5

The Theory of the Simplex Method

Chapter 4 introduced the basic mechanics of the simplex method. Now we shall delve a little more deeply into this algorithm by examining some of its underlying theory. The first section further develops the general geometric and algebraic properties that form the foundation of the simplex method. We then describe the *matrix form* of the simplex method (called the *revised simplex method*), which streamlines the procedure considerably for computer implementation. Next we present a fundamental insight about a property of the simplex method that enables us to deduce how changes that are made in the original model get carried along to the final simplex tableau. This insight will provide the key to the important topics of Chap. 6 (duality theory and sensitivity analysis).

5.1 FOUNDATIONS OF THE SIMPLEX METHOD

Section 4.1 introduced *corner-point feasible (CPF) solutions* and the key role they play in the simplex method. These geometric concepts were related to the algebra of the simplex method in Secs. 4.2 and 4.3. However, all this was done in the context of the Wyndor Glass Co. problem, which has only *two decision variables* and so has a straightforward geometric interpretation. How do these concepts generalize to higher dimensions when we deal with larger problems? We address this question in this section.

We begin by introducing some basic terminology for any linear programming problem with n decision variables. While we are doing this, you may find it helpful to refer to Fig. 5.1 (which repeats Fig. 4.1) to interpret these definitions in two dimensions ($n = 2$).

Terminology

It may seem intuitively clear that optimal solutions for any linear programming problem must lie on the boundary of the feasible region, and in fact this is a general property. Because boundary is a geometric concept, our initial definitions clarify how the boundary of the feasible region is identified algebraically.

> The **constraint boundary equation** for any constraint is obtained by replacing its $\leq$, $=$, or $\geq$ sign by an $=$ sign.

Consequently, the form of a constraint boundary equation is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ for functional constraints and $x_j = 0$ for nonnegativity constraints. Each such

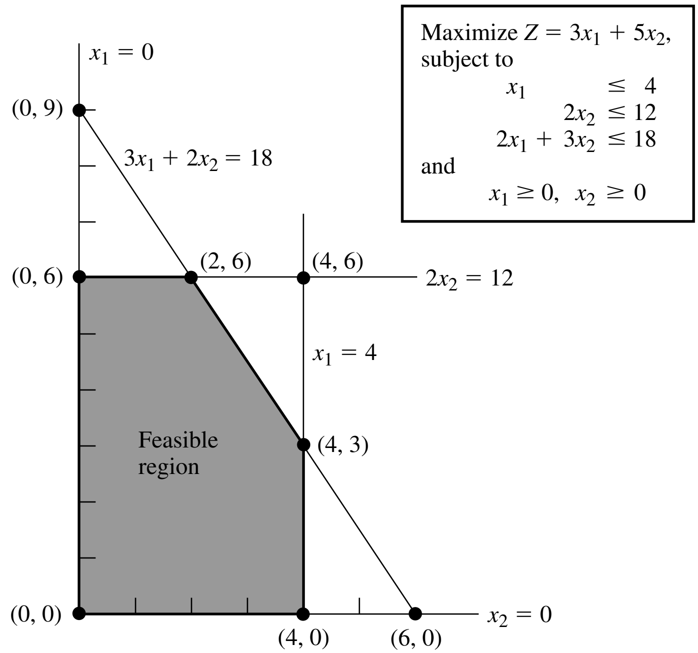


FIGURE 5.1
Constraint boundaries, constraint boundary equations, and corner-point solutions for the Wyndor Glass Co. problem.

equation defines a "flat" geometric shape (called a **hyperplane**) in n-dimensional space, analogous to the line in two-dimensional space and the plane in three-dimensional space. This hyperplane forms the **constraint boundary** for the corresponding constraint. When the constraint has either a $\leq$ or a $\geq$ sign, this *constraint boundary* separates the points that satisfy the constraint (all the points on one side up to and including the constraint boundary) from the points that violate the constraint (all those on the other side of the constraint boundary). When the constraint has an $=$ sign, only the points on the constraint boundary satisfy the constraint.

For example, the Wyndor Glass Co. problem has five constraints (three functional constraints and two nonnegativity constraints), so it has the five *constraint boundary equations* shown in Fig. 5.1. Because $n = 2$, the hyperplanes defined by these constraint boundary equations are simply lines. Therefore, the constraint boundaries for the five constraints are the five lines shown in Fig. 5.1.

The **boundary** of the feasible region contains just those feasible solutions that satisfy one or more of the constraint boundary equations.

Geometrically, any point on the boundary of the feasible region lies on one or more of the hyperplanes defined by the respective constraint boundary equations. Thus, in Fig. 5.1, the boundary consists of the five darker line segments.

Next, we give a general definition of *CPF solution* in n-dimensional space.

A **corner-point feasible (CPF) solution** is a feasible solution that does not lie on *any* line segment[1] connecting two *other* feasible solutions.

[1]An algebraic expression for a line segment is given in Appendix 2.

As this definition implies, a feasible solution that *does* lie on a line segment connecting two other feasible solutions is *not* a CPF solution. To illustrate when $n = 2$, consider Fig. 5.1. The point (2, 3) is *not* a CPF solution, because it lies on various such line segments, e.g., the line segment connecting (0, 3) and (4, 3). Similarly, (0, 3) is *not* a CPF solution, because it lies on the line segment connecting (0, 0) and (0, 6). However, (0, 0) *is* a CPF solution, because it is impossible to find two *other* feasible solutions that lie on completely opposite sides of (0, 0). (Try it.)

When the number of decision variables n is greater than 2 or 3, this definition for *CPF solution* is not a very convenient one for identifying such solutions. Therefore, it will prove most helpful to interpret these solutions algebraically. For the Wyndor Glass Co. example, each CPF solution in Fig. 5.1 lies at the intersection of two ($n = 2$) constraint lines; i.e., it is the *simultaneous solution* of a system of two constraint boundary equations. This situation is summarized in Table 5.1, where **defining equations** refer to the constraint boundary equations that yield (define) the indicated CPF solution.

> For any linear programming problem with n decision variables, each CPF solution lies at the intersection of n constraint boundaries; i.e., it is the *simultaneous solution* of a system of n constraint boundary equations.

However, this is not to say that *every* set of n constraint boundary equations chosen from the $n + m$ constraints (n nonnegativity and m functional constraints) yields a CPF solution. In particular, the simultaneous solution of such a system of equations might violate one or more of the other m constraints not chosen, in which case it is a corner-point *infeasible* solution. The example has three such solutions, as summarized in Table 5.2. (Check to see why they are infeasible.)

Furthermore, a system of n constraint boundary equations might have no solution at all. This occurs twice in the example, with the pairs of equations (1) $x_1 = 0$ and $x_1 = 4$ and (2) $x_2 = 0$ and $2x_2 = 12$. Such systems are of no interest to us.

The final possibility (which never occurs in the example) is that a system of n constraint boundary equations has multiple solutions because of redundant equations. You need not be concerned with this case either, because the simplex method circumvents its difficulties.

TABLE 5.1 Defining equations for each CPF solution for the Wyndor Glass Co. problem

CPF Solution	Defining Equations
(0, 0)	$x_1 = 0$ $x_2 = 0$
(0, 6)	$x_1 = 0$ $2x_2 = 12$
(2, 6)	$2x_2 = 12$ $3x_1 + 2x_2 = 18$
(4, 3)	$3x_1 + 2x_2 = 18$ $x_1 = 4$
(4, 0)	$x_1 = 4$ $x_2 = 0$

TABLE 5.2 Defining equations for each corner-point infeasible solution for the Wyndor Glass Co. problem

Corner-Point Infeasible Solution	Defining Equations
(0, 9)	$x_1 = 0$ $3x_1 + 2x_2 = 18$
(4, 6)	$2x_2 = 12$ $x_1 = 4$
(6, 0)	$3x_1 + 2x_2 = 18$ $x_2 = 0$

To summarize for the example, with five constraints and two variables, there are 10 pairs of constraint boundary equations. Five of these pairs became defining equations for CPF solutions (Table 5.1), three became defining equations for corner-point infeasible solutions (Table 5.2), and each of the final two pairs had no solution.

Adjacent CPF Solutions

Section 4.1 introduced adjacent CPF solutions and their role in solving linear programming problems. We now elaborate.

Recall from Chap. 4 that (when we ignore slack, surplus, and artificial variables) each iteration of the simplex method moves from the current CPF solution to an *adjacent* one. What is the *path* followed in this process? What really is meant by *adjacent* CPF solution? First we address these questions from a geometric viewpoint, and then we turn to algebraic interpretations.

These questions are easy to answer when $n = 2$. In this case, the *boundary* of the feasible region consists of several connected *line segments* forming a *polygon,* as shown in Fig. 5.1 by the five darker line segments. These line segments are the *edges* of the feasible region. Emanating from each CPF solution are *two* such edges leading to an adjacent CPF solution at the other end. (Note in Fig. 5.1 how each CPF solution has two adjacent ones.) The path followed in an iteration is to move along one of these edges from one end to the other. In Fig. 5.1, the first iteration involves moving along the edge from (0, 0) to (0, 6), and then the next iteration moves along the edge from (0, 6) to (2, 6). As Table 5.1 illustrates, each of these moves to an adjacent CPF solution involves just one change in the set of defining equations (constraint boundaries on which the solution lies).

When $n = 3$, the answers are slightly more complicated. To help you visualize what is going on, Fig. 5.2 shows a three-dimensional drawing of a typical feasible region when $n = 3$, where the dots are the CPF solutions. This feasible region is a *polyhedron* rather than the polygon we had with $n = 2$ (Fig. 5.1), because the constraint boundaries now are *planes* rather than lines. The faces of the polyhedron form the *boundary* of the feasible region, where each face is the portion of a constraint boundary that satisfies the other constraints as well. Note that each CPF solution lies at the intersection of three constraint boundaries (sometimes including some of the $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ constraint boundaries for the nonnegativity

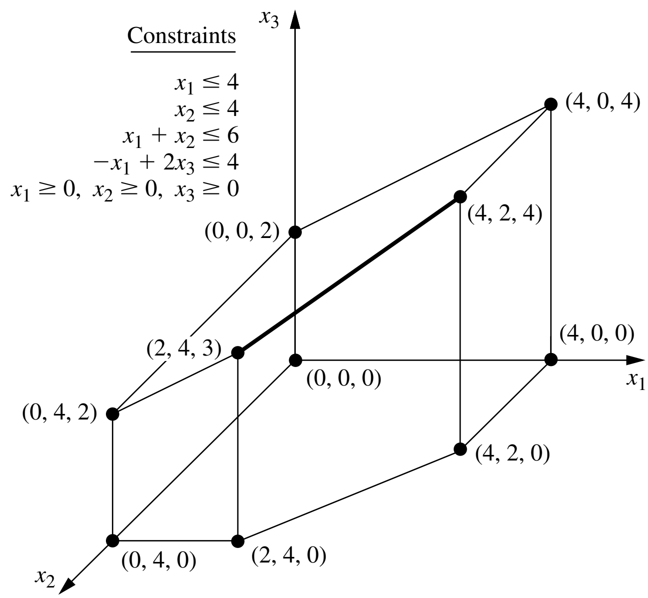


FIGURE 5.2
Feasible region and CPF solutions for a three-variable linear programming problem.

constraints), and the solution also satisfies the other constraints. Such intersections that do not satisfy one or more of the other constraints yield corner-point *infeasible* solutions instead.

