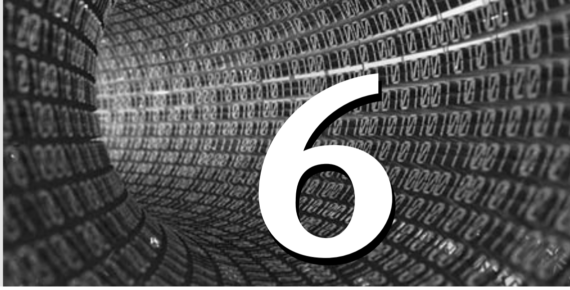

Eigenvalues and Eigenvectors

6.1 Definitions

Consider the matrix $\mathbf{A}$ and the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Forming the products $\mathbf{A}\mathbf{x}_1$, $\mathbf{A}\mathbf{x}_2$, and $\mathbf{A}\mathbf{x}_3$, we obtain

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix}, \quad \mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

But

$$\begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{x}_1, \quad \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix} = 3\mathbf{x}_2, \quad \text{and} \quad \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{x}_3;$$

hence,

$$\begin{aligned} \mathbf{A}\mathbf{x}_1 &= 2\mathbf{x}_1, \\ \mathbf{A}\mathbf{x}_2 &= 3\mathbf{x}_2, \\ \mathbf{A}\mathbf{x}_3 &= 1\mathbf{x}_3. \end{aligned}$$

That is, multiplying $\mathbf{A}$ by any one of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, or $\mathbf{x}_3$ is equivalent to simply multiplying the vector by a suitable scalar.

Definition 1 A nonzero vector $\mathbf{x}$ is an *eigenvector* (or characteristic vector) of a square matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Then λ is an *eigenvalue* (or characteristic value) of $\mathbf{A}$.

Thus, in the above example, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are eigenvectors of $\mathbf{A}$ and 2, 3, 1 are eigenvalues of $\mathbf{A}$.

Note that eigenvectors and eigenvalues are only defined for square matrices. Furthermore, note that the zero vector can *not* be an eigenvector even though $\mathbf{A} \cdot \mathbf{0} = \lambda \cdot \mathbf{0}$ for every scalar λ. An eigenvalue, however, can be zero.

Example 1 Show that

$$\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

is an eigenvector of

$$\mathbf{A} = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}.$$

Solution

$$\mathbf{Ax} = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ and $\lambda = 0$ is an eigenvalue. ■

Example 2 Is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

an eigenvector of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}?$$

Solution

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Thus, if $\mathbf{x}$ is to be an eigenvector of $\mathbf{A}$, there must exist a scalar λ such that $\mathbf{Ax} = \lambda\mathbf{x}$, or such that

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}.$$

It is quickly verified that no such λ exists, hence $\mathbf{x}$ is not an eigenvector of $\mathbf{A}$. ■

Problems 6.1

1. Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -4 & 7 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

(e) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, (f) $\begin{bmatrix} -4 \\ -4 \end{bmatrix}$, (g) $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$, (h) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

2. What are the eigenvalues that correspond to the eigenvectors found in Problem 1?

3. Determine which of the following vectors are eigenvectors for

$$\mathbf{B} = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

(e) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$, (f) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, (g) $\begin{bmatrix} -4 \\ -6 \end{bmatrix}$, (h) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

4. What are the eigenvalues that correspond to the eigenvectors found in Problem 3?

5. Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$,

(e) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, (g) $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, (h) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

6. What are the eigenvalues that correspond to the eigenvectors found in Problem 5?

7. Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$,

(d) $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

8. What are the eigenvalues that correspond to the eigenvectors found in Problem 7?

6.2 Eigenvalues

Let $\mathbf{x}$ be an eigenvector of the matrix $\mathbf{A}$. Then there must exist an eigenvalue λ such that

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{1}$$

or, equivalently,

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{2}$$

CAUTION. We could not have written (2) as $(\mathbf{A} - \lambda)\mathbf{x} = \mathbf{0}$ since the term $\mathbf{A} - \lambda$ would require subtracting a scalar from a matrix, an operation which is not defined. The quantity $\mathbf{A} - \lambda\mathbf{I}$, however, is defined since we are now subtracting one matrix from another.

Define a new matrix

$$\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}. \tag{3}$$

Then (2) may be rewritten as

$$\mathbf{Bx} = \mathbf{0}, \tag{4}$$

is a linear homogeneous system of equations for the unknown $\mathbf{x}$. If $\mathbf{B}$ has an inverse, then we can solve Eq. (4) for $\mathbf{x}$, obtaining $\mathbf{x} = \mathbf{B}^{-1}\mathbf{0}$, or $\mathbf{x} = \mathbf{0}$. This result, however, is absurd since $\mathbf{x}$ is an eigenvector and cannot be zero. Thus, it follows that $\mathbf{x}$ will be an eigenvector of $\mathbf{A}$ if and only if $\mathbf{B}$ does not have an inverse. But if a square matrix does not have an inverse, then its determinant must be zero (Theorem 1 of Section 5.5). Therefore, $\mathbf{x}$ will be an eigenvector of $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{5}$$

Equation (5) is called the *characteristic equation of* $\mathbf{A}$. The roots of (5) determine the eigenvalues of $\mathbf{A}$.

Example 1 Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}.$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$. The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, or $\lambda^2 - 4\lambda - 5 = 0$. Solving for λ, we have that $\lambda = -1, 5$; hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$, $\lambda_2 = 5$. ■

Example 2 Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}.$$

Solution

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix},$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(1-\lambda) + 2 = \lambda^2 - 2\lambda + 3.$$

The characteristic equation is $\lambda^2 - 2\lambda + 3 = 0$; hence, solving for λ by the quadratic formula, we have that $\lambda_1 = 1 + \sqrt{2}\,i$, $\lambda_2 = 1 - \sqrt{2}\,i$ which are eigenvalues of $\mathbf{A}$. ■

NOTE: Even if the elements of a matrix are real, the eigenvalues may be complex.

Example 3 Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix}.$$

Solution

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t-\lambda & 2t \\ 2t & -t-\lambda \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (t-\lambda)(-t-\lambda) - 4t^2 = \lambda^2 - 5t^2.$$

The characteristic equation is $\lambda^2 - 5t^2 = 0$, hence, the eigenvalues are $\lambda_1 = \sqrt{5}t$, $\lambda_2 = -\sqrt{5}t$.

NOTE: If the matrix $\mathbf{A}$ depends on a parameter (in this case the parameter is t), then the eigenvalues may also depend on the parameter. ■

Example 4 Find the eigenvalues for

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 1 \\ 3 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}.$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)[(2-\lambda)(-2-\lambda)+3] = (1-\lambda)(\lambda^2-1).$$

The characteristic equation is $(1-\lambda)(\lambda^2-1) = 0$; hence, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. ■

NOTE: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_k$. When this happens, the eigenvalue is said to be of *multiplicity k*. Thus, in Example 4, $\lambda = 1$ is an eigenvalue of multiplicity 2 while, $\lambda = -1$ is an eigenvalue of multiplicity 1.

From the definition of the characteristic Equation (5), it can be shown that if $\mathbf{A}$ is an $n \times n$ matrix then the characteristic equation of $\mathbf{A}$ is an nth degree polynomial in λ. It follows from the Fundamental Theorem of Algebra, that the characteristic equation has n roots, counting multiplicity. Hence, $\mathbf{A}$ has exactly n eigenvalues, counting multiplicity. (See Examples 1 and 4).

In general, it is very difficult to find the eigenvalues of a matrix. First the characteristic equation must be obtained, and for matrices of high order this is a lengthy task. Then the characteristic equation must be solved for its roots. If the equation is of high order, this can be an impossibility in practice. For example, the reader is invited to find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}.$$

For these reasons, eigenvalues are rarely found by the method just given, and numerical techniques are used to obtain approximate values (see Sections 6.6 and 10.4).

