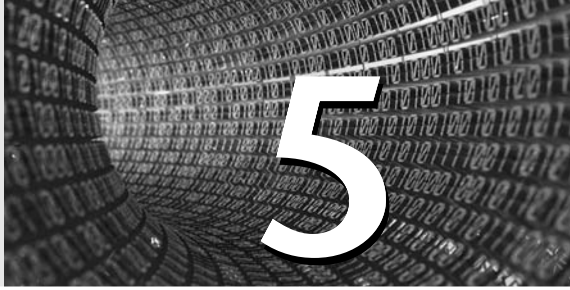

Determinants

5.1 Introduction

Every square matrix has associated with it a scalar called its *determinant*. To be extremely rigorous we would have to define this scalar in terms of permutations on positive integers. However, since in practice it is difficult to apply a definition of this sort, other procedures have been developed which yield the determinant in a more straightforward manner. In this chapter, therefore, we concern ourselves solely with those methods that can be applied easily. We note here for reference that determinants are only defined for square matrices.

Given a square matrix $\mathbf{A}$, we use $\det(\mathbf{A})$ or $|\mathbf{A}|$ to designate its determinant. If the matrix can actually be exhibited, we then designate the determinant of $\mathbf{A}$ by replacing the brackets by vertical straight lines. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1}$$

then

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}. \tag{2}$$

We cannot overemphasize the fact that (1) and (2) represent entirely different animals. (1) represents a matrix, a rectangular array, an entity unto itself while (2) represents a scalar, a number associated with the matrix in (1). There is absolutely no similarity between the two other than form!

We are now ready to calculate determinants.

Definition 1 The determinant of a 1×1 matrix $[a]$ is the scalar a.

Thus, the determinant of the matrix $[5]$ is 5 and the determinant of the matrix $[-3]$ is -3.

Definition 2 The determinant of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the scalar $ad - bc$.

Example 1 Find $\det(\mathbf{A})$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = (1)(3) - (2)(4) = 3 - 8 = -5. \quad \blacksquare$$

Example 2 Find $|\mathbf{A}|$ if

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}.$$

Solution

$$|\mathbf{A}| = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (2)(3) - (-1)(4) = 6 + 4 = 10. \quad \blacksquare$$

We now could proceed to give separate rules which would enable one to compute determinants of 3×3, 4×4, and higher order matrices. This is unnecessary. In the next section, we will give a method that enables us to reduce all determinants of order $n (n > 2)$ (if $\mathbf{A}$ has order $n \times n$ then $\det(\mathbf{A})$ is said to have order n) to a sum of determinants of order 2.

Problems 5.1

In Problems 1 through 18, find the determinants of the given matrices.

1. $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, **2.** $\begin{bmatrix} 3 & -4 \\ 5 & 6 \end{bmatrix}$, **3.** $\begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$,

4. $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, **5.** $\begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix}$, **6.** $\begin{bmatrix} 5 & 6 \\ 7 & -8 \end{bmatrix}$,

7. $\begin{bmatrix} 1 & -1 \\ 2 & 7 \end{bmatrix}$, **8.** $\begin{bmatrix} -2 & -3 \\ -4 & 4 \end{bmatrix}$, **9.** $\begin{bmatrix} 3 & -1 \\ -3 & 8 \end{bmatrix}$,

10. $\begin{bmatrix} 0 & 1 \\ -2 & 6 \end{bmatrix}$, **11.** $\begin{bmatrix} -2 & 3 \\ -4 & -4 \end{bmatrix}$, **12.** $\begin{bmatrix} 9 & 0 \\ 2 & 0 \end{bmatrix}$,

13. $\begin{bmatrix} 12 & 20 \\ -3 & -5 \end{bmatrix}$, **14.** $\begin{bmatrix} -36 & -3 \\ -12 & -1 \end{bmatrix}$, **15.** $\begin{bmatrix} -8 & -3 \\ -7 & 9 \end{bmatrix}$,

16. $\begin{bmatrix} t & 2 \\ 3 & 4 \end{bmatrix}$, **17.** $\begin{bmatrix} 2t & 3 \\ -2 & t \end{bmatrix}$, **18.** $\begin{bmatrix} 3t & -t^2 \\ 2 & t \end{bmatrix}$.

19. Find t so that

$$\begin{vmatrix} t & 2t \\ 1 & t \end{vmatrix} = 0.$$

20. Find t so that

$$\begin{vmatrix} t-2 & t \\ 3 & t+2 \end{vmatrix} = 0.$$

21. Find λ so that

$$\begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = 0.$$

22. Find λ so that

$$\begin{vmatrix} 1-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix} = 0.$$

23. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 1.

24. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 2.

25. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 4.

26. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 7.

27. Find $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{AB}|$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}.$$

What is the relationship between these three determinants?

28. Interchange the rows for each of the matrices given in Problems 1 through 15, and calculate the new determinants. How do they compare with the determinants of the original matrices?

29. The second elementary row operation is to multiply any row of a matrix by a nonzero constant. Apply this operation to the matrices given in Problems 1 through 15 for any constants of your choice, and calculate the new determinants. How do they compare with the determinants of the original matrix?

30. Redo Problem 29 for the third elementary row operation.

31. What is the determinant of a 2×2 matrix if one row or one column contains only zero entries?

32. What is the relationship between the determinant of a 2×2 matrix and its transpose?

33. What is the determinant of a 2×2 matrix if one row is a linear combination of the other row?

5.2 Expansion by Cofactors

Definition 1 Given a matrix $\mathbf{A}$, a *minor* is the determinant of any square submatrix of $\mathbf{A}$.

