_i. These variables (which can never be negative) will "pick up the slack" in the relationships (27) and (28) and convert these inequalities into equations. That is, (27) and (28) can now be written respectively as:

$$3x_1 + 10x_2 + s_1 = 33{,}000 \tag{31}$$

and

$$5x_1 + 8x_2 + s_2 = 42{,}000. \tag{32}$$

We also incorporate these slack variables into our objective function (26), rewriting it as:

$$-7x_1 - 22x_2 + 0s_1 + 0s_2 + z = 0. \tag{33}$$

Finally, we rewrite (27) and (28) as

$$3x_1 + 10x_2 + s_1 + 0s_2 + 0z = 33{,}000 \tag{34}$$

$$5x_1 + 8x_2 + 0s_1 + 1s_2 + 0z = 42{,}000. \tag{35}$$

■

Remark 1 Admittedly, the Equations (33) through (35) seem somewhat strange. However, the reader will soon see why we have recast these equations as they now appear.

We are now ready to put these last three equations into a table known as the *initial tableau*. This is nothing more than a kind of augmented matrix. To do this, we merely "detach" the coefficients of the five unknowns ($x_1, x_2, s_1, s_2,$ and z) and form the following table:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 3 & 10 & 1 & 0 & 0 & 33{,}000 \\ 5 & 8 & 0 & 1 & 0 & 42{,}000 \\ \hline -7 & -22 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \tag{36}$$

Remark 2 Note that the objective function equation—here, Equation (33)—is in the bottom row. Also, unless otherwise stipulated, we shall always assume that the decision variables—that is, x_1 and x_2—are non-negative. Notice, too, the *vertical bar* that appears to the *left* of the rightmost column and the *horizontal bar* placed *above* the bottom row. Finally, we point out that the entry in the last row and last column is always zero for this initial tableau. These conventions will assist us in interpreting the end state of the Simplex Method.

Before continuing with the Simplex Method, let us consider another example.

Example 2 Put the following maximization problem into the initial tableau form: $z = 4x_1 + 7x_2 + 9x_3$, where $x_1 + x_2 + 6x_3 \leq 50, 2x_1 + 3x_2 \leq 40$, and $4x_1 + 9x_2 + 3x_3 \leq 10$.

Note that we have three independent (decision) variables (the x_i) and that the three constraints will give us three slack variables (the s_i). These lead us to the following four equations:

$$-4x_1 - 7x_2 - 9x_3 + 0s_1 + 0s_2 + 0s_3 + z = 0 \tag{37}$$

$$x_1 + x_2 + 6x_3 + s_1 + 0s_2 + 0s_3 + 0z = 50 \tag{38}$$

$$2x_1 + 0x_2 + 3x_3 + 0s_1 + s_2 + 0s_3 + 0z = 40 \tag{39}$$

$$4x_1 + 9x_2 + 3x_3 + 0s_1 + 0s_2 + s_3 + 0z = 10. \tag{40}$$

The initial tableau for this example is given below:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{rrrrrrr|r} 1 & 1 & 6 & 1 & 0 & 0 & 0 & 50 \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & 40 \\ 4 & 9 & 3 & 0 & 0 & 1 & 0 & 10 \\ \hline -4 & -7 & -9 & 0 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \tag{41}$$

■

We will now outline the steps in the Simplex Method:

- Change all inequalities into equations via the use of slack variables.
- Rewrite the objective function, z, in terms of slack variables, setting one side of the equation equal to zero and keeping the coefficient of z equal to $+1$.
- The number of equations should equal the *sum* of the constraints plus one (the equation given by the objective function).
- Form the initial tableau, listing the constraints above the objective function, labeling the columns, beginning with the decision variables, followed by the slack variables, with z represented by the last column before the vertical bar. The last column should have all the "constants."
- Locate the *most negative number* in the last row. If more than one equally negative number is present, arbitrarily choose any one of them. Call this number k. This column will be called the *work column*.
- Consider each *positive* element in the work column. Divide each of these elements into the corresponding row entry element in the last column. The ratio that is the *smallest* will be used as the work column's *pivot*. If there is more than one smallest ratio, arbitrarily choose any one of them.
- Use elementary row operations (see Chapter 2) to change the pivot element to 1, unless it is already 1.
- Use elementary row operations to transform all the other elements in the work column to 0.
- A column is *reduced* when all the elements are 0, with the exception of the pivot, which is 1.
- Repeat the process until there are *no negative elements in the last row*.
- We are then able to *determine the answers* from this final tableau.