Problems 6.2

In Problems 1 through 35, find the eigenvalues of the given matrices.

1. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, **2.** $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **3.** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$,

4. $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$, **5.** $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, **6.** $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$,

7. $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$, **8.** $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$, **9.** $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$,

10. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **11.** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, **12.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

13. $\begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$, **14.** $\begin{bmatrix} 4 & 10 \\ 9 & -5 \end{bmatrix}$, **15.** $\begin{bmatrix} 5 & 10 \\ 9 & -4 \end{bmatrix}$,

16. $\begin{bmatrix} 0 & t \\ 2t & -t \end{bmatrix}$, **17.** $\begin{bmatrix} 0 & 2t \\ -2t & 4t \end{bmatrix}$, **18.** $\begin{bmatrix} 4\theta & 2\theta \\ -\theta & \theta \end{bmatrix}$,

19. $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$, **20.** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, **21.** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$,

22. $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, **23.** $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **24.** $\begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$,

25. $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, **26.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, **27.** $\begin{bmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$,

28. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, **29.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, **30.** $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}$,

31. $\begin{bmatrix} 1 & 5 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, **32.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, **33.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$,

34. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, **35.** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix}$.

36. Consider the matrix

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$

Use mathematical induction to prove that

$$\det(\mathbf{C} - \lambda\mathbf{I}) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0).$$

Deduce that the characteristic equation for this matrix is

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

The matrix **C** is called the *companion matrix* for this characteristic equation.

37. Show that if λ is an eigenvalue of **A**, then $k\lambda$ is an eigenvalue of $k\mathbf{A}$, where k denotes an arbitrary scalar.

38. Show that if $\lambda \neq 0$ is an eigenvalue of **A**, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$, providing the inverse exists.

39. Show that if λ is an eigenvalue of **A**, then it is also an eigenvalue of $\mathbf{A}^{\mathsf{T}}$.

6.3 Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogeneous equations. If an eigenvalue λ_i is substituted into (2), the corresponding eigenvector $\mathbf{x}_i$ is the solution of

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathrm{x}_i = \mathbf{0}. \tag{6}$$

Example 1 Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution The eigenvalues of **A** have already been found to be $\lambda_1 = -1,\ \lambda_2 = 5$ (see Example 1 of Section 6.2). We first calculate the eigenvectors corresponding to λ_1. From (6),

$$(\mathbf{A} - (-1)\mathbf{I})\mathbf{x}_1 = \mathbf{0}. \tag{7}$$

If we designate the unknown vector $\mathbf{x}_1$ by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

Eq. (7) becomes

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

or, equivalently,

$$\begin{aligned} 2x_1 + 2y_1 &= 0, \\ 4x_1 + 4y_1 &= 0. \end{aligned}$$

A nontrivial solution to this set of equations is $x_1 = -y_1$, y_1 arbitrary; hence, the eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -y_1 \\ y_1 \end{bmatrix} = y_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad y_1 \text{ arbitrary.}$$

By choosing different values of y_1, different eigenvectors for $\lambda_1 = -1$ can be obtained. Note, however, that any two such eigenvectors would be scalar multiples of each other, hence linearly dependent. Thus, there is only one linearly independent eigenvector corresponding to $\lambda_1 = -1$. For convenience we choose $y_1 = 1$, which gives us the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Many times, however, the scalar y_1 is chosen in such a manner that the resulting eigenvector becomes a unit vector. If we wished to achieve this result for the above vector, we would have to choose $y_1 = 1/\sqrt{2}$.

Having found an eigenvector corresponding to $\lambda_1 = -1$, we proceed to find an eigenvector $\mathbf{x}_2$ corresponding to $\lambda_2 = 5$. Designating the unknown vector $\mathbf{x}_2$ by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

and substituting it with λ_2 into (6), we obtain

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} -4x_2 + 2y_2 &= 0, \\ 4x_2 - 2y_2 &= 0. \end{aligned}$$

A nontrivial solution to this set of equations is $x_2 = \frac{1}{2}y_2$, where y_2 is arbitrary; hence

$$\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2/2 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

For convenience, we choose $y_2 = 2$, thus

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In order to check whether or not $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda_2 = 5$, we need only check if $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$:

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_2\mathbf{x}_2.$$

Again note that $\mathbf{x}_2$ is *not* unique! Any scalar multiple of $\mathbf{x}_2$ is also an eigenvector corresponding to λ_2. However, in this case, there is just one *linearly independent* eigenvector corresponding to λ_2. ■

Example 2 Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix}.$$

Solution By using the method of the previous section, we find the eigenvalues to be $\lambda_1 = 2$, $\lambda_2 = i$, $\lambda_3 = -i$. We first calculate the eigenvectors corresponding to $\lambda_1 = 2$. Designate $\mathbf{x}_1$ by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

Then (6) becomes

$$\left\{ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} 0 &= 0, \\ 5z_1 &= 0, \\ -y_1 - 4z_1 &= 0. \end{aligned}$$

A nontrivial solution to this set of equations is $y_1 = z_1 = 0$, x_1 arbitrary; hence

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We now find the eigenvectors corresponding to $\lambda_2 = i$. If we designate $\mathbf{x}_2$ by

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$

Eq. (6) becomes

$$\begin{bmatrix} 2-i & 0 & 0 \\ 0 & 2-i & 5 \\ 0 & -1 & -2-i \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} (2-i)x_2 &= 0, \\ (2-i)y_2 + 5z_2 &= 0, \\ -y_2 + (-2-i)z_2 &= 0. \end{aligned}$$

A nontrivial solution to this set of equations is $x_2 = 0$, $y_2 = (-2-i)z_2$, z_2 arbitrary; hence,

$$\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ (-2-i)z_2 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} 0 \\ -2-i \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to $\lambda_3 = -i$ are found in a similar manner to be

$$\mathbf{x}_3 = z_3 \begin{bmatrix} 0 \\ -2-i \\ 1 \end{bmatrix}, \quad z_3 \text{ arbitrary.} \quad \blacksquare$$

It should be noted that even if a mistake is made in finding the eigenvalues of a matrix, the error will become apparent when the eigenvectors corresponding to the incorrect eigenvalue are found. For instance, suppose that λ_1 in Example 2 was calculated erroneously to be 3. If we now try to find $\mathbf{x}_1$ we obtain the equations.

$$\begin{aligned} -x_1 &= 0, \\ -y_1 + 5z_1 &= 0, \\ -y_1 - 5z_1 &= 0. \end{aligned}$$

The only solution to this set of equations is $x_1 = y_1 = z_1 = 0$, hence

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, by definition, an eigenvector cannot be the zero vector. Since every eigenvalue must have a corresponding eigenvector, there is a mistake. A quick check shows that all the calculations above are valid, hence the error must lie in the value of the eigenvalue.

Problems 6.3

In Problems 1 through 23, find an eigenvector corresponding to each eigenvalue of the given matrix.

1. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, **2.** $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **3.** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$,

4. $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$, **5.** $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$, **6.** $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$,

7. $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$, **8.** $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$, **9.** $\begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$,

10. $\begin{bmatrix} 4 & 10 \\ 9 & -5 \end{bmatrix}$, **11.** $\begin{bmatrix} 0 & t \\ 2t & -t \end{bmatrix}$, **12.** $\begin{bmatrix} 4\theta & 2\theta \\ -\theta & \theta \end{bmatrix}$,

13. $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$, **14.** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, **15.** $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 2 \\ -1 & 0 & 3 \end{bmatrix}$,

16. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **17.** $\begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$, **18.** $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$,

19. $\begin{bmatrix} 1 & 5 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, **20.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, **21.** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 0 & 1 & 5 \end{bmatrix}$,

22. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, **23.** $\begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 \\ 0 & 2 & 0 & 4 \end{bmatrix}$.