The darker line segment in Fig. 5.2 depicts the path of the simplex method on a typical iteration. The point (2, 4, 3) is the *current* CPF solution to begin the iteration, and the point (4, 2, 4) will be the new CPF solution at the end of the iteration. The point (2, 4, 3) lies at the intersection of the $x_2 = 4$, $x_1 + x_2 = 6$, and $-x_1 + 2x_3 = 4$ constraint boundaries, so these three equations are the *defining equations* for this CPF solution. If the $x_2 = 4$ defining equation were removed, the intersection of the other two constraint boundaries (planes) would form a line. One segment of this line, shown as the dark line segment from (2, 4, 3) to (4, 2, 4) in Fig. 5.2, lies on the boundary of the feasible region, whereas the rest of the line is infeasible. This line segment is an edge of the feasible region, and its endpoints (2, 4, 3) and (4, 2, 4) are adjacent CPF solutions.

For $n = 3$, all the *edges* of the feasible region are formed in this way as the feasible segment of the line lying at the intersection of two constraint boundaries, and the two endpoints of an edge are *adjacent* CPF solutions. In Fig. 5.2 there are 15 edges of the feasible region, and so there are 15 pairs of adjacent CPF solutions. For the current CPF solution (2, 4, 3), there are three ways to remove one of its three defining equations to obtain an intersection of the other two constraint boundaries, so there are three edges emanating from (2, 4, 3). These edges lead to (4, 2, 4), (0, 4, 2), and (2, 4, 0), so these are the CPF solutions that are adjacent to (2, 4, 3).

For the next iteration, the simplex method chooses one of these three edges, say, the darker line segment in Fig. 5.2, and then moves along this edge away from (2, 4, 3) until it reaches the first new constraint boundary, $x_1 = 4$, at its other endpoint. [We cannot continue farther along this line to the next constraint boundary, $x_2 = 0$, because this leads to a corner-point infeasible solution—(6, 0, 5).] The intersection of this first new constraint boundary with the two constraint boundaries forming the edge yields the *new* CPF solution (4, 2, 4).

When $n > 3$, these same concepts generalize to higher dimensions, except the constraint boundaries now are *hyperplanes* instead of planes. Let us summarize.

> Consider any linear programming problem with n decision variables and a bounded feasible region. A CPF solution lies at the intersection of n constraint boundaries (and satisfies the other constraints as well). An **edge** of the feasible region is a feasible line segment that lies at the intersection of $n - 1$ constraint boundaries, where each endpoint lies on one additional constraint boundary (so that these endpoints are CPF solutions). Two CPF solutions are **adjacent** if the line segment connecting them is an edge of the feasible region. Emanating from each CPF solution are n such edges, each one leading to one of the n adjacent CPF solutions. Each iteration of the simplex method moves from the current CPF solution to an adjacent one by moving along one of these n edges.

When you shift from a geometric viewpoint to an algebraic one, *intersection of constraint boundaries* changes to *simultaneous solution of constraint boundary equations.* The n constraint boundary equations yielding (defining) a CPF solution are its defining equations, where deleting one of these equations yields a line whose feasible segment is an edge of the feasible region.

We next analyze some key properties of CPF solutions and then describe the implications of all these concepts for interpreting the simplex method. However, while the above summary is fresh in your mind, let us give you a preview of its implications. When the simplex method chooses an entering basic variable, the geometric interpretation is that it is choosing one of the edges emanating from the current CPF solution to move along. Increasing this variable from zero (and simultaneously changing the values of the other basic variables accordingly) corresponds to moving along this edge. Having one of the basic variables (the leaving basic variable) decrease so far that it reaches zero corresponds to reaching the first new constraint boundary at the other end of this edge of the feasible region.

Properties of CPF Solutions

We now focus on three key properties of CPF solutions that hold for *any* linear programming problem that has feasible solutions and a bounded feasible region.

> **Property 1:** (*a*) If there is exactly one optimal solution, then it must be a CPF solution. (*b*) If there are multiple optimal solutions (and a bounded feasible region), then at least two must be adjacent CPF solutions.

Property 1 is a rather intuitive one from a geometric viewpoint. First consider Case (*a*), which is illustrated by the Wyndor Glass Co. problem (see Fig. 5.1) where the one optimal solution (2, 6) is indeed a CPF solution. Note that there is nothing special about this example that led to this result. For any problem having just one optimal solution, it always is possible to keep raising the objective function line (hyperplane) until it just touches one point (the optimal solution) at a corner of the feasible region.

We now give an algebraic proof for this case.

> **Proof of Case (*a*) of Property 1:** We set up a *proof by contradiction* by assuming that there is exactly one optimal solution and that it is *not* a CPF solution.

We then show below that this assumption leads to a contradiction and so cannot be true. (The solution assumed to be optimal will be denoted by $\mathbf{x}^*$, and its objective function value by Z^*.)

Recall the definition of *CPF solution* (a feasible solution that does not lie on any line segment connecting two other feasible solutions). Since we have assumed that the optimal solution $\mathbf{x}^*$ is not a CPF solution, this implies that there must be two other feasible solutions such that the line segment connecting them contains the optimal solution. Let the vectors $\mathbf{x}'$ and $\mathbf{x}''$ denote these two other feasible solutions, and let Z_1 and Z_2 denote their respective objective function values. Like each other point on the line segment connecting $\mathbf{x}'$ and $\mathbf{x}''$,

$$\mathbf{x}^* = \alpha\mathbf{x}'' + (1 - \alpha)\mathbf{x}'$$

for some value of α such that $0 < \alpha < 1$. Thus,

$$Z^* = \alpha Z_2 + (1 - \alpha)Z_1.$$

Since the weights α and $1 - \alpha$ add to 1, the only possibilities for how Z^*, Z_1, and Z_2 compare are (1) $Z^* = Z_1 = Z_2$, (2) $Z_1 < Z^* < Z_2$, and (3) $Z_1 > Z^* > Z_2$. The first possibility implies that $\mathbf{x}'$ and $\mathbf{x}''$ also are optimal, which contradicts the assumption that there is exactly one optimal solution. Both the latter possibilities contradict the assumption that $\mathbf{x}^*$ (not a CPF solution) is optimal. The resulting conclusion is that it is impossible to have a single optimal solution that is not a CPF solution.

Now consider Case (*b*), which was demonstrated in Sec. 3.2 under the definition of *optimal solution* by changing the objective function in the example to $Z = 3x_1 + 2x_2$ (see Fig. 3.5 on page 35). What then happens when you are solving graphically is that the objective function line keeps getting raised until it contains the line segment connecting the two CPF solutions (2, 6) and (4, 3). The same thing would happen in higher dimensions except that an objective function *hyperplane* would keep getting raised until it contained the line segment(s) connecting two (or more) adjacent CPF solutions. As a consequence, *all* optimal solutions can be obtained as weighted averages of optimal CPF solutions. (This situation is described further in Probs. 4.5-5 and 4.5-6.)

The real significance of Property 1 is that it greatly simplifies the search for an optimal solution because now only CPF solutions need to be considered. The magnitude of this simplification is emphasized in Property 2.

Property 2: There are only a *finite* number of CPF solutions.

This property certainly holds in Figs. 5.1 and 5.2, where there are just 5 and 10 CPF solutions, respectively. To see why the number is finite in general, recall that each CPF solution is the simultaneous solution of a system of n out of the $m + n$ constraint boundary equations. The number of different combinations of $m + n$ equations taken n at a time is

$$\binom{m+n}{n} = \frac{(m+n)!}{m!n!},$$

which is a finite number. This number, in turn, in an *upper bound* on the number of CPF solutions. In Fig. 5.1, $m = 3$ and $n = 2$, so there are 10 different systems of two equations, but only half of them yield CPF solutions. In Fig. 5.2, $m = 4$ and $n = 3$, which gives 35 different systems of three equations, but only 10 yield CPF solutions.

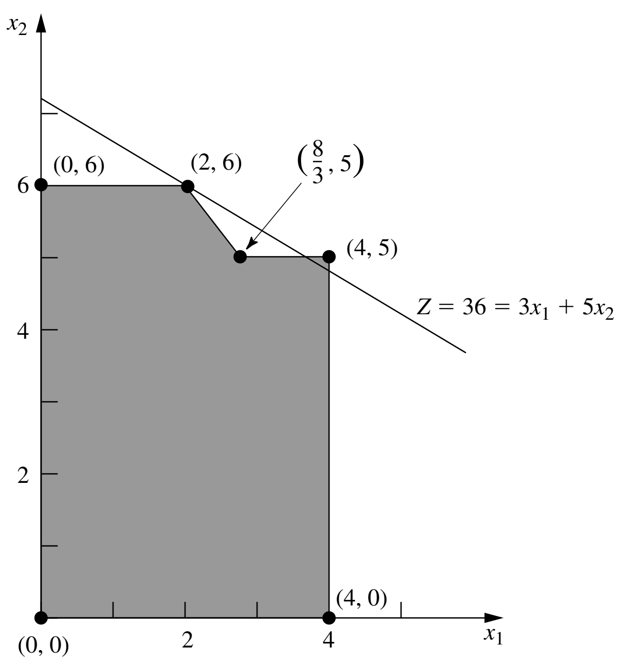
Property 2 suggests that, in principle, an optimal solution can be obtained by exhaustive enumeration; i.e., find and compare all the finite number of CPF solutions. Unfortunately, there are finite numbers, and then there are finite numbers that (for all practical purposes) might as well be infinite. For example, a rather small linear programming problem with only $m = 50$ and $n = 50$ would have $100!/(50!)^2 \approx 10^{29}$ systems of equations to be solved! By contrast, the simplex method would need to examine only approximately 100 CPF solutions for a problem of this size. This tremendous savings can be obtained because of the optimality test given in Sec. 4.1 and restated here as Property 3.

Property 3: If a CPF solution has no *adjacent* CPF solutions that are *better* (as measured by Z), then there are no *better* CPF solutions anywhere. Therefore, such a CPF solution is guaranteed to be an *optimal* solution (by Property 1), assuming only that the problem possesses at least one optimal solution (guaranteed if the problem possesses feasible solutions and a bounded feasible region).

To illustrate Property 3, consider Fig. 5.1 for the Wyndor Glass Co. example. For the CPF solution (2, 6), its adjacent CPF solutions are (0, 6) and (4, 3), and neither has a better value of Z than (2, 6) does. This outcome implies that none of the other CPF solutions—(0, 0) and (4, 0)—can be better than (2, 6), so (2, 6) must be optimal.

By contrast, Fig. 5.3 shows a feasible region that can *never* occur for a linear programming problem but that does violate Property 3. The problem shown is identical to the Wyndor Glass Co. example (including the same objective function) *except* for the en-

FIGURE 5.3
Modification of the Wyndor Glass Co. problem that violates both linear programming and Property 3 for CPF solutions in linear programming.



largement of the feasible region to the right of $(\frac{8}{3}, 5)$. Consequently, the adjacent CPF solutions for (2, 6) now are (0, 6) and $(\frac{8}{3}, 5)$, and again neither is better than (2, 6). However, another CPF solution (4, 5) now is better than (2, 6), thereby violating Property 3. The reason is that the boundary of the feasible region goes down from (2, 6) to $(\frac{8}{3}, 5)$ and then "bends outward" to (4, 5), beyond the objective function line passing through (2, 6).

The key point is that the kind of situation illustrated in Fig. 5.3 can never occur in linear programming. The feasible region in Fig. 5.3 implies that the $2x_2 \leq 12$ and $3x_1 + 2x_2 \leq 18$ constraints apply for $0 \leq x_1 \leq \frac{8}{3}$. However, under the condition that $\frac{8}{3} \leq x_1 \leq 4$, the $3x_1 + 2x_2 \leq 18$ constraint is dropped and replaced by $x_2 \leq 5$. Such "conditional constraints" just are not allowed in linear programming.