That is, given a square matrix $\mathbf{A}$, a minor is the determinant of any matrix formed from $\mathbf{A}$ by the removal of an equal number of rows and columns. As an example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

then

$$\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

are both minors because

$$\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

are both submatrices of $\mathbf{A}$, while

$$\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 2 \end{vmatrix}$$

are not minors because

$$\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$$

is not a submatrix of $\mathbf{A}$ and $[1 \quad 2]$, although a submatrix of $\mathbf{A}$, is not square.

A more useful concept for our immediate purposes, since it will enable us to calculate determinants, is that of the cofactor of an element of a matrix.

Definition 2 Given a matrix $\mathbf{A} = [a_{ij}]$, the *cofactor of the element* a_{ij} is a scalar obtained by multiplying together the term $(-1)^{i+j}$ and the minor obtained from $\mathbf{A}$ by removing the ith row and jth column.

In other words, to compute the cofactor of the element a_{ij} we first form a submatrix of $\mathbf{A}$ by crossing out both the row and column in which the element a_{ij} appears. We then find the determinant of the submatrix and finally multiply it by the number $(-1)^{i+j}$.

Example 1 Find the cofactor of the element 4 in the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution We first note that 4 appears in the (2, 1) position. The submatrix obtained by crossing out the second row and first column is

$$\begin{bmatrix} \not{1} & 2 & 3 \\ \not{4} & \not{5} & \not{6} \\ \not{7} & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix},$$

which has a determinant equal to $(2)(9) - (3)(8) = -6$. Since 4 appears in the $(2, 1)$ position, $i = 2$ and $j = 1$. Thus, $(-1)^{i+j} = (-1)^{2+1} = (-1)^3 = (-1)$. The cofactor of 4 is $(-1)(-6) = 6$. ■

Example 2 Using the same A as in Example 1, find the cofactor of the element 9.

Solution The element 9 appears in the (3, 3) position. Thus, crossing out the third row and third column, we obtain the submatrix

$$\begin{bmatrix} 1 & 2 & \not{3} \\ 4 & 5 & \not{6} \\ \not{7} & \not{8} & \not{9} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}.$$

which has a determinant equal to $(1)(5) - (2)(4) = -3$. Since, in this case, $i = j = 3$, the cofactor of 9 is $(-1)^{3+3}(-3) = (-1)^6(-3) = -3$. ■

We now have enough tools at hand to find the determinant of any matrix.

Expansion by Cofactors. To find the determinant of a matrix $\mathbf{A}$ of arbitrary order, (a) pick any one row or any one column of the matrix (dealer's choice), (b) for each element in the row or column chosen, find its cofactor, (c) multiply each element in the row or column chosen by its cofactor and sum the results. This sum is the determinant of the matrix.

Example 3 Find det(**A**) if

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & 1 \\ 3 & -6 & 4 \end{bmatrix}.$$

Solution In this example, we expand by the second column.

$$\begin{aligned} |\mathbf{A}| &= (5)(\text{cofactor of } 5) + (2)(\text{cofactor of } 2) + (-6)(\text{cofactor of } -6) \\ &= (5)(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} 3 & 0 \\ 3 & 4 \end{vmatrix} + (-6)(-1)^{3+2}\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} \\ &= 5(-1)(-4-3) + (2)(1)(12-0) + (-6)(-1)(3-0) \\ &= (-5)(-7) + (2)(12) + (6)(3) = 35 + 24 + 18 = 77. \quad \blacksquare \end{aligned}$$

Example 4 Using the **A** of Example 3 and expanding by the first row, find det(**A**).

Solution

$$\begin{aligned} |\mathbf{A}| &= 3(\text{cofactor of } 3) + 5(\text{cofactor of } 5) + 0(\text{cofactor of } 0) \\ &= (3)(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + 0 \\ &= (3)(1)(8+6) + (5)(-1)(-4-3) \\ &= (3)(14) + (-5)(-7) = 42 + 35 = 77. \quad \blacksquare \end{aligned}$$

The previous examples illustrate two important properties of the method. First, the value of the determinant is the same regardless of which row or column we choose to expand by and second, expanding by a row or column that contains zeros significantly reduces the number of computations involved.

Example 5 Find det(**A**) if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -1 & 4 & 1 & 0 \\ 3 & 0 & 4 & 1 \\ -2 & 1 & 1 & 3 \end{bmatrix}.$$

Solution We first check to see which row or column contains the most zeros and expand by it. Thus, expanding by the second column gives

$$
\begin{aligned}
|\mathbf{A}| &= 0(\text{cofactor of } 0) + 4(\text{cofactor of } 4) + 0(\text{cofactor of } 0) + 1(\text{cofactor of } 1) \\
&= 0 + 4(-1)^{2+2}\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} + 0 + 1(-1)^{4+2}\begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} \\
&= 4\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix}.
\end{aligned}
$$

Using expansion by cofactors on each of the determinants of order 3 yields

$$
\begin{aligned}
\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} &= 1(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} \\
&= -22 \quad \text{(expanding by the first row)}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} &= 2(-1)^{1+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + 0 + 1(-1)^{3+3}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} \\
&= -8 \quad \text{(expanding by the third column).}
\end{aligned}
$$

Hence,

$$|\mathbf{A}| = 4(-22) - 8 = -88 - 8 = -96. \quad \blacksquare$$

For $n \times n$ matrices with $n > 3$, expansion by cofactors is an inefficient procedure for calculating determinants. It simply takes too long. A more elegant method, based on elementary row operations, is given in Section 5.4 for matrices whose elements are all numbers.

Problems 5.2

In Problems 1 through 22, use expansion by cofactors to evaluate the determinants of the given matrices.

1. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{bmatrix}$, **2.** $\begin{bmatrix} 3 & 2 & -2 \\ 1 & 0 & 4 \\ 2 & 0 & -3 \end{bmatrix}$, **3.** $\begin{bmatrix} 1 & -2 & -2 \\ 7 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$,

4. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$, **5.** $\begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & 4 \\ -2 & 2 & 7 \end{bmatrix}$, **6.** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 3 \\ 4 & 5 & 0 \end{bmatrix}$,

7. $\begin{bmatrix} 2 & 1 & -9 \\ 3 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, **8.** $\begin{bmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{bmatrix}$, **9.** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 4 \\ 3 & 5 & 1 \end{bmatrix}$,

10. $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$, **11.** $\begin{bmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{bmatrix}$, **12.** $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 5 & 1 \\ 2 & -5 & -1 \end{bmatrix}$,

13. $\begin{bmatrix} -4 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix}$, **14.** $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 1 \\ 5 & 3 & 8 \end{bmatrix}$, **15.** $\begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$,

16. $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 1 & -5 & 0 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & -2 & 1 \end{bmatrix}$, **17.** $\begin{bmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{bmatrix}$,

18. $\begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & 5 & 2 & -1 \\ -2 & -2 & 1 & 3 \\ -3 & 4 & -1 & 8 \end{bmatrix}$, **19.** $\begin{bmatrix} -1 & 3 & 2 & -2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{bmatrix}$,

20. $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ -3 & 4 & 0 & 8 \end{bmatrix}$, **21.** $\begin{bmatrix} 1 & 2 & 1 & -1 \\ 4 & 0 & 3 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & -2 & 1 & 1 \end{bmatrix}$,

22. $\begin{bmatrix} 11 & 1 & 0 & 9 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$.

23. Use the results of Problems 1, 13, and 16 to develop a theorem about the determinants of triangular matrices.

24. Use the results of Problems 3, 20, and 22 to develop a theorem regarding determinants of matrices containing a zero row or column.

25. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 2.

26. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 3.

27. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 4.

28. Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 5.

5.3 Properties of Determinants

In this section, we list some useful properties of determinants. For the sake of expediency, we only give proofs for determinants of order three, keeping in mind that these proofs may be extended in a straightforward manner to determinants of higher order.

Property 1 *If one row of a matrix consists entirely of zeros, then the determinant is zero.*

Proof. Expanding by the zero row, we immediately obtain the desired result. □

Property 2 *If two rows of a matrix are interchanged, the determinant changes sign.*

Proof. Consider

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Expanding by the third row, we obtain

$$|\mathbf{A}| = a_{31}(a_{12}\,a_{23} - a_{13}\,a_{22}) - a_{32}(a_{11}\,a_{23} - a_{13}\,a_{21}) \\ + a_{33}(a_{11}\,a_{22} - a_{12}\,a_{21}).$$

Now consider the matrix **B** obtained from **A** by interchanging the second and third rows:

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Expanding by the second row, we find that

$$|\mathbf{B}| = -a_{31}(a_{12}\,a_{23} - a_{13}\,a_{22}) + a_{32}(a_{11}\,a_{23} - a_{13}\,a_{21}) \\ - a_{33}(a_{11}\,a_{22} - a_{12}\,a_{21}).$$

Thus, $|\mathbf{B}| = -|\mathbf{A}|$. Through similar reasoning, one can demonstrate that the result is valid regardless of which two rows are interchanged. □

Property 3 *If two rows of a determinant are identical, the determinant is zero.*

Proof. If we interchange the two identical rows of the matrix, the matrix remains unaltered; hence the determinant of the matrix remains constant. From Property 2, however, by interchanging two rows of a matrix, we change the sign of the determinant. Thus, the determinant must on one hand remain the same while on the other hand change the sign. The only way both of these conditions can be met simultaneously is for the determinant to be zero. □

Property 4 *If the matrix* **B** *is obtained from the matrix* **A** *by multiplying every element in one row of* **A** *by the scalar* λ, *then* $|\mathbf{B}| = \lambda|\mathbf{A}|$.

Proof.

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \lambda a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \lambda a_{13}\begin{vmatrix} a_{12} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \lambda\left(a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

$$= \lambda\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

In essence, Property 4 shows us how to multiply a scalar times a determinant. We know from Chapter 1 that multiplying a scalar times a matrix simply multiplies every element of the matrix by that scalar. Property 4, however, implies that multiplying a scalar times a determinant simply multiplies *one* row of the determinant by the scalar. Thus, while in matrices

$$8\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 24 & 32 \end{bmatrix},$$

in determinants we have

$$8\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 24 & 32 \end{vmatrix},$$

or alternatively

$$8\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4(2)\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4\begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 12 & 16 \end{vmatrix}.$$

□

Property 5 *For an* $n \times n$ *matrix* **A** *and any scalar* λ, $\det(\lambda\mathbf{A}) = \lambda^n \det(\mathbf{A})$.

Proof. This proof makes continued use of Property 4.

$$\det(\lambda\mathbf{A}) = \det\left\{\lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix}\right\}$$

$$= \begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{vmatrix} = \lambda\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{vmatrix}$$

$$= (\lambda)(\lambda)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{vmatrix} = \lambda(\lambda)(\lambda)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \lambda^3 \det(\mathbf{A}).$$

Note that for a 3×3 matrix, $n = 3$. □

Property 6 *If a matrix* $\mathbf{B}$ *is obtained from a matrix* $\mathbf{A}$ *by adding to one row of* $\mathbf{A}$*, a scalar times another row of* $\mathbf{A}$*, then* $|\mathbf{A}| = |\mathbf{B}|$.