24. Find unit eigenvectors (i.e., eigenvectors whose magnitudes equal unity) for the matrix in Problem 1.

25. Find unit eigenvectors for the matrix in Problem 2.

26. Find unit eigenvectors for the matrix in Problem 3.

27. Find unit eigenvectors for the matrix in Problem 13.

28. Find unit eigenvectors for the matrix in Problem 14.

29. Find unit eigenvectors for the matrix in Problem 16.

30. A nonzero vector **x** is a left eigenvector for a matrix **A** if there exists a scalar λ such that $\mathbf{xA} = \lambda\mathbf{x}$. Find a set of left eigenvectors for the matrix in Problem 1.

31. Find a set of left eigenvectors for the matrix in Problem 2.

32. Find a set of left eigenvectors for the matrix in Problem 3.

33. Find a set of left eigenvectors for the matrix in Problem 4.

34. Find a set of left eigenvectors for the matrix in Problem 13.

35. Find a set of left eigenvectors for the matrix in Problem 14.

36. Find a set of left eigenvectors for the matrix in Problem 16.

37. Find a set of left eigenvectors for the matrix in Problem 18.

38. Prove that if **x** is a right eigenvector of a symmetric matrix **A**, then $\mathbf{x}^{\mathrm{T}}$ is a left eigenvector of **A**.

39. A left eigenvector for a given matrix is known to be $[1 \quad 1]$. Find another left eigenvector for the same matrix satisfying the property that the sum of the vector components must equal unity.

40. A left eigenvector for a given matrix is known to be $[2 \quad 3]$. Find another left eigenvector for the same matrix satisfying the property that the sum of the vector components must equal unity.

41. A left eigenvector for a given matrix is known to be $[1 \quad 2 \quad 5]$. Find another left eigenvector for the same matrix satisfying the property that the sum of the vector components must equal unity.

42. A Markov chain (see Problem 16 of Section 1.1 and Problem 16 of Section 1.6) is *regular* if some power of the transition matrix contains only positive elements. If the matrix itself contains only positive elements then the power is one, and the matrix is automatically regular. Transition matrices that are regular always have an eigenvalue of unity. They also have limiting distribution vectors denoted by $\mathbf{x}^{(\infty)}$, where the ith component of $\mathbf{x}^{(\infty)}$ represents the probability of an object being in state i after a large number of time periods have elapsed. The limiting distribution $\mathbf{x}^{(\infty)}$ is a left eigenvector of the transition matrix corresponding to the eigenvalue of unity, and having the sum of its components equal to one.

(a) Find the limiting distribution vector for the Markov chain described in Problem 16 of Section 1.1.

(b) Ultimately, what is the probability that a family will reside in the city?

43. Find the limiting distribution vector for the Markov chain described in Problem 17 of Section 1.1. What is the probability of having a Republican mayor over the long run?

44. Find the limiting distribution vector for the Markov chain described in Problem 18 of Section 1.1. What is the probability of having a good harvest over the long run?

45. Find the limiting distribution vector for the Markov chain described in Problem 19 of Section 1.1. Ultimately, what is the probability that a person will use Brand Y?

6.4 Properties of Eigenvalues and Eigenvectors

Definition 1 The *trace* of a matrix $\mathbf{A}$, designated by $\text{tr}(\mathbf{A})$, is the sum of the elements on the main diagonal.

Example 1 Find the $\text{tr}(\mathbf{A})$ if

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & 1 \\ 1 & -1 & -5 \end{bmatrix}.$$

Solution $\text{tr}(\mathbf{A}) = 3 + 4 + (-5) = 2.$ ■

Property 1 *The sum of the eigenvalues of a matrix equals the trace of the matrix.*

Proof. See Problem 20. □

Property 1 provides us with a quick and useful procedure for checking eigenvalues.

Example 2 Verify Property 1 for

$$\mathbf{A} = \begin{bmatrix} 11 & 3 \\ -5 & -5 \end{bmatrix}.$$

Solution The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 10$, $\lambda_2 = -4$.

$$\text{tr}(\mathbf{A}) = 11 + (-5) = 6 = \lambda_1 + \lambda_2. \quad \blacksquare$$

Property 2 *A matrix is singular if and only if it has a zero eigenvalue.*

Proof. A matrix $\mathbf{A}$ has a zero eigenvalue if and only if $\det(\mathbf{A} - 0\mathbf{I}) = 0$, or (since $0\mathbf{I} = \mathbf{0}$) if and only if $\det(\mathbf{A}) = 0$. But $\det(\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is singular, thus, the result is immediate. □

Property 3 *The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.*

Proof. See Problem 15. □

Example 3 Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}.$$

Solution Since $\mathbf{A}$ is lower triangular, the eigenvalues must be $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. ■

Property 4 *If λ is an eigenvalue of $\mathbf{A}$ and if $\mathbf{A}$ is invertible, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$.*

Proof. Since $\mathbf{A}$ is invertible, Property 2 implies that $\lambda \neq 0$; hence $1/\lambda$ exists. Since λ is an eigenvalue of $\mathbf{A}$ there must exist an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Premultiplying both sides of this equation by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x}$$

or, equivalently, $\mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}$. Thus, $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$. □

Observation 1 If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ and if $\mathbf{A}$ is invertible, then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ corresponding to the eigenvalue $1/\lambda$.

Property 5 *If λ is an eigenvalue of $\mathbf{A}$, then $\alpha\lambda$ is an eigenvalue of $\alpha\mathbf{A}$ where α is any arbitrary scalar.*

Proof. If λ is an eigenvalue of $\mathbf{A}$, then there must exist an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Multiplying both sides of this equation by α, we obtain $(\alpha\mathbf{A})\mathbf{x} = (\alpha\lambda)\mathbf{x}$ which implies Property 5. □

Observation 2 If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector of $\alpha\mathbf{A}$ corresponding to eigenvalue $\alpha\lambda$.

Property 6 *If λ is an eigenvalue of $\mathbf{A}$, then λ^k is an eigenvalue of $\mathbf{A}^k$, for any positive integer k.*

Proof. We prove the result for the special cases $k = 2$ and k equals 3. Other cases are handled by mathematical induction. (See Problem 16.) If λ is an eigenvalue of $\mathbf{A}$, there must exist an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Then,

$$\mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x},$$

which implies that λ^2 is an eigenvalue of $\mathbf{A}^2$. As a result, we also have that

$$\mathbf{A}^3\mathbf{x} = \mathbf{A}(\mathbf{A}^2\mathbf{x}) = \mathbf{A}(\lambda^2\mathbf{x}) = \lambda^2(\mathbf{Ax}) = \lambda^2(\lambda\mathbf{x}) = \lambda^3\mathbf{x},$$

which implies that λ^3 is an eigenvalue of $\mathbf{A}^3$. □

Observation 3 If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector $\mathbf{A}^k$ corresponding to the eigenvalue λ^k, for any positive integer k.

Property 7 *If λ is an eigenvalue of $\mathbf{A}$, then for any scalar c, $\lambda - c$ is an eigenvalue of $\mathbf{A} - c\mathbf{I}$.*

Proof. If λ is an eigenvalue of $\mathbf{A}$, then there exists an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Consequently,

$$\mathbf{Ax} - c\mathbf{x} = \lambda\mathbf{x} - c\mathbf{x},$$

or

$$(\mathbf{A} - c\mathbf{I})\mathbf{x} = (\lambda - c)\mathbf{x}.$$

Thus, $\lambda - c$ is an eigenvalue of $\mathbf{A} - c\mathbf{I}$. □

Observation 4 If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector $\mathbf{A} - c\mathbf{I}$ corresponding to the eigenvalue $\lambda - c$.