The basic reason that Property 3 holds for any linear programming problem is that the feasible region always has the property of being a *convex set,* as defined in Appendix 2 and illustrated in several figures there. For two-variable linear programming problems, this convex property means that the *angle* inside the feasible region at *every* CPF solution is less than 180°. This property is illustrated in Fig. 5.1, where the angles at (0, 0), (0, 6), and (4, 0) are 90° and those at (2, 6) and (4, 3) are between 90° and 180°. By contrast, the feasible region in Fig. 5.3 is *not* a convex set, because the angle at $(\frac{8}{3}, 5)$ is more than 180°. This is the kind of "bending outward" at an angle greater than 180° that can never occur in linear programming. In higher dimensions, the same intuitive notion of "never bending outward" continues to apply.

To clarify the significance of a convex feasible region, consider the objective function hyperplane that passes through a CPF solution that has no adjacent CPF solutions that are better. [In the original Wyndor Glass Co. example, this hyperplane is the objective function line passing through (2, 6).] All these adjacent solutions [(0, 6) and (4, 3) in the example] must lie either on the hyperplane or on the unfavorable side (as measured by Z) of the hyperplane. The feasible region being convex means that its boundary cannot "bend outward" beyond an adjacent CPF solution to give another CPF solution that lies on the favorable side of the hyperplane. So Property 3 holds.

Extensions to the Augmented Form of the Problem

For any linear programming problem in our standard form (including functional constraints in $\leq$ form), the appearance of the functional constraints after slack variables are introduced is as follows:

$$
\begin{array}{lllll}
(1) & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} & & & = b_1 \\
(2) & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + x_{n+2} & & = b_2 \\
\cdots & \cdots & & & \\
(m) & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & + x_{n+m} & = b_m,
\end{array}
$$

where $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are the slack variables. For other linear programming problems, Sec. 4.6 described how essentially this same appearance (proper form from Gaussian elimination) can be obtained by introducing artificial variables, etc. Thus, the original solutions $(x_1, x_2, \ldots, x_n)$ now are augmented by the corresponding values of the slack or artificial variables $(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$ and perhaps some surplus variables as well. This augmentation led in Sec. 4.2 to defining **basic solutions** as *augmented corner-point solutions* and **basic feasible solutions (BF solutions)** as *augmented CPF so-*

lutions. Consequently, the preceding three properties of CPF solutions also hold for BF solutions.

Now let us clarify the algebraic relationships between basic solutions and corner-point solutions. Recall that each corner-point solution is the simultaneous solution of a system of n constraint boundary equations, which we called its *defining equations*. The key question is: How do we tell whether a particular constraint boundary equation is one of the defining equations when the problem is in augmented form? The answer, fortunately, is a simple one. Each constraint has an **indicating variable** that completely indicates (by whether its value is zero) whether that constraint's boundary equation is satisfied by the current solution. A summary appears in Table 5.3. For the type of constraint in each row of the table, note that the corresponding constraint boundary equation (fourth column) is satisfied if and only if this constraint's indicating variable (fifth column) equals zero. In the last row (functional constraint in $\geq$ form), the indicating variable $\bar{x}_{n+i} - x_{s_i}$ actually is the difference between the artificial variable $\bar{x}_{n+i}$ and the surplus variable x_{s_i}.

Thus, whenever a constraint boundary equation is one of the defining equations for a corner-point solution, its indicating variable has a value of zero in the augmented form of the problem. Each such indicating variable is called a *nonbasic variable* for the corresponding basic solution. The resulting conclusions and terminology (already introduced in Sec. 4.2) are summarized next.

> Each **basic solution** has m **basic variables,** and the rest of the variables are **nonbasic variables** set equal to zero. (The number of nonbasic variables equals n plus the number of surplus variables.) The values of the **basic variables** are given by the simultaneous solution of the system of m equations for the problem in augmented form (after the nonbasic variables are set to zero). This basic solution is the augmented corner-point solution whose n defining equations are those indicated by the nonbasic variables. In particular, whenever an indicating variable in the fifth column of Table 5.3 is a nonbasic variable, the constraint boundary equation in the fourth column is a defining equation for the corner-point solution. (For functional constraints in $\geq$ form, at least one of the two supplementary variables $\bar{x}_{n+i}$ and x_{s_i} always is a nonbasic variable, but the constraint boundary equation becomes a defining equation only if *both* of these variables are nonbasic variables.)

TABLE 5.3 Indicating variables for constraint boundary equations*

Type of Constraint	Form of Constraint	Constraint in Augmented Form	Constraint Boundary Equation	Indicating Variable
Nonnegativity	$x_j \geq 0$	$x_j \geq 0$	$x_j = 0$	x_j
Functional ($\leq$)	$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$	$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i$	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	x_{n+i}
Functional ($=$)	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j + \bar{x}_{n+i} = b_i$	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\bar{x}_{n+i}$
Functional ($\geq$)	$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$\sum_{j=1}^{n} a_{ij}x_j + \bar{x}_{n+i} - x_{s_i} = b_i$	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\bar{x}_{n+i} - x_{s_i}$

*Indicating variable $= 0 \Rightarrow$ constraint boundary equation satisfied;
indicating variable $\neq 0 \Rightarrow$ constraint boundary equation violated.

Now consider the basic *feasible* solutions. Note that the only requirements for a solution to be feasible in the augmented form of the problem are that it satisfy the system of equations and that *all* the variables be *nonnegative.*

A **BF solution** is a basic solution where all m basic variables are nonnegative (≥ 0). A BF solution is said to be **degenerate** if any of these m variables equals zero.

Thus, it is possible for a variable to be zero and still not be a nonbasic variable for the current BF solution. (This case corresponds to a CPF solution that satisfies another constraint boundary equation in addition to its n defining equations.) Therefore, it is necessary to keep track of which is the current set of nonbasic variables (or the current set of basic variables) rather than to rely upon their zero values.

We noted earlier that not every system of n constraint boundary equations yields a corner-point solution, because either the system has no solution or it has multiple solutions. For analogous reasons, not every set of n nonbasic variables yields a basic solution. However, these cases are avoided by the simplex method.

To illustrate these definitions, consider the Wyndor Glass Co. example once more. Its constraint boundary equations and indicating variables are shown in Table 5.4.

Augmenting each of the CPF solutions (see Table 5.1) yields the BF solutions listed in Table 5.5. This table places adjacent BF solutions next to each other, except for the pair consisting of the first and last solutions listed. Notice that in each case the nonbasic variables necessarily are the indicating variables for the defining equations. Thus, adjacent BF solutions differ by having just one different nonbasic variable. Also notice that each BF solution is the simultaneous solution of the system of equations for the problem in augmented form (see Table 5.4) when the nonbasic variables are set equal to zero.

Similarly, the three corner-point *infeasible* solutions (see Table 5.2) yield the three basic *infeasible* solutions shown in Table 5.6.

The other two sets of nonbasic variables, (1) x_1 and x_3 and (2) x_2 and x_4, do not yield a basic solution, because setting either pair of variables equal to zero leads to having no solution for the system of Eqs. (1) to (3) given in Table 5.4. This conclusion parallels the observation we made early in this section that the corresponding sets of constraint boundary equations do not yield a solution.

TABLE 5.4 Indicating variables for the constraint boundary equations of the Wyndor Glass Co. problem*

Constraint	Constraint in Augmented Form	Constraint Boundary Equation	Indicating Variable
$x_1 \geq 0$	$x_1 \geq 0$	$x_1 = 0$	x_1
$x_2 \geq 0$	$x_2 \geq 0$	$x_2 = 0$	x_2
$x_1 \leq 4$	(1) $x_1 + x_3 = 4$	$x_1 = 4$	x_3
$2x_2 \leq 12$	(2) $2x_2 + x_4 = 12$	$2x_2 = 12$	x_4
$3x_1 + x_2 \leq 18$	(3) $3x_1 + 2x_2 + x_5 = 18$	$3x_1 + 2x_2 = 18$	x_5

*Indicating variable $= 0 \Rightarrow$ constraint boundary equation satisfied;
indicating variable $\neq 0 \Rightarrow$ constraint boundary equation violated.

TABLE 5.5 BF solutions for the Wyndor Glass Co. problem

CPF Solution	Defining Equations	BF Solution	Nonbasic Variables
(0, 0)	$x_1 = 0$ $x_2 = 0$	(0, 0, 4, 12, 18)	x_1 x_2
(0, 6)	$x_1 = 0$ $2x_2 = 12$	(0, 6, 4, 0, 6)	x_1 x_4
(2, 6)	$2x_2 = 12$ $3x_1 + 2x_2 = 18$	(2, 6, 2, 0, 0)	x_4 x_5
(4, 3)	$3x_1 + 2x_2 = 18$ $x_1 = 4$	(4, 3, 0, 6, 0)	x_5 x_3
(4, 0)	$x_1 = 4$ $x_2 = 0$	(4, 0, 0, 12, 6)	x_3 x_2

The *simplex method* starts at a BF solution and then iteratively moves to a better adjacent BF solution until an optimal solution is reached. At each iteration, how is the adjacent BF solution reached?

For the original form of the problem, recall that an adjacent CPF solution is reached from the current one by (1) deleting one constraint boundary (defining equation) from the set of n constraint boundaries defining the current solution, (2) moving away from the current solution in the feasible direction along the intersection of the remaining $n - 1$ constraint boundaries (an edge of the feasible region), and (3) stopping when the *first* new constraint boundary (defining equation) is reached.

Equivalently, in our new terminology, the simplex method reaches an adjacent BF solution from the current one by (1) deleting one variable (the entering basic variable) from the set of n nonbasic variables defining the current solution, (2) moving away from the current solution by *increasing* this one variable from zero (and adjusting the other basic variables to still satisfy the system of equations) while keeping the remaining $n - 1$ nonbasic variables at zero, and (3) stopping when the *first* of the basic variables (the leaving basic variable) reaches a value of zero (its constraint boundary). With either interpretation, the choice among the n alternatives in step 1 is made by selecting the one that would give the best rate of improvement in Z (per unit increase in the entering basic variable) during step 2.

TABLE 5.6 Basic infeasible solutions for the Wyndor Glass Co. problem

Corner-Point Infeasible Solution	Defining Equations	Basic Infeasible Solution	Nonbasic Variables
(0, 9)	$x_1 = 0$ $3x_1 + 2x_2 = 18$	(0, 9, 4, −6, 0)	x_1 x_5
(4, 6)	$2x_2 = 12$ $x_1 = 4$	(4, 6, 0, 0, −6)	x_4 x_3
(6, 0)	$3x_1 + 2x_2 = 18$ $x_2 = 0$	(6, 0, −2, 12, 0)	x_5 x_2

TABLE 5.7 Sequence of solutions obtained by the simplex method for the Wyndor Glass Co. problem

Iteration	CPF Solution	Defining Equations	BF Solution	Nonbasic Variables	Functional Constraints in Augmented Form
0	(0, 0)	$x_1 = 0$ $x_2 = 0$	(0, 0, 4, 12, 18)	$x_1 = 0$ $x_2 = 0$	$x_1 + \mathbf{x_3} = 4$ $2x_2 + \mathbf{x_4} = 12$ $3x_1 + 2x_2 + \mathbf{x_5} = 18$
1	(0, 6)	$x_1 = 0$ $2x_2 = 12$	(0, 6, 4, 0, 6)	$x_1 = 0$ $x_4 = 0$	$x_1 + \mathbf{x_3} = 4$ $2\mathbf{x_2} + x_4 = 12$ $3x_1 + 2\mathbf{x_2} + \mathbf{x_5} = 18$
2	(2, 6)	$2x_2 = 12$ $3x_1 + 2x_2 = 18$	(2, 6, 2, 0, 0)	$x_4 = 0$ $x_5 = 0$	$\mathbf{x_1} + \mathbf{x_3} = 4$ $2\mathbf{x_2} + x_4 = 12$ $3\mathbf{x_1} + 2\mathbf{x_2} + x_5 = 18$

Table 5.7 illustrates the close correspondence between these geometric and algebraic interpretations of the simplex method. Using the results already presented in Secs. 4.3 and 4.4, the fourth column summarizes the sequence of BF solutions found for the Wyndor Glass Co. problem, and the second column shows the corresponding CPF solutions. In the third column, note how each iteration results in deleting one constraint boundary (defining equation) and substituting a new one to obtain the new CPF solution. Similarly, note in the fifth column how each iteration results in deleting one nonbasic variable and substituting a new one to obtain the new BF solution. Furthermore, the nonbasic variables being deleted and added are the indicating variables for the defining equations being deleted and added in the third column. The last column displays the initial system of equations [excluding Eq. (0)] for the augmented form of the problem, with the current basic variables shown in bold type. In each case, note how setting the nonbasic variables equal to zero and then solving this system of equations for the basic variables must yield the same solution for (x_1, x_2) as the corresponding pair of defining equations in the third column.