Proof. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + \lambda a_{11} & a_{32} + \lambda a_{12} & a_{33} + \lambda a_{13} \end{bmatrix},$$

where $\mathbf{B}$ has been obtained from $\mathbf{A}$ by adding λ times the first row of $\mathbf{A}$ to the third row of $\mathbf{A}$. Expanding $|\mathbf{B}|$ by its third row, we obtain

$$|\mathbf{B}| = (a_{31} + \lambda a_{11})\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - (a_{32} + \lambda a_{12})\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$+ (a_{33} + \lambda a_{13})\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$+ \lambda\left\{a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\}.$$

The first three terms of this sum are exactly $|\mathbf{A}|$ (expand $|\mathbf{A}|$ by its third row), while the last three terms of the sum are

$$\lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

(expand this determinant by its third row). Thus, it follows that

$$|\mathbf{B}| = |\mathbf{A}| + \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}.$$

From Property 3, however, this second determinant is zero since its first and third rows are identical, hence $|\mathbf{B}| = |\mathbf{A}|$.

The same type of argument will quickly show that this result is valid regardless of the two rows chosen. □

Example 1 Without expanding, show that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{vmatrix} a-r & b-s & c-t \\ r+2x & s+2y & t+2z \\ x & y & z \end{vmatrix}.$$

Solution Using Property 6, we have that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{vmatrix} a-r & b-s & c-t \\ r & s & t \\ x & y & z \end{vmatrix}, \quad \left\{\begin{array}{l}\text{by adding to the first} \\ \text{row } (-1) \text{ times the} \\ \text{second row}\end{array}\right.$$

$$= \begin{vmatrix} a-r & b-s & c-t \\ r+2x & s+2y & t+2z \\ x & y & z \end{vmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } (2) \text{ times the} \\ \text{third row}\end{array}\right.$$ ■

Property 7 $\det(\mathbf{A}) = \det(\mathbf{A}^\mathsf{T})$.

Proof. If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^\mathsf{T} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Expanding $\det(\mathbf{A}^{\mathsf{T}})$ by the first column, it follows that

$$\begin{aligned}\left|\mathbf{A}^{\mathsf{T}}\right| &= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32}\end{vmatrix} \\ &= a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}(a_{21}a_{33} - a_{31}\,a_{23}) + a_{13}(a_{21}\,a_{32} - a_{31}\,a_{22}).\end{aligned}$$

This, however, is exactly the expression we would obtain if we expand $\det(\mathbf{A})$ by the first row. Thus $\left|\mathbf{A}^{\mathsf{T}}\right| = |\mathbf{A}|$. □

It follows from Property 7 that any property about determinants dealing with row operations is equally true for column operations (the analogous elementary row operation applied to columns), because a row operation on $\mathbf{A}^{\mathsf{T}}$ is the same as a column operation on $\mathbf{A}$. Thus, if one column of a matrix consists entirely of zeros, then its determinant is zero; if two columns of a matrix are interchanged, the determinant changes the sign; if two columns of a matrix are identical, its determinant is zero; multiplying a determinant by a scalar is equivalent to multiplying one column of the matrix by that scalar and then calculating the new determinant; and the third elementary column operation when applied to a matrix does not change its determinant.

Property 8 *The determinant of a triangular matrix, either upper or lower, is the product of the elements on the main diagonal.*

Proof. See Problem 2. □

Property 9 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are of the same order, then* $\det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{AB})$.

Because of its difficulty, the proof of Property 9 is omitted here.

Example 2 Show that Property 9 is valid for

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -1 \\ 7 & 4 \end{bmatrix}.$$

Solution $|\mathbf{A}| = 5$, $|\mathbf{B}| = 31$.

$$\mathbf{AB} = \begin{bmatrix} 33 & 10 \\ 34 & 15 \end{bmatrix} \quad \text{thus} \quad |\mathbf{AB}| = 155 = |\mathbf{A}||\mathbf{B}|. \quad \blacksquare$$

Problems 5.3

1. Prove that the determinant of a diagonal matrix is the product of the elements on the main diagonal.

2. Prove that the determinant of an upper or lower triangular matrix is the product of the elements on the main diagonal.

3. Without expanding, show that

$$\begin{vmatrix} a+x & r-x & x \\ b+y & s-y & y \\ c+z & t-z & z \end{vmatrix} = \begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}.$$

4. Verify Property 5 for $\lambda = -3$ and

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 5 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix}.$$

5. Verify Property 9 for

$$\mathbf{A} = \begin{vmatrix} 6 & 1 \\ 1 & 2 \end{vmatrix} \quad \text{and} \quad \mathbf{B} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}.$$

6. Without expanding, show that

$$\begin{vmatrix} 2a & 3r & x \\ 4b & 6s & 2y \\ -2c & -3t & -z \end{vmatrix} = -12\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}.$$

7. Without expanding, show that

$$\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix} = 5\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}.$$

8. Without expanding, show that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -\begin{vmatrix} a & x & r \\ b & y & s \\ c & z & t \end{vmatrix}.$$

9. Without expanding, show that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -\frac{1}{4}\begin{vmatrix} 2a & 4b & 2c \\ -r & -2s & -t \\ x & 2y & z \end{vmatrix}.$$

10. Without expanding, show that

$$\begin{vmatrix} a-3x & b-3y & c-3z \\ a+5x & b+5y & c+5z \\ x & y & z \end{vmatrix} = 0.$$

11. Without expanding, show that

$$\begin{vmatrix} 2a & 3a & c \\ 2r & 3r & t \\ 2x & 3x & z \end{vmatrix} = 0.$$

12. Prove that if one column of a square matrix is a linear combination of another column, then the determinant of that matrix is zero.