Let us illustrate this by returning to Example 1, where the initial tableau is given by

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 10 & 1 & 0 & 0 & 33{,}000 \\ 5 & 8 & 0 & 1 & 0 & 42{,}000 \\ \hline -7 & -22 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \tag{42}$$

We first note that -22 is the *most negative number* in the last row of (42). So the "x_2" column is our *work column*.

We next divide 33,000 by $10 = 3300$ and 42,000 by $8 = 5250$; since 3300 is the *lesser positive number*, we will use 10 as the *pivot*. Note that we have put a carat

(∧) over the 10 to signify it is the pivot element.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 3 & \hat{10} & 1 & 0 & 0 & 33{,}000 \\ 5 & 8 & 0 & 1 & 0 & 42{,}000 \\ \hline -7 & -22 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \qquad (43)$$

We now divide every element in the row containing the pivot by 10.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 0.3 & \hat{1} & 0.1 & 0 & 0 & 3300 \\ 5 & 8 & 0 & 1 & 0 & 42{,}000 \\ \hline -7 & -22 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \qquad (44)$$

Next, we use elementary row operations; we multiply the first row by −8 and add it to the second row and multiply the first row by 22 and add it to the third row. This will give us a 0 for every element (other than the pivot) in the work column.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 0.3 & \hat{1} & 0.1 & 0 & 0 & 3300 \\ 2.6 & 0 & -0.8 & 1 & 0 & 15{,}600 \\ \hline -0.4 & 0 & 2.2 & 0 & 1 & 72{,}600 \end{array}\right]. \end{array} \qquad (45)$$

And now we repeat the process because we still have a negative entry in the last row; that is, −0.4 is in the "x_1" column. Hence, this becomes our new work column.

Dividing 3300 by 0.3 yields 11,000; dividing 15,600 by 2.6 gives us 6000; since 6000 is the lesser of the two positive ratios, we will use the 2.6 entry as the pivot (again denoting it with a carat, and removing the carat from our first pivot).

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 0.3 & 1 & 0.1 & 0 & 0 & 3300 \\ \hat{2.6} & 0 & -0.8 & 1 & 0 & 15{,}600 \\ \hline -0.4 & 0 & 2.2 & 0 & 1 & 72{,}600 \end{array}\right]. \end{array} \qquad (46)$$

Dividing each element in this row by 2.6 gives us the following tableau:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 0.3 & 1 & 0.1 & 0 & 0 & 3300 \\ \hat{1} & 0 & -.31 & .38 & 0 & 6000 \\ \hline -0.4 & 0 & 2.2 & 0 & 1 & 72{,}600 \end{array}\right]. \end{array} \qquad (47)$$

Using our pivot and elementary row operations, we transform every other element in this work column to 0. That is, we multiply each element in the second row by

−0.3 and add the row to the first row and we multiply every element in the second row by 0.4 and add the row to the last row. This gives us the following tableau:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 0.19 & -0.12 & 0 & 1500 \\ 1 & 0 & -.31 & .38 & 0 & 6000 \\ \hline 0 & 0 & 2.08 & 0.15 & 1 & 75{,}000 \end{array}\right]. \end{array} \tag{48}$$

We are now *finished* with the process, because there are no negative elements in the last row. We *interpret* this final tableau as follows:

- $x_1 = 6000$ (note the "1" in the x_1 column and the "0" in the x_2 column).
- $x_2 = 1500$ (note the "0" in the x_1 column and the "1" in the x_2 column).
- Both slack variables equal 0. To verify this, please see Equations (31) and (32) and substitute our values for x_1 and x_2 into these equations.
- The maximum value of z is 75,000 (found in the lower right-hand corner box).

We now give another example.

Example 3 Maximize $z = x_1 + 2x_2$, subject to the constraints $4x_1 + 2x_2 \leq 40$ and $3x_1 + 4x_2 \leq 60$.