5.2 THE REVISED SIMPLEX METHOD

The simplex method as described in Chap. 4 (hereafter called the *original simplex method*) is a straightforward algebraic procedure. However, this way of executing the algorithm (in either algebraic or tabular form) is not the most efficient computational procedure for computers because it computes and stores many numbers that are not needed at the current iteration and that may not even become relevant for decision making at subsequent iterations. The only pieces of information relevant at each iteration are the coefficients of the nonbasic variables in Eq. (0), the coefficients of the entering basic variable in the other equations, and the right-hand sides of the equations. It would be very useful to have a procedure that could obtain this information efficiently without computing and storing all the other coefficients.

As mentioned in Sec. 4.8, these considerations motivated the development of the *revised simplex method.* This method was designed to accomplish exactly the same things as the original simplex method, but in a way that is more efficient for execution on a computer. Thus, it is a streamlined version of the original procedure. It computes and stores only the information that is currently needed, and it carries along the essential data in a more compact form.

The revised simplex method explicitly uses *matrix* manipulations, so it is necessary to describe the problem in matrix notation. (See Appendix 4 for a review of matrices.) To help you distinguish between matrices, vectors, and scalars, we consistently use **BOLDFACE CAPITAL** letters to represent matrices, **boldface lowercase** letters to represent vectors, and *italicized* letters in ordinary print to represent scalars. We also use a boldface zero (**0**) to denote a *null vector* (a vector whose elements all are zero) in either column or row form (which one should be clear from the context), whereas a zero in ordinary print (0) continues to represent the number zero.

Using matrices, our standard form for the general linear programming model given in Sec. 3.2 becomes

$$\begin{array}{|l|}\hline \text{Maximize} \quad Z = \mathbf{cx}, \\ \text{subject to} \\ \mathbf{Ax} \le \mathbf{b} \quad \text{and} \quad \mathbf{x} \ge \mathbf{0}, \\ \hline \end{array}$$

where **c** is the row vector

$$\mathbf{c} = [c_1, c_2, \ldots, c_n],$$

x, **b**, and **0** are the column vectors such that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and **A** is the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

To obtain the *augmented form* of the problem, introduce the column vector of slack variables

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

so that the constraints become

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \qquad \text{and} \qquad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \ge \mathbf{0},$$

where **I** is the $m \times m$ identity matrix, and the null vector **0** now has $n + m$ elements. (We comment at the end of the section about how to deal with problems that are not in our standard form.)

Solving for a Basic Feasible Solution

Recall that the general approach of the simplex method is to obtain a sequence of *improving BF solutions* until an optimal solution is reached. One of the key features of the revised simplex method involves the way in which it solves for each new BF solution after identifying its basic and nonbasic variables. Given these variables, the resulting basic solution is the solution of the m equations

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b},$$

in which the n *nonbasic variables* from the $n + m$ elements of

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix}$$

are set equal to zero. Eliminating these n variables by equating them to zero leaves a set of m equations in m unknowns (the *basic variables*). This set of equations can be denoted by

$$\mathbf{B}\mathbf{x}_B = \mathbf{b},$$

where the **vector of basic variables**

$$\mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix}$$

is obtained by eliminating the nonbasic variables from

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix},$$

and the **basis matrix**

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \dots & \dots & \dots & \dots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{bmatrix}$$

is obtained by eliminating the columns corresponding to coefficients of nonbasic variables from [**A**, **I**]. (In addition, the elements of $\mathbf{x}_B$ and, therefore, the columns of **B** may be placed in a different order when the simplex method is executed.)

The simplex method introduces only basic variables such that **B** is *nonsingular,* so that $\mathbf{B}^{-1}$ always will exist. Therefore, to solve $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, both sides are premultiplied by $\mathbf{B}^{-1}$:

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.$$

Since $\mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$, the desired solution for the basic variables is

$$\boxed{\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.}$$

Let $\mathbf{c}_B$ be the vector whose elements are the objective function coefficients (including zeros for slack variables) for the corresponding elements of $\mathbf{x}_B$. The value of the objective function for this basic solution is then

$$\boxed{Z = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}.}$$

Example. To illustrate this method of solving for a BF solution, consider again the Wyndor Glass Co. problem presented in Sec. 3.1 and solved by the original simplex method in Table 4.8. In this case,

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Referring to Table 4.8, we see that the sequence of BF solutions obtained by the simplex method (original or revised) is the following:

Iteration 0

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0.$$

Iteration 1

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

so

$$\begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 5, 0], \quad \text{so} \quad Z = [0, 5, 0]\begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.$$

Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

so

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 5, 3], \qquad \text{so} \qquad Z = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

Matrix Form of the Current Set of Equations

The last preliminary before we summarize the revised simplex method is to show the matrix form of the set of equations appearing in the simplex tableau for any iteration of the original simplex method.

For the *original* set of equations, the matrix form is

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.$$

This set of equations also is exhibited in the first simplex tableau of Table 5.8.

The algebraic operations performed by the simplex method (multiply an equation by a constant and add a multiple of one equation to another equation) are expressed in ma-

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Coefficient of: Original Variables	Coefficient of: Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	(1, 2, . . . m)	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

trix form by premultiplying both sides of the original set of equations by the appropriate matrix. This matrix would have the same elements as the identity matrix, *except* that each multiple for an algebraic operation would go into the spot needed to have the matrix multiplication perform this operation. Even after a series of algebraic operations over several iterations, we still can deduce what this matrix must be (symbolically) for the entire series by using what we already know about the right-hand sides of the new set of equations. In particular, after any iteration, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$, so the right-hand sides of the new set of equations have become

$$\begin{bmatrix} Z \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

Because we perform the same series of algebraic operations on *both* sides of the original set of operations, we use this same matrix that premultiplies the original right-hand side to premultiply the original left-hand side. Consequently, since

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix},$$

the desired matrix form of the *set of equations after any iteration* is

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

The second simplex tableau of Table 5.8 also exhibits this same set of equations.

Example. To illustrate this matrix form for the current set of equations, we will show how it yields the final set of equations resulting from iteration 2 for the Wyndor Glass Co. problem. Using the $\mathbf{B}^{-1}$ and $\mathbf{c}_B$ given for iteration 2 at the end of the preceding subsection, we have

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [0, \tfrac{3}{2}, 1],$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = [0, 0].$$

Also, by using the values of $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ calculated at the end of the preceding subsection, these results give the following set of equations:

$$\left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ \hline 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix},$$

as shown in the final simplex tableau in Table 4.8.

The Overall Procedure

There are two key implications from the matrix form of the current set of equations shown at the bottom of Table 5.8. The first is that *only* $\mathbf{B}^{-1}$ needs to be derived to be able to calculate all the numbers in the simplex tableau from the original parameters ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}_B$) of the problem. (This implication is the essence of the **fundamental insight** described in the next section.) The second is that *any one* of these numbers can be obtained *individually,* usually by performing *only* a vector multiplication (one row times one column) instead of a complete matrix multiplication. Therefore, the *required numbers* to perform an iteration of the simplex method can be obtained as needed *without* expending the computational effort to obtain *all* the numbers. These two key implications are incorporated into the following summary of the overall procedure.

Summary of the Revised Simplex Method.

1. *Initialization:* Same as for the original simplex method.

2. *Iteration:*

Step 1 Determine the entering basic variable: Same as for the original simplex method.

Step 2 Determine the leaving basic variable: Same as for the original simplex method, except calculate only the numbers required to do this [the coefficients of the entering basic variable in every equation but Eq. (0), and then, for each strictly positive coefficient, the right-hand side of that equation].[1]

Step 3 Determine the new BF solution: Derive $\mathbf{B}^{-1}$ and set $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

3. *Optimality test:* Same as for the original simplex method, except calculate only the numbers required to do this test, i.e., the coefficients of the *nonbasic variables* in Eq. (0).

In step 3 of an iteration, $\mathbf{B}^{-1}$ could be derived each time by using a standard computer routine for inverting a matrix. However, since $\mathbf{B}$ (and therefore $\mathbf{B}^{-1}$) changes so little from one iteration to the next, it is much more efficient to derive the new $\mathbf{B}^{-1}$ (denote it by $\mathbf{B}_{\text{new}}^{-1}$) from the $\mathbf{B}^{-1}$ at the preceding iteration (denote it by $\mathbf{B}_{\text{old}}^{-1}$). (For the initial BF solution,

[1]Because the value of $\mathbf{x}_B$ is the entire vector of right-hand sides except for Eq. (0), the relevant right-hand sides need not be calculated here if $\mathbf{x}_B$ was calculated in step 3 of the preceding iteration.

$\mathbf{B} = \mathbf{I} = \mathbf{B}^{-1}$.) One method for doing this derivation is based directly upon the interpretation of the elements of $\mathbf{B}^{-1}$ [the coefficients of the slack variables in the current Eqs. (1), (2), . . . , (m)] presented in the next section, as well as upon the procedure used by the original simplex method to obtain the new set of equations from the preceding set.

To describe this method formally, let

x_k = entering basic variable,

a'_{ik} = coefficient of x_k in current Eq. (i), for $i = 1, 2, \ldots, m$ (calculated in step 2 of an iteration),

r = number of equation containing the leaving basic variable.

Recall that the new set of equations [excluding Eq. (0)] can be obtained from the preceding set by subtracting a'_{ik}/a'_{rk} times Eq. (r) from Eq. (i), for all $i = 1, 2, \ldots, m$ except $i = r$, and then dividing Eq. (r) by a'_{rk}. Therefore, the element in row i and column j of $\mathbf{B}_{\text{new}}^{-1}$ is

$$(\mathbf{B}_{\text{new}}^{-1})_{ij} = \begin{cases} (\mathbf{B}_{\text{old}}^{-1})_{ij} - \dfrac{a'_{ik}}{a'_{rk}}(\mathbf{B}_{\text{old}}^{-1})_{rj} & \text{if } i \neq r, \\ \dfrac{1}{a'_{rk}}(\mathbf{B}_{\text{old}}^{-1})_{rj} & \text{if } i = r. \end{cases}$$

These formulas are expressed in matrix notation as

$$\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1},$$

where matrix $\mathbf{E}$ is an identity matrix except that its rth column is replaced by the vector

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}, \qquad \text{where} \qquad \eta_i = \begin{cases} -\dfrac{a'_{ik}}{a'_{rk}} & \text{if } i \neq r, \\ \dfrac{1}{a'_{rk}} & \text{if } i = r. \end{cases}$$

Thus, $\mathbf{E} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{r-1}, \boldsymbol{\eta}, \mathbf{U}_{r+1}, \ldots, \mathbf{U}_m]$, where the m elements of each of the $\mathbf{U}_i$ column vectors are 0 except for a 1 in the ith position.