13. Prove that if $\mathbf{A}$ is invertible, then $\det\left(\mathbf{A}^{-1}\right) = 1/\det(\mathbf{A})$.

5.4 Pivotal Condensation

Properties 2, 4, and 6 of the previous section describe the effects on the determinant of a matrix of applying elementary row operations to the matrix itself. They comprise part of an efficient algorithm for calculating determinants of matrices whose elements are numbers. The technique is known as *pivotal condensation*: A given matrix is transformed into row-reduced form using elementary row operations. A record is kept of the changes to the determinant as a result of Properties 2, 4, and 6. Once the transformation is complete, the row-reduced matrix is in upper triangular form, and its determinant is found easily by Property 8. In fact, since a row-reduced matrix has either unity elements or zeros on its main diagonal, its determinant will be unity if all its diagonal elements are unity, or zero if any one diagonal element is zero.

Example 1 Use pivotal condensation to evaluate

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & -1 & 1 \end{vmatrix}.$$

Solution

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 8 \\ 3 & -1 & 1 \end{vmatrix} \quad \left\{\begin{array}{l}\text{Property 6: adding to} \\ \text{the second row (2)} \\ \text{times the first row}\end{array}\right.$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 8 \\ 0 & -7 & -8 \end{vmatrix} \quad \left\{\begin{array}{l}\text{Property 6: adding to} \\ \text{the third row (−3)} \\ \text{times the first row}\end{array}\right.$$

$$= 7\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{8}{7} \\ 0 & -7 & -8 \end{vmatrix} \quad \left\{\begin{array}{l}\text{Property 4: applied} \\ \text{to the second row}\end{array}\right.$$

$$= 7\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 0 \end{vmatrix} \quad \begin{cases} \text{Property 6: adding} \\ \text{to the third row (7)} \\ \text{times the second row} \end{cases}$$

$$= 7(0) = 0. \quad \{\text{Property 8} \quad \blacksquare$$

Example 2 Use pivotal condensation to evaluate

$$\begin{vmatrix} 0 & -1 & 4 \\ 1 & -5 & 1 \\ -6 & 2 & -3 \end{vmatrix}.$$

Solution

$$\begin{vmatrix} 0 & -1 & 4 \\ 1 & -5 & 1 \\ -6 & 2 & -3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -5 & 1 \\ 0 & -1 & 4 \\ -6 & 2 & -3 \end{vmatrix} \quad \begin{cases} \text{Property 2: interchanging} \\ \text{the first and second rows} \end{cases}$$

$$= (-1)\begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & 4 \\ 0 & -28 & 3 \end{vmatrix} \quad \begin{cases} \text{Property 6: adding} \\ \text{to the third row (6)} \\ \text{times the first row} \end{cases}$$

$$= (-1)(-1)\begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & -4 \\ 0 & -28 & 3 \end{vmatrix} \quad \begin{cases} \text{Property 4: applied} \\ \text{to the second row} \end{cases}$$

$$= \begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -109 \end{vmatrix} \quad \begin{cases} \text{Property 6: adding} \\ \text{to the third row (28)} \\ \text{times the second row} \end{cases}$$

$$= (-109)\begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{vmatrix} \quad \begin{cases} \text{Property 4: applied} \\ \text{to the third row} \end{cases}$$

$$= (-109)(1) = -109. \quad \{\text{Property 8} \quad \blacksquare$$

Pivotal condensation is easily coded for implementation on a computer. Although shortcuts can be had by creative individuals evaluating determinants by hand, this rarely happens. The orders of most matrices that occur in practice are too large and, therefore, too time consuming to consider hand calculations in the evaluation of their determinants. In fact, such determinants can bring computer algorithms to their knees. As a result, calculating determinants is avoided whenever possible.

Still, when determinants are evaluated by hand, appropriate shortcuts are taken, as illustrated in the next two examples. The general approach involves operating on a matrix so that one row or one column is transformed into a new row or column containing at most one nonzero element. Expansion by cofactors is then applied to that row or column.

Example 3 Evaluate

$$\begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix}.$$

Solution

$$\begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix} = \begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ 0 & 3 & 3 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding (1) times the} \\ \text{first row to the third row} \\ \text{(Property 6)}\end{array}\right.$$

$$= \begin{vmatrix} 10 & -6 & -3 \\ 6 & -5 & -2 \\ 0 & 3 & 0 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } (-1) \text{ times the} \\ \text{second column to the} \\ \text{third column (Property 6)}\end{array}\right.$$

$$= -3\begin{vmatrix} 10 & -3 \\ 6 & -2 \end{vmatrix} \quad \{\text{by expansion by cofactors}$$

$$= -3(-20 + 18) = 6. \quad \blacksquare$$

Example 4 Evaluate

$$\begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 1 & 4 & 1 \\ 3 & -2 & 3 & 5 \\ 9 & 7 & 0 & 2 \end{vmatrix}.$$

Solution Since the third column already contains two zeros, it would seem advisable to work on that one.