Property 8 *If λ is an eigenvalue of $\mathbf{A}$, then λ is an eigenvalue of $\mathbf{A}^T$.*

Proof. Since λ is an eigenvalue of $\mathbf{A}$, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Hence

$$\begin{aligned} 0 = |\mathbf{A} - \lambda\mathbf{I}| &= \left|\left(\mathbf{A}^\mathsf{T}\right)^\mathsf{T} - \lambda\mathbf{I}^\mathsf{T}\right| && \{\text{Property 1 (Section 1.4)} \\ &= \left|\left(\mathbf{A}^\mathsf{T} - \lambda\mathbf{I}\right)^\mathsf{T}\right| && \{\text{Property 3 (Section 1.4)} \\ &= \left|\mathbf{A}^\mathsf{T} - \lambda\mathbf{I}\right| && \{\text{Property 7 (Section 5.3)} \end{aligned}$$

Thus, $\det\left(\mathbf{A}^\mathsf{T} - \lambda\mathbf{I}\right) = 0$, which implies that λ is an eigenvalue of $\mathbf{A}^\mathsf{T}$. □

Property 9 *The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.*

Proof. See Problem 21. □

Example 4 Verify Property 9 for the matrix **A** given in Example 2:

Solution For this **A**, $\lambda_1 = 10,\ \lambda_2 = -4, \det(\mathbf{A}) = -55 + 15 = -40 = \lambda_1\lambda_2$. ■

Problems 6.4

1. One eigenvalue of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$$

is known to be 2. Determine the second eigenvalue by inspection.

2. One eigenvalue of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}$$

is known to be 0.7574, rounded to four decimal places. Determine the second eigenvalue by inspection.

3. Two eigenvalues of a 3×3 matrix are known to be 5 and 8. What can be said about the remaining eigenvalue if the trace of the matrix is -4?

4. Redo Problem 3 if the determinant of the matrix is -4 instead of its trace.

5. The determination of a 4×4 matrix **A** is 144 and two of its eigenvalues are known to be -3 and 2. What can be said about the remaining eigenvalues?

6. A 2×2 matrix **A** is known to have the eigenvalues -3 and 4. What are the eigenvalues of (a) $2\mathbf{A}$, (b) $5\mathbf{A}$, (c) $\mathbf{A} - 3\mathbf{I}$, and (d) $\mathbf{A} + 4\mathbf{I}$?

7. A 3×3 matrix **A** is known to have the eigenvalues -2, 2, and 4. What are the eigenvalues of (a) $\mathbf{A}^2$, (b) $\mathbf{A}^3$, (c) $-3\mathbf{A}$, and (d) $\mathbf{A} + 3\mathbf{I}$?

8. A 2×2 matrix **A** is known to have the eigenvalues -1 and 1. Find a matrix in terms of **A** that has for its eigenvalues:

(a) -2 and 2, (b) -5 and 5,
(c) 1 and 1, (d) 2 and 4.

9. A 3×3 matrix **A** is known to have the eigenvalues 2, 3, and 4. Find a matrix in terms of **A** that has for its eigenvalues:

(a) 4, 6, and 8, (b) 4, 9, and 16,
(c) 8, 27, and 64, (d) 0, 1, and 2.

10. Verify Property 1 for

$$\mathbf{A} = \begin{bmatrix} 12 & 16 \\ -3 & -7 \end{bmatrix}.$$

11. Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ -1 & 2 & -1 \\ 2 & 1 & 7 \end{bmatrix}.$$

12. Show that if λ is an eigenvalue of $\mathbf{A}$, then it is also an eigenvalue for $\mathbf{S}^{-1}\mathbf{AS}$ for any nonsingular matrix $\mathbf{S}$.

13. Show by example that, in general, an eigenvalue of $\mathbf{A} + \mathbf{B}$ is not the sum of an eigenvalue of $\mathbf{A}$ with an eigenvalue of $\mathbf{B}$.

14. Show by example that, in general, an eigenvalue of $\mathbf{AB}$ is not the product of an eigenvalue of $\mathbf{A}$ with an eigenvalue of $\mathbf{B}$.

15. Prove Property 3.

16. Use mathematical induction to complete the proof of Property 6.

17. The determinant of $\mathbf{A} - \lambda\mathbf{I}$ is known as the characteristic polynomial of $\mathbf{A}$. For an $n \times n$ matrix $\mathbf{A}$, it has the form

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0),$$

where $a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are constants that depend on the elements of $\mathbf{A}$. Show that $(-1)^n a_0 = \det(\mathbf{A})$.

18. (Problem 17 continued) Convince yourself by considering arbitrary 3×3 and 4×4 matrices that $a_{n-1} = \mathrm{tr}(\mathbf{A})$.

19. Assume that $\mathbf{A}$ is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where some or all of the eigenvalues may be equal. Since each eigenvalue $\lambda_i (i = 1, 2, \ldots, n)$ is a root of the characteristic polynomial, $(\lambda - \lambda_i)$ must be a factor of that polynomial. Deduce that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

20. Use the results of Problems 18 and 19 to prove Property 1.

21. Use the results of Problems 17 and 19 to prove Property 9.

22. Show, by example, that an eigenvector of $\mathbf{A}$ need not be an eigenvector of $\mathbf{A}^\mathsf{T}$.

23. Prove that an eigenvector of $\mathbf{A}$ is a left eigenvector of $\mathbf{A}^\mathsf{T}$.

6.5 Linearly Independent Eigenvectors

Since every eigenvalue has an infinite number of eigenvectors associated with it (recall that if $\mathbf{x}$ is an eigenvector, then any scalar multiple of $\mathbf{x}$ is also an eigenvector), it becomes academic to ask how many different eigenvectors can a matrix have? The answer is clearly an infinite number. A more revealing question is how many linearly independent eigenvectors can a matrix have? Theorem 4 of Section 2.6 provides us with a partial answer.

Theorem 1 *In an n-dimensional vector space, every set of $n+1$ vectors is linearly dependent.*

Therefore, since all of the eigenvectors of an $n \times n$ matrix must be n-dimensional (why?), it follows from Theorem 1 that an $n \times n$ matrix can have *at most n* linearly independent eigenvectors. The following three examples shed more light on the subject.

Example 1 Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = \lambda_3 = 2$, therefore $\lambda = 2$ is an eigenvalue of multiplicity 3. If we designate the unknown eigenvector $\mathbf{x}$ by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then Eq. (6) gives rise to the three equations

$$\begin{aligned} y &= 0, \\ z &= 0, \\ 0 &= 0. \end{aligned}$$

Thus, $y = z = 0$ and x is arbitrary; hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Setting $x = 1$, we see that $\lambda = 2$ generates only one linearly independent eigenvector,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad \blacksquare$$

Example 2 Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution Again, the eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 2$, therefore $\lambda = 2$ is an eigenvalue of multiplicity 3. Designate the unknown eigenvector $\mathbf{x}$ by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Equation (6) now gives rise to the equations

$$\begin{aligned} y &= 0, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

Thus, $y = 0$ and both x and z are arbitrary; hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since x and z can be chosen arbitrarily, we can first choose $x = 1$ and $z = 0$ to obtain

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and then choose $x = 0$ and $z = 1$ to obtain

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathbf{x}_1$ and $\mathbf{x}_2$ can easily be shown to be linearly independent vectors, hence we see that $\lambda = 2$ generates the two linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad \blacksquare$$

Example 3 Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution Again the eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 2$ so again $\lambda = 2$ is an eigenvalue of multiplicity three. Designate the unknown eigenvector $\mathbf{x}$ by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Equation (6) gives rise to the equations

$$\begin{aligned} 0 &= 0, \\ 0 &= 0, \\ 0 &= 0, \end{aligned}$$

Thus, x, y, and z are all arbitrary; hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since x, y, and z can be chosen arbitrarily, we can first choose $x = 1,\ y = z = 0$, then choose $x = z = 0,\ y = 1$ and finally choose $y = x = 0,\ z = 1$ to generate the three linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In this case we see that three linearly independent eigenvectors are generated by $\lambda = 2$. (Note that, from Theorem 1, this is the maximal number that could be generated.) ■