Example. We shall illustrate the revised simplex method by applying it to the Wyndor Glass Co. problem. The initial basic variables are the slack variables

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Iteration 1

Because the initial $\mathbf{B}^{-1} = \mathbf{I}$, no calculations are needed to obtain the numbers required to identify the entering basic variable x_2 ($-c_2 = -5 < -3 = -c_1$) and the leaving basic variable x_4 ($a_{12} = 0$, $b_2/a_{22} = \frac{12}{2} < \frac{18}{2} = b_3/a_{32}$, so $r = 2$). Thus, the new set of basic variables is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}.$$

To obtain the new $\mathbf{B}^{-1}$,

$$\boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a_{12}}{a_{22}} \\ \dfrac{1}{a_{22}} \\ -\dfrac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{2} \\ -1 \end{bmatrix},$$

so

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

so that

$$\mathbf{x}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}.$$

To test whether this solution is optimal, we calculate the coefficients of the nonbasic variables (x_1 and x_4) in Eq. (0). Performing only the relevant parts of the matrix multiplications, we obtain

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix} - [3, —] = [-3, —],$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 0] \begin{bmatrix} — & 0 & — \\ — & \frac{1}{2} & — \\ — & -1 & — \end{bmatrix} = [—, \tfrac{5}{2}, —],$$

so the coefficients of x_1 and x_4 are -3 and $\frac{5}{2}$, respectively. Since x_1 has a negative coefficient, this solution is *not* optimal.

Iteration 2

Using these coefficients of the nonbasic variables in Eq. (0), since only x_1 has a negative coefficient, we begin the next iteration by identifying x_1 as the entering basic variable. To determine the leaving basic variable, we must calculate the other coefficients of x_1:

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix} = \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix}.$$

By using the *right side* column for the current BF solution (the value of $\mathbf{x}_B$) just given for iteration 1, the ratios $4/1 > 6/3$ indicate that x_5 is the leaving basic variable, so the new set of basic variables is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \qquad \text{with} \qquad \boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a'_{11}}{a'_{31}} \\ -\dfrac{a'_{21}}{a'_{31}} \\ \dfrac{1}{a'_{31}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{3} \\ 0 \\ \dfrac{1}{3} \end{bmatrix}.$$

Therefore, the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

so that

$$\mathbf{x}_B = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}.$$

Applying the optimality test, we find that the coefficients of the nonbasic variables (x_4 and x_5) in Eq. (0) are

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} — & \frac{1}{3} & -\frac{1}{3} \\ — & \frac{1}{2} & 0 \\ — & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [—, \tfrac{3}{2}, 1].$$

Because both coefficients ($\frac{3}{2}$ and 1) are nonnegative, the current solution ($x_1 = 2$, $x_2 = 6$, $x_3 = 2$, $x_4 = 0$, $x_5 = 0$) is optimal and the procedure terminates.

General Observations

The preceding discussion was limited to the case of linear programming problems fitting our standard form given in Sec. 3.2. However, the modifications for other forms are relatively straightforward. The initialization would be conducted just as it would for the original simplex method (see Sec. 4.6). When this step involves introducing artificial variables to obtain an initial BF solution (and thereby to obtain an *identity matrix* as the *initial basis matrix*), these variables are included among the m elements of x_s.

Let us summarize the advantages of the revised simplex method over the original simplex method. One advantage is that the number of arithmetic computations may be reduced. This is especially true when the **A** matrix contains a large number of zero elements (which is usually the case for the large problems arising in practice). The amount of information that must be stored at each iteration is less, sometimes considerably so. The revised simplex method also permits the control of the rounding errors inevitably generated by computers. This control can be exercised by periodically obtaining the current $\mathbf{B}^{-1}$ by directly inverting $\mathbf{B}$. Furthermore, some of the postoptimality analysis problems discussed in Sec. 4.7 can be handled more conveniently with the revised simplex method. For all these reasons, the revised simplex method is usually preferable to the original simplex method for computer execution.

5.3 A FUNDAMENTAL INSIGHT

We shall now focus on a property of the simplex method (in any form) that has been revealed by the revised simplex method in the preceding section.[1] This fundamental insight provides the key to both duality theory and sensitivity analysis (Chap. 6), two very important parts of linear programming.

The insight involves the coefficients of the *slack* variables and the information they give. It is a direct result of the initialization, where the ith slack variable x_{n+i} is given a coefficient of $+1$ in Eq. (i) and a coefficient of 0 in *every other equation* [including Eq. (0)] for $i = 1, 2, \ldots, m$, as shown by the null vector $\mathbf{0}$ and the identity matrix $\mathbf{I}$ in the *slack variables* column for iteration 0 in Table 5.8. (For most of this section, we are assuming that the problem is in *our standard form,* with $b_i \geq 0$ for all $i = 1, 2, \ldots, m$, so that no additional adjustments are needed in the initialization.) The other key factor is that subsequent iterations change the initial equations *only* by

1. Multiplying (or dividing) an *entire* equation by a nonzero constant
2. Adding (or subtracting) a multiple of one *entire* equation to another *entire* equation

As already described in the preceding section, a sequence of these kinds of elementary algebraic operations is equivalent to premultiplying the initial simplex tableau by some matrix. (See Appendix 4 for a review of matrices.) The consequence can be summarized as follows.

Verbal description of fundamental insight: After any iteration, the coefficients of the *slack* variables in each equation immediately reveal how that equation has been obtained from the *initial* equations.

As one example of the importance of this insight, recall from Table 5.8 that the matrix formula for the optimal solution obtained by the simplex method is

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b},$$

where $\mathbf{x}_B$ is the vector of basic variables, $\mathbf{B}^{-1}$ is the matrix of coefficients of slack variables for rows 1 to m of the final tableau, and $\mathbf{b}$ is the vector of original right-hand sides (resource availabilities). (We soon will denote this particular $\mathbf{B}^{-1}$ by $\mathbf{S}^*$.) Postoptimality analysis normally includes an investigation of possible changes in $\mathbf{b}$. By using this formula, you can see exactly how the optimal BF solution changes (or whether it becomes infeasible because of negative variables) as a function of $\mathbf{b}$. You do *not* have to reapply the simplex method over and over for each new $\mathbf{b}$, because the coefficients of the slack

[1]However, since some instructors do not cover the preceding section, we have written this section in a way that can be understood without first reading Sec. 5.2. It is helpful to take a brief look at the matrix notation introduced at the beginning of Sec. 5.2, including the resulting key equation, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

variables tell all! In a similar fashion, this fundamental insight provides a tremendous computational saving for the rest of sensitivity analysis as well.

To spell out the how and the why of this insight, let us look again at the Wyndor Glass Co. example. (The OR Tutor also includes another demonstration example.)

Example. Table 5.9 shows the relevant portion of the simplex tableau for demonstrating this fundamental insight. Light lines have been drawn around the coefficients of the slack variables in all the tableaux in this table because these are the crucial coefficients for applying the insight. To avoid clutter, we then identify the pivot row and pivot column by a single box around the pivot number only.

Iteration 1

To demonstrate the fundamental insight, our focus is on the algebraic operations performed by the simplex method while using Gaussian elimination to obtain the new BF solution. If we do all the algebraic operations with the *old* row 2 (the pivot row) rather than the new one, then the algebraic operations spelled out in Chap. 4 for iteration 1 are

$$\begin{aligned} \text{New row 0} &= \text{old row 0} + \left(\tfrac{5}{2}\right)(\text{old row 2}), \\ \text{New row 1} &= \text{old row 1} + (0)(\text{old row 2}), \\ \text{New row 2} &= \left(\tfrac{1}{2}\right)(\text{old row 2}), \\ \text{New row 3} &= \text{old row 3} + (-1)(\text{old row 2}). \end{aligned}$$

TABLE 5.9 **Simplex tableaux without leftmost columns for the Wyndor Glass Co. problem**

Iteration	Coefficient of: x_1	x_2	x_3	x_4	x_5	Right Side
0	−3	−5	0	0	0	0
	1	0	1	0	0	4
	0	[2]	0	1	0	12
	3	2	0	0	1	18
1	−3	0	0	$\frac{5}{2}$	0	30
	1	0	1	0	0	4
	0	1	0	$\frac{1}{2}$	0	6
	[3]	0	0	−1	1	6
2	0	0	0	$\frac{3}{2}$	1	36
	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	0	1	0	$\frac{1}{2}$	0	6
	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Ignoring row 0 for the moment, we see that these algebraic operations amount to premultiplying rows 1 to 3 of the initial tableau by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Rows 1 to 3 of the initial tableau are

$$\text{Old rows 1–3} = \left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right],$$

where the third, fourth, and fifth columns (the coefficients of the slack variables) form an *identity matrix.* Therefore,

$$\text{New rows 1–3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 3 & 0 & 0 & -1 & 1 & 6 \end{array}\right].$$

Note how the first matrix is reproduced exactly in the box below it as the coefficients of the slack variables in rows 1 to 3 of the new tableau, because the coefficients of the slack variables in rows 1 to 3 of the initial tableau form an identity matrix. Thus, just as stated in the verbal description of the fundamental insight, the coefficients of the slack variables in the new tableau do indeed provide a record of the algebraic operations performed.

This insight is not much to get excited about after just one iteration, since you can readily see from the initial tableau what the algebraic operations had to be, but it becomes invaluable after all the iterations are completed.

For row 0, the algebraic operation performed amounts to the following matrix calculations, where now our focus is on the vector $[0, \frac{5}{2}, 0]$ that premultiplies rows 1 to 3 of the initial tableau.

$$\text{New row 0} = [-3, \quad -5 \quad 0, \quad 0, \quad 0 \quad 0] + [0, \quad \tfrac{5}{2}, \quad 0] \left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right]$$

$$= [-3, \quad 0, \quad \boxed{0, \quad \tfrac{5}{2}, \quad 0,} \quad 30].$$

Note how this vector is reproduced exactly in the box below it as the coefficients of the slack variables in row 0 of the new tableau, just as was claimed in the statement of the fundamental insight. (Once again, the reason is the identity matrix for the coefficients of the slack variables in rows 1 to 3 of the initial tableau, along with the zeros for these coefficients in row 0 of the initial tableau.)

Iteration 2

The algebraic operations performed on the second tableau of Table 5.9 for iteration 2 are

$$\begin{aligned}
\text{New row 0} &= \text{old row 0} + (1)(\text{old row 3}),\\
\text{New row 1} &= \text{old row 1} + (-\tfrac{1}{3})(\text{old row 3}),\\
\text{New row 2} &= \text{old row 2} + (0)(\text{old row 3}),\\
\text{New row 3} &= (\tfrac{1}{3})(\text{old row 3}).
\end{aligned}$$

Ignoring row 0 for the moment, we see that these operations amount to premultiplying rows 1 to 3 of this tableau by the matrix

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Writing this second tableau as the matrix product shown for iteration 1 (namely, the corresponding matrix times rows 1 to 3 of the initial tableau) then yields

$$\text{Final rows 1–3} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right]$$

$$= \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right]$$

$$= \begin{bmatrix} 0 & 0 & \boxed{\begin{matrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{matrix}} & \begin{matrix} 2 \\ 6 \\ 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} & & \end{bmatrix}.$$

The first two matrices shown on the first line of these calculations summarize the algebraic operations of the second and first iterations, respectively. Their product, shown as the first matrix on the second line, then combines the algebraic operations of the two iterations. Note how this matrix is reproduced exactly in the box below it as the coefficients of the slack variables in rows 1 to 3 of the new (final) tableau shown on the third line. What this portion of the tableau reveals is how the *entire* final tableau (except row 0) has been obtained from the initial tableau, namely,

$$\begin{aligned}
\text{Final row 1} &= (1)(\text{initial row 1}) + (\tfrac{1}{3})(\text{initial row 2}) + (-\tfrac{1}{3})(\text{initial row 3}),\\
\text{Final row 2} &= (0)(\text{initial row 1}) + (\tfrac{1}{2})(\text{initial row 2}) + (0)(\text{initial row 3}),\\
\text{Final row 3} &= (0)(\text{initial row 1}) + (-\tfrac{1}{3})(\text{initial row 2}) + (\tfrac{1}{3})(\text{initial row 3}).
\end{aligned}$$

To see why these multipliers of the initial rows are correct, you would have to trace through all the algebraic operations of both iterations. For example, why does final row 1 include $(\frac{1}{3})$(initial row 2), even though a multiple of row 2 has never been added directly to row 1? The reason is that initial row 2 was subtracted from initial row 3 in iteration 1, and then $(\frac{1}{3})$(old row 3) was subtracted from old row 1 in iteration 2.