$$\begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 1 & 4 & 1 \\ 3 & -2 & 3 & 5 \\ 9 & 7 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 1 & 4 & 1 \\ 3 & -\frac{11}{4} & 0 & \frac{17}{4} \\ 9 & 7 & 0 & 2 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } \left(-\frac{3}{4}\right) \text{ times} \\ \text{the second row to} \\ \text{the third row.}\end{array}\right.$$

$$= -4\begin{vmatrix} 3 & -1 & 2 \\ 3 & -\frac{11}{4} & \frac{17}{4} \\ 9 & 7 & 2 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by expansion} \\ \text{by cofactors}\end{array}\right.$$

$$= -4\left(\frac{1}{4}\right)\begin{vmatrix} 3 & -1 & 2 \\ 12 & -11 & 17 \\ 9 & 7 & 2 \end{vmatrix} \quad \{\text{by Property 4}$$

$$= (-1)\begin{vmatrix} 3 & -1 & 2 \\ 0 & -7 & 9 \\ 9 & 7 & 2 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } (-4) \text{ times} \\ \text{the first row to the} \\ \text{second row}\end{array}\right.$$

$$= (-1)\begin{vmatrix} 3 & -1 & 2 \\ 0 & -7 & 9 \\ 0 & 10 & -4 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } (-3) \text{ times} \\ \text{the first row to the} \\ \text{third row}\end{array}\right.$$

$$= (-1)(3)\begin{vmatrix} -7 & 9 \\ 10 & -4 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by expansion by} \\ \text{cofactors}\end{array}\right.$$

$$= (-3)(28 - 90) = 186. \quad \blacksquare$$

Problems 5.4

In Problems 1 through 18, evaluate the determinants of the given matrices.

1. $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 3 & 3 \\ 2 & 5 & 0 \end{bmatrix}$, **2.** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, **3.** $\begin{bmatrix} 3 & -4 & 2 \\ -1 & 5 & 7 \\ 1 & 9 & -6 \end{bmatrix}$,

4. $\begin{bmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{bmatrix}$, **5.** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 4 \\ 3 & 5 & 1 \end{bmatrix}$, **6.** $\begin{bmatrix} 2 & 1 & -9 \\ 3 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$,

7. $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$, **8.** $\begin{bmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{bmatrix}$, **9.** $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 5 & 1 \\ 2 & -5 & -1 \end{bmatrix}$,

10. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$, **11.** $\begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & 4 \\ -2 & 2 & 7 \end{bmatrix}$, **12.** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 3 \\ 4 & 5 & 0 \end{bmatrix}$,

13. $\begin{bmatrix} 3 & 5 & 4 & 6 \\ -2 & 1 & 0 & 7 \\ -5 & 4 & 7 & 2 \\ 8 & -3 & 1 & 1 \end{bmatrix}$, **14.** $\begin{bmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{bmatrix}$,

15. $\begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & 5 & 2 & -1 \\ -2 & -2 & 1 & 3 \\ -3 & 4 & -1 & 8 \end{bmatrix}$, **16.** $\begin{bmatrix} -1 & 3 & 2 & -2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{bmatrix}$,

17. $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ -3 & 4 & 0 & 8 \end{bmatrix}$, **18.** $\begin{bmatrix} -2 & 0 & 1 & 3 \\ 4 & 0 & 2 & -2 \\ -3 & 1 & 0 & 1 \\ 5 & 4 & 1 & 7 \end{bmatrix}$.

19. What can you say about the determinant of an $n \times n$ matrix that has rank less than n?

20. What can you say about the determinant of a singular matrix?

5.5 Inversion

As an immediate consequence of Theorem 1 of Section 3.2 and the method of pivotal condensation, we have:

Theorem 1 *A square matrix has an inverse if and only if its determinant is not zero.*

In this section, we develop a method to calculate inverses of nonsingular matrices using determinants. For matrices with order greater than 3×3, this method is less efficient than the one described in Section 3.2, and is generally avoided.

Definition 1 The *cofactor matrix* associated with an $n \times n$ matrix $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^c$ obtained from $\mathbf{A}$ by replacing each element of $\mathbf{A}$ by its cofactor.

Example 1 Find $\mathbf{A}^c$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution

$$\mathbf{A}^c = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -2 & 4 \\ 1 & 6 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} \end{bmatrix},$$

$$\mathbf{A}^c = \begin{bmatrix} 18 & 16 & -11 \\ 0 & 16 & -8 \\ -6 & -16 & 17 \end{bmatrix}. \quad \blacksquare$$

If $\mathbf{A} = [a_{ij}]$, we will use the notation $\mathbf{A}^c = [a^c_{ij}]$ to represent the cofactor matrix. Thus a^c_{ij} represents the cofactor of a_{ij}.

Definition 2 The *adjugate* of an $n \times n$ matrix $\mathbf{A}$ is the transpose of the cofactor matrix of $\mathbf{A}$.

Thus, if we designate the adjugate of $\mathbf{A}$ by $\mathbf{A}^a$, we have that $\mathbf{A}^a = (\mathbf{A}^c)^\mathsf{T}$.

Example 2 Find $\mathbf{A}^a$ for the $\mathbf{A}$ given in Example 1.

Solution

$$\mathbf{A}^a = \begin{bmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{bmatrix}. \quad \blacksquare$$

The importance of the adjugate is given in the following theorem, which is proved in the Final Comments to this chapter.

Theorem 2 $\mathbf{A}\mathbf{A}^a = \mathbf{A}^a\mathbf{A} = |\mathbf{A}|\mathbf{I}$.

If $|\mathbf{A}| \neq 0$, *we may divide by it in Theorem 2 and obtain*

$$\mathbf{A}\left(\frac{\mathbf{A}^a}{|\mathbf{A}|}\right) = \left(\frac{\mathbf{A}^a}{|\mathbf{A}|}\right)\mathbf{A} = \mathbf{I}.$$

Thus, using the definition of the inverse, we have

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a \quad \textit{if } |\mathbf{A}| \neq 0.$$

That is, if $|\mathbf{A}| \neq 0$, *then* $\mathbf{A}^{-1}$ *may be obtained by dividing the adjugate of* $\mathbf{A}$ *by the determinant of* $\mathbf{A}$.

Example 3 Find $\mathbf{A}^{-1}$ for the $\mathbf{A}$ given in Example 1.