The preceding examples are illustrations of

Theorem 2 *If λ is an eigenvalue of multiplicity k of an $n \times n$ matrix $\mathbf{A}$, then the number of linearly independent eigenvectors of $\mathbf{A}$ associated with λ is given by $\rho = n - r(\mathbf{A} - \lambda\mathbf{I})$. Furthermore, $1 \le \rho \le k$.*

Proof. Let $\mathbf{x}$ be an n-dimensional vector. If $\mathbf{x}$ is an eigenvector, then it must satisfy the vector equation $\mathbf{Ax} = \lambda\mathbf{x}$ or, equivalently, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. This system is homogeneous, hence consistent, so by Theorem 2 of Section 2.7, we have that the solution vector $\mathbf{x}$ will be in terms of $n - r(\mathbf{A} - \lambda\mathbf{I})$ arbitrary unknowns. Since these unknowns can be picked independently of each other, it follows that the number of linearly independent eigenvectors of $\mathbf{A}$ associated with λ is also $\rho = n - r(\mathbf{A} - \lambda\mathbf{I})$. We defer a proof that $1 \leq \rho \leq k$ until Chapter 9. □

In Example 1, $\mathbf{A}$ is 3×3; hence $n = 3$, and $r(\mathbf{A} - 2\mathbf{I}) = 2$. Thus, there should be $3 - 2 = 1$ linearly independent eigenvector associated with $\lambda = 2$ which is indeed the case. In Example 2, once again $n = 3$ but $r(\mathbf{A} - 2\mathbf{I}) = 1$. Thus, there should be $3 - 1 = 2$ linearly independent eigenvectors associated with $\lambda = 2$ which also is the case.

The next theorem gives the relationship between eigenvectors that correspond to different eigenvalues.

Theorem *Eigenvectors corresponding to distinct (that is, different) eigenvalues are linearly independent.*

Proof. For the sake of clarity, we consider the case of three distinct eigenvectors and leave the more general proof as an exercise (see Problem 17). Therefore, let $\lambda_1, \lambda_2, \lambda_3$, be distinct eigenvalues of the matrix $\mathbf{A}$ and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be the associated eigenvectors. That is

$$\begin{aligned} \mathbf{Ax}_1 &= \lambda_1\mathbf{x}_1, \\ \mathbf{Ax}_2 &= \lambda_2\mathbf{x}_2, \\ \mathbf{Ax}_3 &= \lambda_3\mathbf{x}_3, \end{aligned} \tag{8}$$

and $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

Since we want to show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, we must show that the only solution to

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0} \tag{9}$$

is $c_1 = c_2 = c_3 = 0$. By premultiplying (9) by $\mathbf{A}$, we obtain

$$c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + c_3\mathbf{Ax}_3 = \mathbf{A} \cdot \mathbf{0} = \mathbf{0}.$$

It follows from (8), therefore, that

$$c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + c_3\lambda_3\mathbf{x}_3 = \mathbf{0}. \tag{10}$$

By premultiplying (10) by $\mathbf{A}$ and again using (8), we obtain

$$c_1\lambda_1^2\mathbf{x}_1 + c_2\lambda_2^2\mathbf{x}_2 + c_3\lambda_3^2\mathbf{x}_3 = \mathbf{0}. \tag{11}$$

Equations (9)–(11) can be written in the matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} c_1\mathbf{x}_1 \\ c_2\mathbf{x}_2 \\ c_3\mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Define

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}.$$

It can be shown that $\det(\mathbf{B}) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)$. Thus, since all the eigenvalues are distinct, $\det(\mathbf{B}) \neq 0$ and $\mathbf{B}$ is invertible. Therefore,

$$\begin{bmatrix} c_1\mathbf{x}_1 \\ c_2\mathbf{x}_2 \\ c_3\mathbf{x}_3 \end{bmatrix} = \mathbf{B}^{-1}\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

or

$$\begin{bmatrix} c_1\mathbf{x}_1 = 0 \\ c_2\mathbf{x}_2 = 0 \\ c_3\mathbf{x}_3 = 0 \end{bmatrix} \tag{12}$$

But since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are eigenvectors, they are nonzero, therefore, it follows from (12) that $c_1 = c_2 = c_3 = 0$. This result together with (9) implies Theorem 3.

Theorems 2 and 3 together completely determine the number of linearly independent eigenvectors of a matrix. □

Example 4 Find a set of linearly independent eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Solution The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = 5$. For this matrix, $n = 3$ and $r(\mathbf{A} - 1\mathbf{I}) = 1$, hence $n - r(\mathbf{A} - 1\mathbf{I}) = 2$. Thus, from Theorem 2, we know that $\mathbf{A}$ has two linearly independent eigenvectors corresponding to $\lambda = 1$ and one linearly independent eigenvector corresponding to $\lambda = 5$ (why?). Furthermore, Theorem 3 guarantees that the two eigenvectors corresponding to $\lambda = 1$ will be linearly independent of the eigenvector corresponding to $\lambda = 5$ and vice versa. It only remains to produce these vectors.

For $\lambda = 1$, the unknown vector

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

must satisfy the vector equation $(\mathbf{A} - 1\mathbf{I})\mathbf{x}_1 = 0$, or equivalently, the set of equations

$$\begin{aligned} 0 &= 0, \\ 4x_1 + 2y_1 + 2z_1 &= 0, \\ 4x_1 + 2y_1 + 2z_1 &= 0. \end{aligned}$$

A solution to this equation is $z_1 = -2x_1 - y_1$, x_1, and y_1 arbitrary. Thus,

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ -2x_1 - y_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + y_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

By first choosing $x_1 = 1$, $y_1 = 0$ and then $x_1 = 0$, $y_1 = 1$, we see that $\lambda = 1$ generates the two linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

An eigenvector corresponding to $\lambda_3 = 5$ is found to be

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, **A** possesses the three linearly independent eigenvectors,

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \quad \blacksquare$$

Problems 6.5

In Problems 1–16 find a set of linearly independent eigenvectors for the given matrices.

1. $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, **2.** $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, **3.** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$,

4. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, **5.** $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, **6.** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$,

7. $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, **8.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, **9.** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$,

10. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$, **11.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$, **12.** $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}$,

13. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}$, **14.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 3 \end{bmatrix}$,

15. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$, **16.** $\begin{bmatrix} 3 & 1 & 1 & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

17. The Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

is known to equal the product

$$(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)(x_4 - x_3)(x_4 - x_2) \cdots (x_n - x_1).$$

Using this result, prove Theorem 3 for n distinct eigenvalues.

6.6 Power Methods

The analytic methods described in Sections 6.2 and 6.3 are impractical for calculating the eigenvalues and eigenvectors of matrices of large order. Determining the characteristic equations for such matrices involves enormous effort, while finding its roots algebraically is usually impossible. Instead, iterative methods which lend themselves to computer implementation are used. Ideally, each iteration yields a new approximation, which converges to an eigenvalue and the corresponding eigenvetor.

The *dominant* eigenvalue of a matrix is the one having largest absolute values. Thus, if the eigenvalues of a matrix are 2, 5, and -13, then -13 is the dominant eigenvalue because it is the largest in absolute value. The *power method* is an algorithm for locating the dominant eigenvalue and a corresponding eigenvector for a matrix of real numbers when the following two conditions exist:

Condition 1. The dominant eigenvalue of a matrix is real (not complex) and is strictly greater in absolute values than all other eigenvalues.

Condition 2. If the matrix has order $n \times n$, then it possesses n linearly independent eigenvectors.