However, there is no need for you to trace through. Even when the simplex method has gone through hundreds or thousands of iterations, the coefficients of the slack variables in the final tableau will reveal how this tableau has been obtained from the initial tableau. Furthermore, the same algebraic operations would give these same coefficients even if the values of some of the parameters in the original model (initial tableau) were changed, so these coefficients also reveal how the *rest* of the final tableau changes with changes in the initial tableau.

To complete this story for row 0, the fundamental insight reveals that the entire final row 0 can be calculated from the initial tableau by using just the coefficients of the slack variables in the final row 0—$[0, \frac{3}{2}, 1]$. This calculation is shown below, where the first vector is row 0 of the initial tableau and the matrix is rows 1 to 3 of the initial tableau.

$$\text{Final row } 0 = [-3, \quad -5 \quad 0, \quad 0, \quad 0 \quad 0] + [0, \quad \tfrac{3}{2}, \quad 1]\left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right]$$

$$= [0, \quad 0, \quad \boxed{0, \quad \tfrac{3}{2}, \quad 1,} \quad 36].$$

Note again how the vector premultiplying rows 1 to 3 of the initial tableau is reproduced exactly as the coefficients of the slack variables in the final row 0. These quantities must be identical because of the coefficients of the slack variables in the initial tableau (an identity matrix below a null vector). This conclusion is the row 0 part of the fundamental insight.

Mathematical Summary

Because its primary applications involve the *final* tableau, we shall now give a general mathematical expression for the fundamental insight just in terms of this tableau, using matrix notation. If you have not read Sec. 5.2, you now need to know that the *parameters* of the model are given by the matrix $\mathbf{A} = \|a_{ij}\|$ and the vectors $\mathbf{b} = \|b_i\|$ and $\mathbf{c} = \|c_j\|$, as displayed at the beginning of that section.

The only other notation needed is summarized and illustrated in Table 5.10. Notice how vector $\mathbf{t}$ (representing row 0) and matrix $\mathbf{T}$ (representing the other rows) together correspond to the rows of the initial tableau in Table 5.9, whereas vector $\mathbf{t}^*$ and matrix $\mathbf{T}^*$ together correspond to the rows of the final tableau in Table 5.9. This table also shows these vectors and matrices partitioned into three parts: the coefficients of the original variables, the coefficients of the slack variables (our focus), and the right-hand side. Once again, the notation distinguishes between parts of the initial tableau and the final tableau by using an asterisk only in the latter case.

For the coefficients of the slack variables (the middle part) in the initial tableau of Table 5.10, notice the null vector $\mathbf{0}$ in row 0 and the identity matrix $\mathbf{I}$ below, which provide the keys for the fundamental insight. The vector and matrix in the same location of the final tableau, $\mathbf{y}^*$ and $\mathbf{S}^*$, then play a prominent role in the equations for the fundamental insight. $\mathbf{A}$ and $\mathbf{b}$ in the initial tableau turn into $\mathbf{A}^*$ and $\mathbf{b}^*$ in the final tableau. For row 0 of the final tableau, the coefficients of the decision variables are $\mathbf{z}^* - \mathbf{c}$ (so the vector $\mathbf{z}^*$ is what has been added to the vector of initial coefficients, $-\mathbf{c}$), and the right-hand side Z^* denotes the optimal value of Z.

TABLE 5.10 General notation for initial and final simplex tableaux in matrix form, illustrated by the Wyndor Glass Co. problem

Initial Tableau	
Row 0:	$\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0]$.
Other rows:	$\mathbf{T} = \left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}]$.
Combined:	$\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c:c:c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right]$.
Final Tableau	
Row 0:	$\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*]$.
Other rows:	$\mathbf{T}^* = \left[\begin{array}{cc:ccc:c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*]$.
Combined:	$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c:c:c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right]$.

It is helpful at this point to look back at Table 5.8 in Sec. 5.2 and compare it with Table 5.10. (If you haven't previously studied Sec. 5.2, you will need to read the definition of the basis matrix $\mathbf{B}$ and the vectors $\mathbf{x}_B$ and $\mathbf{c}_B$ given early in that section before looking at Table 5.8.) The notation for the components of the *initial* simplex tableau is the same in the two tables. The lower part of Table 5.8 shows *any* later simplex tableau in matrix form, whereas the lower part of Table 5.10 gives the *final* tableau in matrix form. Note that the matrix $\mathbf{B}^{-1}$ in Table 5.8 is in the same location as $\mathbf{S}^*$ in Table 5.10. Thus,

$$\mathbf{S}^* = \mathbf{B}^{-1}$$

when $\mathbf{B}$ is the basis matrix for the *optimal* solution found by the simplex method.

Referring to Table 5.10 again, suppose now that you are given the initial tableau, $\mathbf{t}$ and $\mathbf{T}$, and just $\mathbf{y}^*$ and $\mathbf{S}^*$ from the final tableau. How can this information alone be used to calculate the rest of the final tableau? The answer is provided by Table 5.8. This table includes some information that is not directly relevant to our current discussion, namely, how $\mathbf{y}^*$ and $\mathbf{S}^*$ themselves can be calculated ($\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$ and $\mathbf{S}^* = \mathbf{B}^{-1}$) by knowing the set of basic variables and so the basis matrix $\mathbf{B}$ for the optimal solution found by the simplex method. However, the lower part of this table also shows how the rest of the final tableau can be obtained from the coefficients of the slack variables, which is summarized as follows.

Fundamental Insight

(1) $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \quad \mathbf{y}^* \quad \mathbf{y}^*\mathbf{b}]$.
(2) $\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \quad \mathbf{S}^* \quad \mathbf{S}^*\mathbf{b}]$.

Thus, by knowing the parameters of the model in the initial tableau (**c**, **A**, and **b**) and *only* the coefficients of the slack variables in the final tableau ($\mathbf{y}^*$ and $\mathbf{S}^*$), these equations enable calculating *all* the other numbers in the final tableau.

We already used these two equations when dealing with iteration 2 for the Wyndor Glass Co. problem in the preceding subsection. In particular, the right-hand side of the expression for final row 0 for iteration 2 is just $\mathbf{t} + \mathbf{y}^*\mathbf{T}$, and the second line of the expression for final rows 1 to 3 is just $\mathbf{S}^*\mathbf{T}$.

Now let us summarize the mathematical logic behind the two equations for the fundamental insight. To derive Eq. (2), recall that the entire sequence of algebraic operations performed by the simplex method (excluding those involving row 0) is equivalent to premultiplying **T** by some matrix, call it **M**. Therefore,

$$\mathbf{T}^* = \mathbf{MT},$$

but now we need to identify **M**. By writing out the component parts of **T** and $\mathbf{T}^*$, this equation becomes

$$\begin{aligned}[\mathbf{A}^* \;\; \mathbf{S}^* \;\; \mathbf{b}^*] &= \mathbf{M}\,[\mathbf{A} \;\; \mathbf{I} \;\; \mathbf{b}] \\ &= [\mathbf{MA} \;\; \mathbf{M} \;\; \mathbf{Mb}].\end{aligned}$$

Because the middle (or any other) component of these equal matrices must be the same, it follows that $\mathbf{M} = \mathbf{S}^*$, so Eq. (2) is a valid equation.

Equation (1) is derived in a similar fashion by noting that the entire sequence of algebraic operations involving row 0 amounts to adding some linear combination of the rows in **T** to **t**, which is equivalent to adding to **t** some *vector* times **T**. Denoting this vector by **v**, we thereby have

$$\mathbf{t}^* = \mathbf{t} + \mathbf{vT},$$

but **v** still needs to be identified. Writing out the component parts of **t** and $\mathbf{t}^*$ yields

$$\begin{aligned}[\mathbf{z}^* - \mathbf{c} \;\; \mathbf{y}^* \;\; Z^*] &= [-\mathbf{c} \;\; \mathbf{0} \;\; 0] + \mathbf{v}\,[\mathbf{A} \;\; \mathbf{I} \;\; \mathbf{b}] \\ &= [-\mathbf{c} + \mathbf{vA} \;\; \mathbf{v} \;\; \mathbf{vb}].\end{aligned}$$

Equating the middle component of these equal vectors gives $\mathbf{v} = \mathbf{y}^*$, which validates Eq. (1).

Adapting to Other Model Forms

Thus far, the fundamental insight has been described under the assumption that the original model is in our standard form, described in Sec. 3.2. However, the above mathematical logic now reveals just what adjustments are needed for other forms of the original model. The key is the identity matrix **I** in the initial tableau, which turns into $\mathbf{S}^*$ in the final tableau. If some artificial variables must be introduced into the initial tableau to serve as initial basic variables, then it is the set of columns (appropriately ordered) for *all* the initial basic variables (both slack and artificial) that forms **I** in this tableau. (The columns for any surplus variables are extraneous.) The *same* columns in the final tableau provide $\mathbf{S}^*$ for the $\mathbf{T}^* = \mathbf{S}^*\mathbf{T}$ equation and $\mathbf{y}^*$ for the $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T}$ equation. If *M*'s were introduced into the preliminary row 0 as coefficients for artificial variables, then the **t** for the $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T}$ equation is the row 0 for the initial tableau after these nonzero coefficients for basic variables are algebraically eliminated. (Alternatively, the preliminary row 0 can be used for **t**, but then these M's must be subtracted from the final row 0 to give $\mathbf{y}^*$.) (See Prob. 5.3-11.)

Applications

The fundamental insight has a variety of important applications in linear programming. One of these applications involves the revised simplex method. As described in the preceding section (see Table 5.8), this method used $\mathbf{B}^{-1}$ and the initial tableau to calculate all the relevant numbers in the current tableau for *every* iteration. It goes even further than the fundamental insight by using $\mathbf{B}^{-1}$ to calculate $\mathbf{y}^*$ itself as $\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$.

Another application involves the interpretation of the *shadow prices* $(y_1^*, y_2^*, \ldots, y_m^*)$ described in Sec. 4.7. The fundamental insight reveals that Z^* (the value of Z for the optimal solution) is

$$Z^* = \mathbf{y}^*\mathbf{b} = \sum_{i=1}^{m} y_i^* b_i,$$

so, e.g.,

$$Z^* = 0b_1 + \frac{3}{2}b_2 + b_3$$

for the Wyndor Glass Co. problem. This equation immediately yields the interpretation for the y_i^* values given in Sec. 4.7.

Another group of extremely important applications involves various *postoptimality tasks* (reoptimization technique, sensitivity analysis, parametric linear programming—described in Sec. 4.7) that investigate the effect of making one or more changes in the original model. In particular, suppose that the simplex method already has been applied to obtain an optimal solution (as well as $\mathbf{y}^*$ and $\mathbf{S}^*$) for the original model, and then these changes are made. If exactly the same sequence of algebraic operations were to be applied to the revised initial tableau, what would be the resulting changes in the final tableau? Because $\mathbf{y}^*$ and $\mathbf{S}^*$ don't change, the fundamental insight reveals the answer immediately.