Solution The determinant of $\mathbf{A}$ is found to be 48. Using the solution to Example 2, we have

$$\mathbf{A}^{-1} = \left(\frac{\mathbf{A}^a}{|\mathbf{A}|}\right) = 1/48\begin{bmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{bmatrix} = \begin{bmatrix} 3/8 & 0 & -1/8 \\ 1/3 & 1/3 & -1/3 \\ -11/48 & -1/6 & 17/48 \end{bmatrix}. \quad \blacksquare$$

Example 4 Find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}.$$

Solution $\det(\mathbf{A}) = -1 \neq 0$, therefore $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^c = \begin{bmatrix} -5 & 4 & -8 \\ 11 & -9 & 17 \\ 6 & -5 & 10 \end{bmatrix}, \quad \mathbf{A}^a = \left(\mathbf{A}^c\right)^{\mathrm{T}} = \begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix},$$

$$\mathbf{A}^{-1} = \frac{\mathbf{A}^a}{|\mathbf{A}|} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}, \quad \blacksquare$$

Example 5 Find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution $|\mathbf{A}| = -2$, therefore $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^c = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}^a = \left(\mathbf{A}^c\right)^{\mathrm{T}} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix},$$

$$\mathbf{A}^{-1} = \frac{\mathbf{A}^a}{|\mathbf{A}|} = \left(-\tfrac{1}{2}\right)\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}. \quad \blacksquare$$

Problems 5.5

In Problems 1 through 15, find the inverses of the given matrices, if they exist.

1. $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, **2.** $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, **3.** $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$,

4. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, **5.** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$, **6.** $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$,

7. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, **8.** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, **9.** $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$,

10. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, **11.** $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$, **12.** $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -2 & -4 \\ 2 & 3 & -1 \end{bmatrix}$,

13. $\begin{bmatrix} 2 & 4 & 3 \\ 3 & -4 & -4 \\ 5 & 0 & -1 \end{bmatrix}$, **14.** $\begin{bmatrix} 5 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$, **15.** $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix}$.

16. Find a formula for the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if its determinant is nonzero.

17. Prove that if **A** and **B** are square matrices of the same order, then the product **AB** is nonsingular if and only if both **A** and **B** are.

18. Prove Theorem 1.

19. What can be said about the rank of a square matrix having a nonzero determinant?

5.6 Cramer's Rule

Cramer's rule is a method, based on determinants, for solving systems of simultaneous linear equations. In this section, we first state the rule, then illustrate its usage by an example, and finally prove its validity using the properties derived in Section 5.3. We also discuss the many limitations of the method.

Cramer's rule states that given a system of simultaneous linear equations in the matrix form $\mathbf{Ax} = \mathbf{b}$ (see Section 1.3), the ith component of $\mathbf{x}$ (or equivalently the ith unknown) is the quotient of two determinants. The determinant in the numerator is the determinant of a matrix obtained from **A** by replacing the ith column of **A** by the vector **b**, while the determinant in the denominator is just $|\mathbf{A}|$ Thus, if we are considering the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned}$$

where x_1, x_2, and x_3 represent the unknowns, then Cramer's rule states that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\mathbf{A}|}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|\mathbf{A}|},$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|\mathbf{A}|}, \quad \text{where} \quad |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Two restrictions on the application of Cramer's rule are immediate. First, the systems under consideration must have exactly the same number of equations as unknowns to insure that all matrices involved are square and hence have determinants. Second, the determinant of the coefficient matrix must not be zero since it appears in the denominator. If $|\mathbf{A}| = 0$, then Cramer's rule cannot be applied.

Example 1 Solve the system

$$\begin{aligned} x + 2y - 3z + w &= -5, \\ y + 3z + w &= 6, \\ 2x + 3y + z + w &= 4, \\ x + z + w &= 1. \end{aligned}$$

Solution

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -5 \\ 6 \\ 4 \\ 1 \end{bmatrix}.$$

Since $|\mathbf{A}| = 20$, Cramer's rule can be applied, and

$$x = \frac{\begin{vmatrix} -5 & -2 & -3 & 1 \\ 6 & 1 & 3 & 1 \\ 4 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix}}{20} = \frac{0}{20} = 0, \quad y = \frac{\begin{vmatrix} 1 & -5 & -3 & 1 \\ 0 & 6 & 3 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{20} = \frac{20}{20} = 1,$$

$$z = \frac{\begin{vmatrix} 1 & 2 & -5 & 1 \\ 0 & 1 & 6 & 1 \\ 2 & 3 & 4 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix}}{20} = \frac{40}{20} = 2, \quad w = \frac{\begin{vmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & 3 & 6 \\ 2 & 3 & 1 & 4 \\ 1 & 0 & 1 & 1 \end{vmatrix}}{20} = \frac{-20}{20} = -1. \quad \blacksquare$$

We now derive Cramer's rule using only those properties of determinants given in Section 5.3. We consider the general system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$x_1|\mathbf{A}| = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21}x_1 & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31}x_1 & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} \quad \{\text{by Property 4 modified to columns}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} \quad \begin{cases} \text{by adding } (x_2) \text{ times} \\ \text{the second column to} \\ \text{the first column} \end{cases}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} \quad \begin{cases} \text{by adding } (x_3) \\ \text{times the third} \\ \text{column to the} \\ \text{first column} \end{cases}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}$$

by making continued use of Property 6 in the obvious manner. We now note that the first column of the new determinant is nothing more than $\mathbf{Ax}$, and since, $\mathbf{Ax} = \mathbf{b}$, the first column reduces to $\mathbf{b}$.