Denote the eigenvalues of a given square matrix $\mathbf{A}$ satisfying Conditions 1 and 2 by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and a set of corresponding eigenvectors by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, respectively. Assume the indexing is such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

Any vector $\mathbf{x}_0$ can be expressed as a linear combination of the eigenvectors of $\mathbf{A}$, so we may write

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Multiplying this equation by $\mathbf{A}^k$, for some large, positive integer k, we get

$$\begin{aligned}\mathbf{A}^k\mathbf{x}_0 &= \mathbf{A}^k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n)\\ &= c_1\mathbf{A}^k\mathbf{v}_1 + c_2\mathbf{A}^k\mathbf{v}_2 + \cdots + c_n\mathbf{A}^k\mathbf{v}_n.\end{aligned}$$

It follows from Property 6 and Observation 3 of Section 6.4 that

$$\begin{aligned}\mathbf{A}^k\mathbf{x}_0 &= c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n\\ &= \lambda_1^k\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n\right]\\ &\approx \lambda_1^k c_1\mathbf{v}_1 \qquad \text{for large } k.\end{aligned}$$

This last pseudo-equality follows from noting that each quotient of eigenvalues is less than unity in absolute value, as a result of indexing the first eigenvalue as the dominant one, and therefore tends to zero as that quotient is raised to successively higher powers.

Thus, $\mathbf{A}^k\mathbf{x}_0$ approaches a scalar multiple of $\mathbf{v}_1$. But any nonzero scalar multiple of an eigenvector is itself an eigenvector, so $\mathbf{A}^k\mathbf{x}_0$ approaches an eigenvector of $\mathbf{A}$ corresponding to the dominant eigenvalue, providing c_1 is not zero. The scalar c_1 will be zero only if $\mathbf{x}_0$ is a linear combination of $\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$.

The power method begins with an initial vector $\mathbf{x}_0$, usually the vector having all ones for its components, and then iteratively calculates the vectors

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0, \\
\mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 = \mathbf{A}^2\mathbf{x}_0, \\
\mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 = \mathbf{A}^3\mathbf{x}_0, \\
&\vdots \\
\mathbf{x}_k &= \mathbf{A}\mathbf{x}_{k-1} = \mathbf{A}^k\mathbf{x}_0.
\end{aligned}$$

As k gets larger, $\mathbf{x}_k$ approaches an eigenvector of $\mathbf{A}$ corresponding to its dominant eigenvalue.

We can even determine the dominant eigenvalue by scaling appropriately. If k is large enough so that $\mathbf{x}_k$ is a good approximation to the eigenvector, say to within acceptable roundoff error, then it follows from Eq. (1) that

$$\mathbf{A}\mathbf{x}_k = \lambda_1\mathbf{x}_k.$$

If $\mathbf{x}_k$ is scaled so that its largest component is unity, then the component of $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k = \lambda_1\mathbf{x}_k$ having the largest absolute value must be λ_1.

We can now formalize the power method. Begin with an initial guess $\mathbf{x}_0$ for the eigenvector, having the property that its largest component in absolute value is unity. Iteratively, calculate $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ by multiplying each successive iterate by $\mathbf{A}$, the matrix of interest. Each time $\mathbf{x}_k (k = 1, 2, 3, \ldots)$ is computed, identify its dominant component and divide each component by it. Redefine this scaled vector as the new $\mathbf{x}_k$. Each $\mathbf{x}_k$ is an estimate of an eigenvector for $\mathbf{A}$ and each dominant component is an estimate for the associated eigenvalue.

Example 1 Find the dominant eigenvalue, and a corresponding eigenvector for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution We initialize $\mathbf{x}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathsf{T}}$. Then

First Iteration

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

$$\lambda \approx 7,$$

$$\mathbf{x}_1 \leftarrow \frac{1}{7}\begin{bmatrix} 3 & 7 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 0.428571 & 1 \end{bmatrix}^{\mathsf{T}}.$$

Second Iteration

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0.428571 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.428571 \\ 4.714286 \end{bmatrix},$$

$$\lambda \approx 4.714286,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{4.714286}\begin{bmatrix} 2.428571 & 4.714286 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 0.515152 & 1 \end{bmatrix}^{\mathsf{T}}.$$

Third Iteration

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0.515152 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.515152 \\ 5.060606 \end{bmatrix},$$

$$\lambda = 5.060606,$$

$$\mathbf{x}_3 \leftarrow \frac{1}{5.060606}[2.515152 \;\; 5.060606]^{\mathsf{T}} = [0.497006 \;\; 1]^{\mathsf{T}}.$$

Fourth Iteration

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0.497006 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.497006 \\ 4.988024 \end{bmatrix},$$

$$\lambda \approx 4.988024,$$

$$\mathbf{x}_4 \leftarrow \frac{1}{4.988024}\begin{bmatrix} 2.497006 & 4.988024 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 0.500600 & 1 \end{bmatrix}^{\mathsf{T}}.$$

The method is converging to the eigenvalue 5 and its corresponding eigenvector $\begin{bmatrix} 0.5 & 1 \end{bmatrix}^{\mathsf{T}}$. ■

Example 2 Find the dominant eigenvalue and a corresponding eigenvector for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -1 & -7 \end{bmatrix}.$$

Solution We initialize $\mathbf{x}_0 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\mathsf{T}}$. Then

First Iteration

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = \begin{bmatrix}1 & 1 & 10\end{bmatrix}^{\mathsf{T}},$$

$$\lambda \approx 10,$$

$$\mathbf{x}_1 \leftarrow \frac{1}{10}\begin{bmatrix}1 & 1 & 10\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}0.1 & 0.1 & 1\end{bmatrix}^{\mathsf{T}}.$$

Second Iteration

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 18 & -1 & -7\end{bmatrix}\begin{bmatrix}0.1\\ 0.1\\ 1\end{bmatrix} = \begin{bmatrix}0.1\\ 1\\ -5.3\end{bmatrix},$$

$$\lambda \approx -5.3,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{-5.3}\begin{bmatrix}0.1 & 1 & -5.3\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}-0.018868 & -0.188679 & 1\end{bmatrix}^{\mathsf{T}}.$$

Third Iteration

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 18 & -1 & -7\end{bmatrix}\begin{bmatrix}-0.018868\\ -0.188679\\ 1\end{bmatrix} = \begin{bmatrix}-0.188679\\ 1\\ -7.150943\end{bmatrix},$$

$$\lambda \approx -7.150943,$$

$$\mathbf{x}_3 \leftarrow \frac{1}{-7.150943}\begin{bmatrix}-0.188679 & 1 & -7.150943\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}0.026385 & -0.139842 & 1\end{bmatrix}^{\mathsf{T}}.$$

Continuing in this manner, we generate Table 6.1, where all entries are rounded to four decimal places. The algorithm is converging to the eigenvalue −6.405125 and its corresponding eigenvector

$$\begin{bmatrix}0.024376 & -0.1561240 & 1\end{bmatrix}^{\mathsf{T}}. \quad \blacksquare$$

Although effective when it converges, the power method has deficiencies. It does not converge to the dominant eigenvalue when that eigenvalue is complex, and it may not converge when there are more than one equally dominant eigenvalues (See Problem 12). Furthermore, the method, in general, cannot be used to locate all the eigenvalues.

A more powerful numerical method is the *inverse power method*, which is the power method applied to the inverse of a matrix. This, of course, adds another assumption: the inverse must exist, or equivalently, the matrix must not have any zero eigenvalues. Since a nonsingular matrix and its inverse share identical

Table 6.1

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.1000	0.1000	1.0000	10.0000
2	−0.0189	−0.1887	1.0000	−5.3000
3	0.0264	−0.1398	1.0000	−7.1509
4	0.0219	−0.1566	1.0000	−6.3852
5	0.0243	−0.1551	1.0000	−6.4492
6	0.0242	−0.1561	1.0000	−6.4078
7	0.0244	−0.1560	1.0000	−6.4084
8	0.0244	−0.1561	1.0000	−6.4056

eigenvectors and reciprocal eigen- values (see Property 4 and Observation 1 of Section 6.4), once we know the eigenvalues and eigenvectors of the inverse of a matrix, we have the analogous information about the matrix itself.