For example, consider the change from $b_2 = 12$ to $b_2 = 13$ as illustrated in Fig. 4.8 for the Wyndor Glass Co. problem. It is not necessary to *solve* for the new optimal solution $(x_1, x_2) = (\frac{5}{3}, \frac{13}{2})$ because the values of the basic variables in the final tableau ($\mathbf{b}^*$) are immediately revealed by the fundamental insight:

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \mathbf{b}^* = \mathbf{S}^*\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 13 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{13}{2} \\ \frac{5}{3} \end{bmatrix}.$$

There is an even easier way to make this calculation. Since the only change is in the *second* component of **b** ($\Delta b_2 = 1$), which gets premultiplied by only the *second* column of $\mathbf{S}^*$, the *change* in $\mathbf{b}^*$ can be calculated as simply

$$\Delta\mathbf{b}^* = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} \Delta b_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix},$$

so the original values of the basic variables in the final tableau ($x_3 = 2$, $x_2 = 6$, $x_1 = 2$) now become

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{13} \\ \frac{13}{2} \\ \frac{5}{3} \end{bmatrix}.$$

(If any of these new values were *negative,* and thus infeasible, then the reoptimization technique described in Sec. 4.7 would be applied, starting from this revised final tableau.) Applying *incremental analysis* to the preceding equation for Z^* also immediately yields

$$\Delta Z^* = \frac{3}{2}\Delta b_2 = \frac{3}{2}.$$

The fundamental insight can be applied to investigating other kinds of changes in the original model in a very similar fashion; it is the crux of the sensitivity analysis procedure described in the latter part of Chap. 6.

You also will see in the next chapter that the fundamental insight plays a key role in the very useful duality theory for linear programming.

5.4 CONCLUSIONS

Although the simplex method is an algebraic procedure, it is based on some fairly simple geometric concepts. These concepts enable one to use the algorithm to examine only a relatively small number of BF solutions before reaching and identifying an optimal solution.

Chapter 4 describes how *elementary algebraic operations* are used to execute the *algebraic form* of the simplex method, and then how the *tableau form* of the simplex method uses the equivalent *elementary row operations* in the same way. Studying the simplex method in these forms is a good way of getting started in learning its basic concepts. However, these forms of the simplex method do not provide the most efficient form for execution on a computer. *Matrix operations* are a faster way of combining and executing elementary algebraic operations or row operations. Therefore, by using the *matrix form* of the simplex method, the revised simplex method provides an effective way of adapting the simplex method for computer implementation.

The final simplex tableau includes complete information on how it can be algebraically reconstructed directly from the initial simplex tableau. This fundamental insight has some very important applications, especially for postoptimality analysis.

SELECTED REFERENCES

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2. Dantzig, G. B., and M. N. Thapa: *Linear Programming 1: Introduction,* Springer, New York, 1997.
3. Schriver, A: *Theory of Linear and Integer Programming,* Wiley, New York, 1986.
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LEARNING AIDS FOR THIS CHAPTER IN YOUR OR COURSEWARE

A Demonstration Example in OR Tutor:

Fundamental Insight

Interactive Routines:

Enter or Revise a General Linear Programming Model
Set Up for the Simplex Method—Interactive Only
Solve Interactively by the Simplex Method

Files (Chapter 3) for Solving the Wyndor Example:

Excel File
LINGO/LINDO File
MPL/CPLEX File

See Appendix 1 for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The demonstration example listed above may be helpful.
I: You can check some of your work by using the interactive routines listed above for the original simplex method.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

5.1-1.* Consider the following problem.

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$\begin{aligned} 2x_1 + x_2 &\le 6 \\ x_1 + 2x_2 &\le 6 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

(a) Solve this problem graphically. Identify the CPF solutions by circling them on the graph.
(b) Identify all the sets of two defining equations for this problem. For each set, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or corner-point infeasible solution.
(c) Introduce slack variables in order to write the functional constraints in augmented form. Use these slack variables to identify the basic solution that corresponds to each corner-point solution found in part (*b*).
(d) Do the following for *each* set of two defining equations from part (*b*): Identify the indicating variable for each defining equation. Display the set of equations from part (*c*) *after* deleting these two indicating (nonbasic) variables. Then use the latter set of equations to solve for the two remaining variables (the basic variables). Compare the resulting basic solution to the corresponding basic solution obtained in part (*c*).
(e) Without executing the simplex method, use its geometric interpretation (and the objective function) to identify the path (sequence of CPF solutions) it would follow to reach the optimal solution. For each of these CPF solutions in turn, identify the following decisions being made for the next iteration: (*i*) which defining equation is being deleted and which is being added; (*ii*) which indicating variable is being deleted (the entering basic variable) and which is being added (the leaving basic variable).

5.1-2. Repeat Prob. 5.1-1 for the model in Prob. 3.1-5.

5.1-3. Consider the following problem.

Maximize $Z = 2x_1 + 3x_2$,

subject to

$$\begin{aligned} -3x_1 + x_2 &\le 1 \\ 4x_1 + 2x_2 &\le 20 \\ 4x_1 - x_2 &\le 10 \\ -x_1 + 2x_2 &\le 5 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

(a) Solve this problem graphically. Identify the CPF solutions by circling them on the graph.
(b) Develop a table giving each of the CPF solutions and the corresponding defining equations, BF solution, and nonbasic variables. Calculate Z for each of these solutions, and use just this information to identify the optimal solution.
(c) Develop the corresponding table for the corner-point infeasible solutions, etc. Also identify the sets of defining equations and nonbasic variables that do not yield a solution.

5.1-4. Consider the following problem.

Maximize $Z = 2x_1 - x_2 + x_3,$

subject to

$$3x_1 + x_2 + x_3 \le 60$$
$$x_1 - x_2 + 2x_3 \le 10$$
$$x_1 + x_2 - x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

After slack variables are introduced and then one complete iteration of the simplex method is performed, the following simplex tableau is obtained.

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
1	Z	(0)	1	0	−1	3	0	2	0	20
	x_4	(1)	0	0	4	−5	1	−3	0	30
	x_1	(2)	0	1	−1	2	0	1	0	10
	x_6	(3)	0	0	2	−3	0	−1	1	10

(a) Identify the CPF solution obtained at iteration 1.
(b) Identify the constraint boundary equations that define this CPF solution.

5.1-5. Consider the three-variable linear programming problem shown in Fig. 5.2.
(a) Construct a table like Table 5.1, giving the set of defining equations for each CPF solution.
(b) What are the defining equations for the corner-point infeasible solution (6, 0, 5)?
(c) Identify one of the systems of three constraint boundary equations that yields neither a CPF solution nor a corner-point infeasible solution. Explain why this occurs for this system.

5.1-6. Consider the linear programming problem given in Table 6.1 as the dual problem for the Wyndor Glass Co. example.
(a) Identify the 10 sets of defining equations for this problem. For each one, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or corner-point infeasible solution.
(b) For each corner-point solution, give the corresponding basic solution and its set of nonbasic variables. (Compare with Table 6.9.)

5.1-7. Consider the following problem.

Minimize $Z = x_1 + 2x_2,$

subject to

$$-x_1 + x_2 \le 15$$
$$2x_1 + x_2 \le 90$$
$$x_2 \ge 30$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

(a) Solve this problem graphically.
(b) Develop a table giving each of the CPF solutions and the corresponding defining equations, BF solution, and nonbasic variables.

5.1-8. Reconsider the model in Problem 4.6-3.
(a) Identify the 10 sets of defining equations for this problem. For each one, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or a corner-point infeasible solution.
(b) For each corner-point solution, give the corresponding basic solution and its set of nonbasic variables.

5.1-9. Reconsider the model in Prob. 3.1-4.
(a) Identify the 15 sets of defining equations for this problem. For each one, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or a corner-point infeasible solution.
(b) For each corner-point solution, give the corresponding basic solution and its set of nonbasic variables.

5.1-10. Each of the following statements is true under most circumstances, but not always. In each case, indicate when the statement will not be true and why.
(a) The best CPF solution is an optimal solution.
(b) An optimal solution is a CPF solution.
(c) A CPF solution is the only optimal solution if none of its adjacent CPF solutions are better (as measured by the value of the objective function).

5.1-11. Consider the original form (before augmenting) of a linear programming problem with n decision variables (each with a nonnegativity constraint) and m functional constraints. Label each of the following statements as true or false, and then justify your

answer with specific references (including page citations) to material in the chapter.
(a) If a feasible solution is optimal, it must be a CPF solution.
(b) The number of CPF solutions is at least

$$\frac{(m+n)!}{m!n!}.$$

(c) If a CPF solution has adjacent CPF solutions that are better (as measured by Z), then one of these adjacent CPF solutions must be an optimal solution.

5.1-12. Label each of the following statements about linear programming problems as true or false, and then justify your answer.
(a) If a feasible solution is optimal but not a CPF solution, then infinitely many optimal solutions exist.
(b) If the value of the objective function is equal at two different feasible points $\mathbf{x}^*$ and $\mathbf{x}^{**}$, then all points on the line segment connecting $\mathbf{x}^*$ and $\mathbf{x}^{**}$ are feasible and Z has the same value at all those points.
(c) If the problem has n variables (before augmenting), then the simultaneous solution of any set of n constraint boundary equations is a CPF solution.

5.1-13. Consider the augmented form of linear programming problems that have feasible solutions and a bounded feasible region. Label each of the following statements as true or false, and then justify your answer by referring to specific statements (with page citations) in the chapter.
(a) There must be at least one optimal solution.
(b) An optimal solution must be a BF solution.
(c) The number of BF solutions is finite.

5.1-14.* Reconsider the model in Prob. 4.6-10. Now you are given the information that the basic variables in the optimal solution are x_2 and x_3. Use this information to identify a system of three constraint boundary equations whose simultaneous solution must be this optimal solution. Then solve this system of equations to obtain this solution.

5.1-15. Reconsider Prob. 4.3-7. Now use the given information and the theory of the simplex method to identify a system of three constraint boundary equations (in x_1, x_2, x_3) whose simultaneous solution must be the optimal solution, without applying the simplex method. Solve this system of equations to find the optimal solution.

5.1-16. Reconsider Prob. 4.3-8. Using the given information and the theory of the simplex method, analyze the constraints of the problem in order to identify a system of three constraint boundary equations whose simultaneous solution must be the optimal solution (not augmented). Then solve this system of equations to obtain this solution.

5.1-17. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 2x_2 + 3x_3,$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\le 4 \\ x_1 + x_2 + x_3 &\le 3 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4 and x_5 be the slack variables for the respective functional constraints. Starting with these two variables as the basic variables for the initial BF solution, you now are given the information that the simplex method proceeds as follows to obtain the optimal solution in two iterations: (1) In iteration 1, the entering basic variable is x_3 and the leaving basic variable is x_4; (2) in iteration 2, the entering basic variable is x_2 and the leaving basic variable is x_5.
(a) Develop a three-dimensional drawing of the feasible region for this problem, and show the path followed by the simplex method.
(b) Give a geometric interpretation of why the simplex method followed this path.
(c) For each of the two edges of the feasible region traversed by the simplex method, give the equation of each of the two constraint boundaries on which it lies, and then give the equation of the additional constraint boundary at each endpoint.
(d) Identify the set of defining equations for each of the three CPF solutions (including the initial one) obtained by the simplex method. Use the defining equations to solve for these solutions.
(e) For each CPF solution obtained in part (*d*), give the corresponding BF solution and its set of nonbasic variables. Explain how these nonbasic variables identify the defining equations obtained in part (*d*).