Following the practice discussed in this section and introducing the slack variables, we have:

$$4x_1 + 2x_2 + s_1 = 40 \tag{49}$$

$$3x_1 + 4x_2 + s_2 = 60 \tag{50}$$

and

$$-x_1 - 2x_2 + z = 0. \tag{51}$$

We form the initial tableau, using coefficients of 0 where needed, as follows:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 4 & 2 & 1 & 0 & 0 & 40 \\ 3 & 4 & 0 & 1 & 0 & 60 \\ \hline -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \tag{52}$$

The second column will be our work column, since −2 is the most negative entry. Dividing 40 by 2 gives 20; dividing 60 by 4 yields 15. Since 15 is a lesser positive ratio than 20, we will use the 4 as the pivot:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 4 & 2 & 1 & 0 & 0 & 40 \\ 3 & \hat{4} & 0 & 1 & 0 & 60 \\ \hline -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \tag{53}$$

Dividing every element of the second row will make our pivoting element 1:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 4 & 2 & 1 & 0 & 0 & 40 \\ 0.75 & \hat{1} & 0 & 0.25 & 0 & 15 \\ \hline -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array} \tag{54}$$

We now use our pivot, along with the proper elementary row operations, to make every other element in the column zero. This leads to the following tableau:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 2.5 & 0 & 1 & -0.5 & 0 & 10 \\ 0.75 & \hat{1} & 0 & 0.25 & 0 & 15 \\ \hline 0.5 & 0 & 0 & 0.5 & 1 & 30 \end{array}\right]. \end{array} \tag{55}$$

Since the last row has no negative entries, we are finished and have the final tableau:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 2.5 & 0 & 1 & -0.5 & 0 & 10 \\ 0.75 & 1 & 0 & 0.25 & 0 & 15 \\ \hline 0.5 & 0 & 0 & 0.5 & 1 & 30 \end{array}\right]. \end{array} \tag{56}$$

This final tableau is a little more complicated to interpret than (48).

First notice the "1" in the second row; this implies that $x_2 = 15$. The corresponding equation represented by this second row thereby reduces to

$$0.75x_1 + 15 + 0.25s_2 = 15. \tag{57}$$

Which forces both x_1 and s_2 to be zero, since neither can be negative. This forces $s_1 = 10$, as we can infer from the equation represented by the first row:

$$0.25x_1 + s_1 - 0.5s_2 = 10. \tag{58}$$

In practice, we are not concerned with the values of the slack variables, so we summarize by simply saying that our answers are $x_1 = 0$ and $x_2 = 15$ with a maximum value of $z = 30$. ■

As we have pointed out, this is a *classic* technique. However, as the number of variables (decision and/or slack) increases, the calculations can be somewhat burdensome. Thankfully, there are many software packages to assist in this matter. Please refer to the Final Comments at the end of this chapter.

One final remark: As is the case with linear programming, if there are an infinite number of optimal solutions, the Simplex Method does not give *all* solutions.

Problems 4.4

Using the Simplex Method, solve the following problems:

1. Section 4.2, Problem (1).
2. Section 4.2, Problem (2).
3. Maximize $z = 3x_1 + 5x_2$, subject to $x_1 + x_2 \leq 6$ and $2x_1 + x_2 \leq 8$.
4. Maximize $z = 8x_1 + x_2$, subject to the same constraints in (3).
5. Maximize $z = x_1 + 12x_2$, subject to the same constraints in (3).
6. Maximize $z = 3x_1 + 6x_2$, subject to the constraints $x_1 + 3x_2 \leq 30$, $2x_1 + 2x_2 \leq 40$, and $3x_1 + x_2 \leq 30$.
7. Consider problem (9) at the end of Section 4.2. Set up the initial tableaus for problems (9a) through (9d).

4.5 Final Comments on Chapter 4

In this chapter we covered two approaches to optimization, the Linear Programming Method and the Simplex Method. Both of these techniques are classical and their geometrics and algebraic simplicity reflect both the beauty and power of mathematics.

Our goal was to introduce the reader to the basics of these "simple" methods. However, he or she should be cautioned with regard to the underlying theory. That is, many times in mathematics we have elegant results (theorems) which are proved using very *deep* and *subtle* mathematical concepts with respect to the proofs of these theorems.

As we mentioned in the last section, the calculations, while not difficult, can be a burden. Calculators and software packages can be of great assistance here.

We close with two observations. Please note that we have considered very special cases where the constraints of the "$\leq$" variety had *positive* quantities on the right-hand side. If this is not the case for all the constraints, then we must use an *enhanced* version of the Simplex Method (see, for example, *Finite Mathematics: A Modeling Approach* by R. Bronson and G. Bronson published by West in 1996).

Similarly, regarding the solving of *minimization* problems via the Simplex Method, we essentially consider the "negation" of the objective function, and then apply a modified version of the Simplex Method. For example, suppose we wanted to minimize $z = 3x_1 + 2x_2$, subject to the same constraints. In this case we would *maximize* $Z = -z = -3x_1 - 2x_2$ while recasting our constraints, and then proceed with the Simplex Method.