The power method applied to the inverse of a matrix $\mathbf{A}$ will generally converge to the dominant eigenvalue of $\mathbf{A}^{-1}$. Its reciprocal will be the eigenvalue of $\mathbf{A}$ having the smallest absolute value. The advantages of the inverse power method are that it converges more rapidly than the power method, and if often can be used to find all real eigenvalues of $\mathbf{A}$; a disadvantage is that it deals with $\mathbf{A}^{-1}$, which is laborious to calculate for matrices of large order. Such a calculation, however, can be avoided using $\mathbf{LU}$ decomposition.

The power method generates the sequence of vectors

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1}.$$

The inverse power method will generate the sequence

$$\mathbf{x}_k = \mathbf{A}^{-1}\mathbf{x}_{k-1},$$

which may be written as

$$\mathbf{A}\mathbf{x}_k = \mathbf{x}_{k-1}.$$

We solve for the unknown vector $\mathbf{x}_k$ using $\mathbf{LU}$-decomposition (see Section 3.5).

Example 3 Use the inverse power method to find an eigenvalue for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

Solution We initialize $\mathbf{x}_0 = [1 \quad 1]^{\mathsf{T}}$. The $\mathbf{LU}$ decomposition for $\mathbf{A}$ has $\mathbf{A} = \mathbf{LU}$ with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

First Iteration. We solve the system $\mathbf{LUx}_1 = \mathbf{x}_0$ by first solving the system $\mathbf{Ly} = \mathbf{x}_0$ for $\mathbf{y}$, and then solving the system $\mathbf{Ux}_1 = \mathbf{y}$ for $\mathbf{x}_1$. Set $\mathbf{y} = [y_1 \quad y_2]^{\mathsf{T}}$ and

$\mathbf{x}_1 = [a \;\; b]^\mathsf{T}$. The first system is

$$\begin{aligned} y_1 + 0y_2 &= 1, \\ y_1 + y_2 \;\; &= 1, \end{aligned}$$

which has as its solution $y_1 = 1$ and $y_2 = 0$. The system $\mathbf{U}\mathbf{x}_1 = \mathbf{y}$ becomes

$$\begin{aligned} 2a + b &= 1, \\ 2b &= 0, \end{aligned}$$

which admits the solution $a = 0.5$ and $b = 0$. Thus,

$$\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{x}_0 = [0.5 \quad 0]^\mathsf{T},$$

$$\lambda \approx 0.5 \quad \text{(an approximation to an eigenvalue for } \mathbf{A}^{-1}\text{)},$$

$$\mathbf{x}_1 \leftarrow \frac{1}{0.5}[0.5 \;\; 0]^\mathsf{T} = [1 \;\; 0]^\mathsf{T}.$$

Second Iteration. We solve the system $\mathbf{L}\mathbf{U}\mathbf{x}_2 = \mathbf{x}_1$ by first solving the system $\mathbf{L}\mathbf{y} = \mathbf{x}_1$ for $\mathbf{y}$, and then solving the system $\mathbf{U}\mathbf{x}_2 = \mathbf{y}$ for $\mathbf{x}_2$. Set $\mathbf{y} = [y_1 \;\; y_2]^\mathsf{T}$ and $\mathbf{x}_2 = [a \;\; b]^\mathsf{T}$. The first system is

$$\begin{aligned} y_1 + 0y_2 &= 1, \\ y_1 + \;\; y_2 &= 0, \end{aligned}$$

which has as its solution $y_1 = 1$ and $y_2 = -1$. The system $\mathbf{U}\mathbf{x}_2 = \mathbf{y}$ becomes

$$\begin{aligned} 2a + b &= 1, \\ 2b &= -1, \end{aligned}$$

which admits the solution $a = 0.75$ and $b = -0.5$. Thus,

$$\mathbf{x}_2 = \mathbf{A}^{-1}\mathbf{x}_1 = [0.75 \quad -0.5]^\mathsf{T},$$

$$\lambda \approx 0.75,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{0.75}\left[0.75 \quad -0.5\right]^\mathsf{T} = \left[1 \quad -0.666667\right]^\mathsf{T}.$$

Third Iteration. We first solve $\mathbf{L}\mathbf{y} = \mathbf{x}_2$ to obtain $\mathbf{y} = \left[1 \quad -1.666667\right]^\mathsf{T}$, and then $\mathbf{U}\mathbf{x}_3 = \mathbf{y}$ to obtain $\mathbf{x}_3 = \left[0.916667 \quad -0.833333\right]^\mathsf{T}$. Then,

$$\lambda \approx 0.916667$$

$$\mathbf{x}_3 \leftarrow \frac{1}{0.916667}\left[0.916667 \quad -0.833333\right]^\mathsf{T} = \left[1 \quad -0.909091\right]^\mathsf{T}.$$

Continuing, we converge to the eigenvalue 1 for $\mathbf{A}^{-1}$ and its reciprocal $1/1 = 1$ for $\mathbf{A}$. The vector approximations are converging to $\left[1 \quad -1\right]^\mathsf{T}$, which is an eigenvector for both $\mathbf{A}^{-1}$ and $\mathbf{A}$. ■

Example 4 Use the inverse power method to find an eigenvalue for

$$\mathbf{A} = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 6 \\ 0 & 6 & 7 \end{bmatrix}.$$

Solution We initialize $\mathbf{x}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$. The **LU** decomposition for **A** has $\mathbf{A} = \mathbf{LU}$ with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.285714 & 1 & 0 \\ 0 & 14 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 7 & 2 & 0 \\ 0 & 0.428571 & 6 \\ 0 & 0 & -77 \end{bmatrix}.$$

First Iteration

Set $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{x}_1 = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathsf{T}}$. The first system is

$$y_1 + 0y_2 + 0y_3 = 1,$$

$$0.285714y_1 + y_2 + 0y_3 = 1,$$

$$0y_1 + 14y_2 + y_3 = 1,$$

which has as its solution $y_1 = 1$, and $y_2 = 0.714286$, and $y_3 = -9$. The system $\mathbf{Ux}_1 = \mathbf{y}$ becomes

$$7a + 2b = 1,$$

$$0.428571b + 6c = 0.714286,$$

$$-77c = -9,$$

which admits the solution $a = 0.134199$, $b = 0.030303$, and $c = 0.116883$. Thus,

$$\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{x}_0 = \begin{bmatrix} 0.134199 & 0.030303 & 0.116833 \end{bmatrix}^{\mathsf{T}},$$

$$\lambda \approx 0.134199 \quad \text{(an approximation to an eigenvalue for } \mathbf{A}^{-1}\text{)},$$

$$\mathbf{x}_1 \leftarrow \frac{1}{0.134199} \begin{bmatrix} 0.134199 & 0.030303 & 0.116833 \end{bmatrix}^{\mathsf{T}}$$

$$= \begin{bmatrix} 1 & 0.225806 & 0.870968 \end{bmatrix}^{\mathsf{T}}.$$

SECOND ITERATION

Solving the system $\mathbf{Ly} = \mathbf{x}_1$ for $\mathbf{y}$, we obtain

$$\mathbf{y} = \begin{bmatrix} 1 & -0.059908 & 1.709677 \end{bmatrix}^{\mathsf{T}}.$$

Then, solving the system $\mathbf{Ux}_2 = \mathbf{y}$ for $\mathbf{x}_2$, we get

$$\mathbf{x}_2 = \begin{bmatrix} 0.093981 & 0.171065 & -0.022204 \end{bmatrix}^{\mathsf{T}}.$$

Therefore,

$$\begin{aligned} \lambda &\approx 0.171065, \\ \mathbf{x}_2 &\leftarrow \frac{1}{0.171065} \begin{bmatrix} 0.093981 & 0.171065 & -0.022204 \end{bmatrix}^{\mathsf{T}}, \\ &= \begin{bmatrix} 0.549388 & 1 & -0.129796 \end{bmatrix}^{\mathsf{T}}. \end{aligned}$$