Thus,

$$x_1|\mathbf{A}| = \begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ b_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

or

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{|\mathbf{A}|}$$

providing $|\mathbf{A}| \neq 0$. This expression is Cramer's rule for obtaining x_1. A similar argument applied to the jth column, instead of the first column, quickly shows that Cramer's rule is valid for every x_j, $j = 1, 2, \ldots, n$.

Although Cramer's rule gives a systematic method for the solution of simultaneous linear equations, the number of computations involved can become awesome if the order of the determinant is large. Thus, for large systems, Cramer's rule is never used. The recommended algorithms include Gaussian elimination (Section 2.3) and **LU** decomposition (Section 3.5).

Problems 5.6

Solve the following systems of equations by Cramer's rule.

1. $\begin{aligned} x + 2y &= -3, \\ 3x + y &= 1. \end{aligned}$

2. $\begin{aligned} 2x + y &= 3, \\ x - y &= 6. \end{aligned}$

3. $\begin{aligned} 4a + 2b &= 0, \\ 5a - 3b &= 10. \end{aligned}$

4. $\begin{aligned} 3s - 4t &= 30, \\ -2s + 3t &= -10. \end{aligned}$

5. $\begin{aligned} 2x - 8y &= 200, \\ -x + 4y &= 150. \end{aligned}$

6. $\begin{aligned} x + y - 2z &= 3, \\ 2x - y + 3z &= 2. \end{aligned}$

7. $\begin{aligned} x + y &= 15, \\ x + z &= 15, \\ y + z &= 10. \end{aligned}$

8. $\begin{aligned} 3x + y + z &= 4, \\ x - y + 2z &= 15, \\ 2x - 2y - z &= 5. \end{aligned}$

9. $\begin{aligned} x + 2y - 2z &= -1, \\ 2x + y + z &= 5, \\ -x + y - z &= -2. \end{aligned}$

10. $\begin{aligned} 2a + 3b - c &= 4, \\ -a - 2b + c &= -2, \\ 3a - b &= 2. \end{aligned}$

11. $\begin{aligned} 2x + 3y + 2z &= 3, \\ 3x + y + 5z &= 2, \\ 7y - 4z &= 5. \end{aligned}$

12. $\begin{aligned} 5r + 8s + t &= 2, \\ 2s + t &= -1, \\ 4r + 3s - t &= 3. \end{aligned}$

13. $\begin{aligned} x + 2y + z + w &= 7, \\ 3x + 4y - 2z - 4w &= 13, \\ 2x + y - z + w &= -4, \\ x - 3y + 4z + 5w &= 0. \end{aligned}$

5.7 Final Comments on Chapter 5

We shall now prove Theorem 2 of Section 5.5 dealing with the product of a matrix with its adjugate. For this proof we will need the following lemma:

Lemma 1 *If each element of one row of a matrix is multiplied by the cofactor of the corresponding element of a different row, the sum is zero.*

Proof. We prove this lemma only for an arbitrary 3×3 matrix $\mathbf{A}$ where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Consider the case in which we multiply every element of the third row by the cofactor of the corresponding element in the second row and then sum the results. Thus,

$$\begin{aligned}
&a_{31}(\text{cofactor of } a_{21}) + a_{32}(\text{cofactor of } a_{22}) + a_{33}(\text{ cofactor of } a_{23}) \\
&\quad = a_{31}(-1)^3 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32}(-1)^4 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{33}(-1)^5 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&\quad = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \left\{\text{from Property 3, Section 5.3}\right.
\end{aligned}$$

□

Note that this property is equally valid if we replace the word row by the word column.

Theorem 1 $\mathbf{AA}^a = |\mathbf{A}|\mathbf{I}$.

Proof. We prove this theorem only for matrices of order 3×3. The proof easily may be extended to cover matrices of any arbitrary order. This extension is left as an exercise for the student.

$$\mathbf{AA}^a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11}^c & a_{21}^c & a_{31}^c \\ a_{12}^c & a_{22}^c & a_{32}^c \\ a_{13}^c & a_{23}^c & a_{33}^c \end{bmatrix}.$$

If we denote this product matrix by $[b_{ij}]$, then

$$\begin{aligned}
b_{11} &= a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c, \\
b_{12} &= a_{11}a_{21}^c + a_{12}a_{22}^c + a_{13}a_{23}^c, \\
b_{23} &= a_{21}a_{31}^c + a_{22}a_{32}^c + a_{23}a_{33}^c, \\
b_{22} &= a_{21}a_{21}^c + a_{22}a_{22}^c + a_{23}a_{23}^c, \\
&\text{etc.}
\end{aligned}$$

We now note that $b_{11} = |\mathbf{A}|$ since it is precisely the term obtained when one computes $\det(\mathbf{A})$ by cofactors, expanding by the first row. Similarly, $b_{22} = |\mathbf{A}|$ since it is precisely the term obtained by computing $\det(\mathbf{A})$ by cofactors after expanding by the second row. It follows from the above lemma that $b_{12} = 0$ and $b_{23} = 0$ since b_{12} is the term obtained by multiplying each element in the first row of $\mathbf{A}$ by the cofactor of the corresponding element in the second row and adding, while b_{23} is the term obtained by multiplying each element in the second row of **A** by the cofactor of the corresponding element in the third row and adding. Continuing this analysis for each b_{ij}, we find that

$$\mathbf{A}\mathbf{A}^a = \begin{bmatrix} |\mathbf{A}| & 0 & 0 \\ 0 & |\mathbf{A}| & 0 \\ 0 & 0 & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}\mathbf{A}^a = |\mathbf{A}|\mathbf{I}.$$ □

Theorem 2 $\mathbf{A}^a\mathbf{A} = |\mathbf{A}|\mathbf{I}$.

Proof. This proof is completely analogous to the previous one and is left as an exercise for the student. □

This page intentionally left blank