THIRD ITERATION

Solving the system $\mathbf{Ly} = \mathbf{x}_2$ for $\mathbf{y}$, we obtain

$$\mathbf{y} = \begin{bmatrix} 0.549388 & 0.843032 & -11.932245 \end{bmatrix}^{\mathsf{T}}.$$

Then, solving the system $\mathbf{Ux}_3 = \mathbf{y}$ for $\mathbf{x}_3$, we get

$$\mathbf{x}_3 = \begin{bmatrix} 0.136319 & -0.202424 & 0.154964 \end{bmatrix}^{\mathsf{T}}.$$

Table 6.2

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	1.0000	0.2258	0.8710	0.1342
2	0.5494	1.0000	−0.1298	0.1711
3	−0.6734	1.0000	−0.7655	−0.2024
4	−0.0404	1.0000	−0.5782	−0.3921
5	−0.2677	1.0000	−0.5988	−0.3197
6	−0.1723	1.0000	−0.6035	−0.3372
7	−0.2116	1.0000	−0.5977	−0.3323
8	−0.1951	1.0000	−0.6012	−0.3336
9	−0.2021	1.0000	−0.5994	−0.3333
10	−0.1991	1.0000	−0.6003	−0.3334
11	−0.2004	1.0000	−0.5999	−0.3333
12	−0.1998	1.0000	−0.6001	−0.3333

Therefore,

$$\lambda \approx -0.202424,$$

$$x_3 \leftarrow \frac{1}{-0.202424}\begin{bmatrix}0.136319 & -0.202424 & 0.154964\end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix}-0.673434 & 1 & -0.765542\end{bmatrix}^{\mathrm{T}}.$$

Continuing in this manner, we generate Table 6.2, where all entries are rounded to four decimal places. The algorithm is converging to the eigenvalue $-1/3$ for $\mathbf{A}^{-1}$ and its reciprocal -3 for $\mathbf{A}$. The vector approximations are converging to $[-0.2 \;\; 1 \;\; -0.6]^{\mathrm{T}}$, which is an eigenvector for both $\mathbf{A}^{-1}$ and $\mathbf{A}$. ■

We can use Property 7 and Observation 4 of Section 6.4 in conjunction with the inverse power method to develop a procedure for finding all eigenvalues and a set of corresponding eigenvectors for a matrix, providing that the eigenvalues are real and distinct, and estimates of their locations are known. The algorithm is known as the *shifted inverse power method*.

If c is an estimate for an eigenvalue of $\mathbf{A}$, then $\mathbf{A} - c\mathbf{I}$ will have an eigenvalue near zero, and its reciprocal will be the dominant eigenvalue of $(\mathbf{A} - c\mathbf{I})^{-1}$. We use the inverse power method with an $\mathbf{LU}$ decomposition of $\mathbf{A} - c\mathbf{I}$ to calculate the dominant eigenvalue λ and its corresponding eigenvector $\mathbf{x}$ for $(\mathbf{A} - c\mathbf{I})^{-1}$. Then $1/\lambda$ and $\mathbf{x}$ are an eigenvalue and eigenvector for $\mathbf{A} - c\mathbf{I}$ while $1/\lambda + c$ and $\mathbf{x}$ are an eigenvalue and eigenvector for $\mathbf{A}$.

Example 5 Find a second eigenvalue for the matrix given in Example 4.

Table 6.3

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.6190	0.7619	1.0000	−0.2917
2	0.4687	0.7018	1.0000	−0.2639
3	0.3995	0.6816	1.0000	−0.2557
4	0.3661	0.6736	1.0000	−0.2526
5	0.3496	0.6700	1.0000	−0.2513
6	0.3415	0.6683	1.0000	−0.2506
7	0.3374	0.6675	1.0000	−0.2503
8	0.3354	0.6671	1.0000	−0.2502
9	0.3343	0.6669	1.0000	−0.2501
10	0.3338	0.6668	1.0000	−0.2500
11	0.3336	0.6667	1.0000	−0.2500

Solution Since we do not have an estimate for any of the eigenvalues, we arbitrarily choose $c = 15$. Then

$$\mathbf{A} - c\mathbf{I} = \begin{bmatrix} -8 & 2 & 0 \\ 2 & -14 & 6 \\ 0 & 6 & -8 \end{bmatrix},$$

which has an **LU** decomposition with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & -0.444444 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} -8 & 2 & 0 \\ 0 & -13.5 & 6 \\ 0 & 0 & -5.333333 \end{bmatrix}.$$

Applying the inverse power method to $\mathbf{A} - 15\mathbf{I}$, we generate Table 6.3, which is converging to $\lambda = -0.25$ and $\mathbf{x} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}^{\mathrm{T}}$. The corresponding eigenvalue of $\mathbf{A}$ is $1/-0.25 + 15 = 11$, with the same eigenvector.

Using the results of Examples 4 and 5, we have two eigenvalues, $\lambda_1 = -3$ and $\lambda_2 = 11$, of the 3×3 matrix defined in Example 4. Since the trace of a matrix equals the sum of the eigenvalues (Property 1 of Section 6.4), we know $7 + 1 + 7 = -3 + 11 + \lambda_3$, so the last eigenvalue is $\lambda_3 = 7$. ■

Problems 6.6

In Problems 1 through 10, use the power method to locate the dominant eigenvalue and a corresponding eigenvector for the given matrices. Stop after five iterations.

1. $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **2.** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, **3.** $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$,

4. $\begin{bmatrix} 0 & 1 \\ -4 & 6 \end{bmatrix}$, **5.** $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$, **6.** $\begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}$,

7. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **8.** $\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 6 \\ 0 & 6 & 7 \end{bmatrix}$, **9.** $\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 11 \end{bmatrix}$,

10. $\begin{bmatrix} 2 & -17 & 7 \\ -17 & -4 & 1 \\ 7 & 1 & -14 \end{bmatrix}$.

11. Use the power method on

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix},$$

and explain why it does not converge to the dominant eigenvalue $\lambda = 3$.

12. Use the power method on

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix},$$

and explain why it does not converge.

13. Shifting can also be used with the power method to locate the next most dominant eigenvalue, if it is real and distinct, once the dominant eigenvalue has been determined. Construct $\mathbf{A} - \lambda\mathbf{I}$, where λ is the dominant eigenvalue of $\mathbf{A}$, and apply the power method to the shifted matrix. If the algorithm converges to μ and $\mathbf{x}$, then $\mu + \lambda$ is an eigenvalue of $\mathbf{A}$ with the corresponding eigenvector $\mathbf{x}$. Apply this shifted power method algorithm to the matrix in Problem 1. Use the results of Problem 1 to determine the appropriate shift.

14. Use the shifted power method as described in Problem 13 to the matrix in Problem 9. Use the results of Problem 9 to determine the appropriate shift.

15. Use the inverse power method on the matrix defined in Example 1. Stop after five iterations.

16. Use the inverse power method on the matrix defined in Problem 3. Take $\mathbf{x}_0 = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^{\mathsf{T}}$ and stop after five iterations.

17. Use the inverse power method on the matrix defined in Problem 5. Stop after five iterations.

18. Use the inverse power method on the matrix defined in Problem 6. Stop after five iterations.

19. Use the inverse power method on the matrix defined in Problem 9. Stop after five iterations.

20. Use the inverse power method on the matrix defined in Problem 10. Stop after five iterations.

21. Use the inverse power method on the matrix defined in Problem 11. Stop after five iterations.

22. Use the inverse power method on the matrix defined in Problem 4. Explain the difficulty, and suggest a way to avoid it.

23. Use the inverse power method on the matrix defined in Problem 2. Explain the difficulty, and suggest a way to avoid it.

24. Can the power method converge to a dominant eigenvalue it that eigenvalue is not distinct?

25. Apply the shifted inverse power method to the matrix defined in Problem 9, with a shift constant of 10.

26. Apply the shifted inverse power method to the matrix defined in Problem 10, with a shift constant of −25.