5.1-18. Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + 2x_3,$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\le 20 \\ x_1 + 2x_2 + x_3 &\le 30 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4 and x_5 be the slack variables for the respective functional constraints. Starting with these two variables as the basic variables for the initial BF solution, you now are given the information that the simplex method proceeds as follows to obtain the optimal solution in two iterations: (1) In iteration 1, the entering basic variable is x_2 and the leaving basic variable is x_5; (2) in iteration 2, the entering basic variable is x_1 and the leaving basic variable is x_4.

Follow the instructions of Prob. 5.1-17 for this situation.

5.1-19. By inspecting Fig. 5.2, explain why Property 1*b* for CPF solutions holds for this problem if it has the following objective function.

(a) Maximize $Z = x_3$.

(b) Maximize $Z = -x_1 + 2x_3$.

5.1-20. Consider the three-variable linear programming problem shown in Fig. 5.2.

(a) Explain in geometric terms why the set of solutions satisfying any individual constraint is a convex set, as defined in Appendix 2.

(b) Use the conclusion in part (*a*) to explain why the entire feasible region (the set of solutions that simultaneously satisfies every constraint) is a convex set.

5.1-21. Suppose that the three-variable linear programming problem given in Fig. 5.2 has the objective function

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + 3x_3.$$

Without using the algebra of the simplex method, apply just its geometric reasoning (including choosing the edge giving the maximum rate of increase of Z) to determine and explain the path it would follow in Fig. 5.2 from the origin to the optimal solution.

5.1-22. Consider the three-variable linear programming problem shown in Fig. 5.2.

(a) Construct a table like Table 5.4, giving the indicating variable for each constraint boundary equation and original constraint.

(b) For the CPF solution (2, 4, 3) and its three adjacent CPF solutions (4, 2, 4), (0, 4, 2), and (2, 4, 0), construct a table like Table 5.5, showing the corresponding defining equations, BF solution, and nonbasic variables.

(c) Use the sets of defining equations from part (*b*) to demonstrate that (4, 2, 4), (0, 4, 2), and (2, 4, 0) are indeed adjacent to (2, 4, 3), but that none of these three CPF solutions are adjacent to each other. Then use the sets of nonbasic variables from part (*b*) to demonstrate the same thing.

5.1-23. The formula for the line passing through (2, 4, 3) and (4, 2, 4) in Fig. 5.2 can be written as

$$(2, 4, 3) + \alpha[(4, 2, 4) - (2, 4, 3)] = (2, 4, 3) + \alpha(2, -2, 1),$$

where $0 \le \alpha \le 1$ for just the line segment between these points. After augmenting with the slack variables x_4, x_5, x_6, x_7 for the respective functional constraints, this formula becomes

$$(2, 4, 3, 2, 0, 0, 0) + \alpha(2, -2, 1, -2, 2, 0, 0).$$

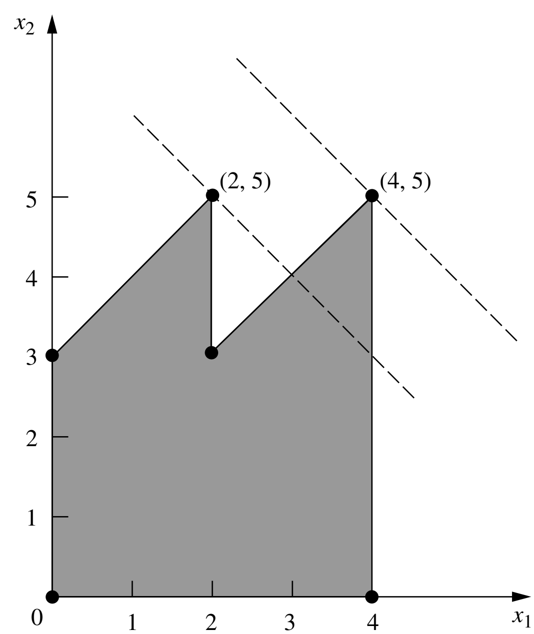
Use this formula directly to answer each of the following questions, and thereby relate the algebra and geometry of the simplex method as it goes through one iteration in moving from (2, 4, 3) to (4, 2, 4). (You are given the information that it is moving along this line segment.)

(a) What is the entering basic variable?

(b) What is the leaving basic variable?

(c) What is the new BF solution?

5.1-24. Consider a two-variable mathematical programming problem that has the feasible region shown on the graph, where the six dots correspond to CPF solutions. The problem has a linear objective function, and the two dashed lines are objective function lines passing through the optimal solution (4, 5) and the second-best CPF solution (2, 5). Note that the nonoptimal solution (2, 5) is better than both of its adjacent CPF solutions, which violates Property 3 in Sec. 5.1 for CPF solutions in linear programming. Demonstrate that this problem *cannot* be a linear programming problem by constructing the feasible region that would result if the six line segments on the boundary were constraint boundaries for linear programming constraints.



5.2-1. Consider the following problem.

$$\text{Maximize} \quad Z = 8x_1 + 4x_2 + 6x_3 + 3x_4 + 9x_5,$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 3x_4 &\le 180 \quad \text{(resource 1)} \\ 4x_1 + 3x_2 + 2x_3 + x_4 + x_5 &\le 270 \quad \text{(resource 2)} \\ x_1 + 3x_2 + x_4 + 3x_5 &\le 180 \quad \text{(resource 3)} \end{aligned}$$

and

$x_j \geq 0, \qquad j = 1, \ldots, 5.$

You are given the facts that the basic variables in the optimal solution are x_3, x_1, and x_5 and that

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}.$$

(a) Use the given information to identify the optimal solution.
(b) Use the given information to identify the shadow prices for the three resources.

I **5.2-2.*** Work through the revised simplex method step by step to solve the following problem.

Maximize $\quad Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5,$

subject to

$$2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \leq 20$$
$$3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \leq 30$$

and

$x_j \geq 0, \qquad j = 1, 2, 3, 4, 5.$

I **5.2-3.** Work through the revised simplex method step by step to solve the model given in Prob. 4.3-4.

5.2-4. Reconsider Prob. 5.1-1. For the sequence of CPF solutions identified in part (*e*), construct the basis matrix **B** for each of the corresponding BF solutions. For each one, invert **B** manually, use this $\mathbf{B}^{-1}$ to calculate the current solution, and then perform the next iteration (or demonstrate that the current solution is optimal).

I **5.2-5.** Work through the revised simplex method step by step to solve the model given in Prob. 4.1-5.

I **5.2-6.** Work through the revised simplex method step by step to solve the model given in Prob. 4.7-6.

I **5.2-7.** Work through the revised simplex method step by step to solve each of the following models:
(a) Model given in Prob. 3.1-5.
(b) Model given in Prob. 4.7-8.

D **5.3-1.*** Consider the following problem.

Maximize $\quad Z = x_1 - x_2 + 2x_3,$

subject to

$$2x_1 - 2x_2 + 3x_3 \leq 5$$
$$x_1 + x_2 - x_3 \leq 3$$
$$x_1 - x_2 + x_3 \leq 2$$

and

$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1				1	1	0	
x_2	(1)	0				1	3	0	
x_6	(2)	0				0	1	1	
x_3	(3)	0				1	2	0	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.
(b) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

D **5.3-2.** Consider the following problem.

Maximize $\quad Z = 4x_1 + 3x_2 + x_3 + 2x_4,$

subject to

$$4x_1 + 2x_2 + x_3 + x_4 \leq 5$$
$$3x_1 + x_2 + 2x_3 + x_4 \leq 4$$

and

$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad x_4 \geq 0.$

Let x_5 and x_6 denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1					1	1	
x_2	(1)	0					1	−1	
x_4	(2)	0					−1	2	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.
(b) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

D **5.3-3.** Consider the following problem.

Maximize $Z = 6x_1 + x_2 + 2x_3,$

subject to

$$2x_1 + 2x_2 + \frac{1}{2}x_3 \le 2$$
$$-4x_1 - 2x_2 - \frac{3}{2}x_3 \le 3$$
$$x_1 + 2x_2 + \frac{1}{2}x_3 \le 1$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1				2	0	2	
x_5	(1)	0				1	1	2	
x_3	(2)	0				−2	0	4	
x_1	(3)	0				1	0	−1	

Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.

D **5.3-4.** Consider the following problem.

Maximize $Z = x_1 - x_2 + 2x_3,$

subject to

$$x_1 + x_2 + 3x_3 \le 15$$
$$2x_1 - x_2 + x_3 \le 2$$
$$-x_1 + x_2 + x_3 \le 4$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After the simplex method is applied, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1				0	$\frac{3}{2}$	$\frac{1}{2}$	
x_4	(1)	0				1	−1	−2	
x_3	(2)	0				0	$\frac{1}{2}$	$\frac{1}{2}$	
x_2	(3)	0				0	$-\frac{1}{2}$	$\frac{1}{2}$	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.
(b) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

D **5.3-5.** Consider the following problem.

Maximize $Z = 20x_1 + 6x_2 + 8x_3,$

subject to

$$8x_1 + 2x_2 + 3x_3 \le 200$$
$$4x_1 + 3x_2 + 3x_3 \le 100$$
$$2x_1 + 3x_2 + x_3 \le 50$$
$$x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4, x_5, x_6, and x_7 denote the slack variables for the first through fourth constraints, respectively. Suppose that after some number of iterations of the simplex method, a portion of the current simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right Side
Z	(0)	1				$\frac{9}{4}$	$\frac{1}{2}$	0	0	
x_1	(1)	0				$\frac{3}{16}$	$-\frac{1}{8}$	0	0	
x_2	(2)	0				$-\frac{1}{4}$	$\frac{1}{2}$	0	0	
x_6	(3)	0				$-\frac{3}{8}$	$\frac{1}{4}$	1	0	
x_7	(4)	0				0	0	0	1	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the current simplex tableau. Show your calculations.
(b) Indicate which of these missing numbers would be generated by the revised simplex method in order to perform the next iteration.
(c) Identify the defining equations of the CPF solution corresponding to the BF solution in the current simplex tableau.

D **5.3-6.** You are using the simplex method to solve the following linear programming problem.

Maximize $Z = 6x_1 + 5x_2 - x_3 + 4x_4$,

subject to

$$3x_1 + 2x_2 - 3x_3 + x_4 \le 120$$
$$3x_1 + 3x_2 + x_3 + 3x_4 \le 180$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0.$$

You have obtained the following final simplex tableau where x_5 and x_6 are the slack variables for the respective constraints.

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	0	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{4}$	Z^*
x_1	(1)	0	1	$\frac{11}{12}$	0	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	b_1^*
x_3	(2)	0	0	$\frac{1}{4}$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	b_2^*

Use the fundamental insight presented in Sec. 5.3 to identify Z^*, b_1^*, and b_2^*. Show your calculations.

D **5.3-7.** Consider the following problem.

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3$,

subject to

$$x_1 + 2x_2 + x_3 \le b$$
$$2x_1 + x_2 + 3x_3 \le 2b$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Note that values have not been assigned to the coefficients in the objective function (c_1, c_2, c_3), and that the only specification for the right-hand side of the functional constraints is that the second one ($2b$) be twice as large as the first (b).

Now suppose that your boss has inserted her best estimate of the values of c_1, c_2, c_3, and b without informing you and then has run the simplex method. You are given the resulting final simplex tableau below (where x_4 and x_5 are the slack variables for the respective functional constraints), but you are unable to read the value of Z^*.

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{7}{10}$	0	0	$\frac{3}{5}$	$\frac{4}{5}$	Z^*
x_2	(1)	0	$\frac{1}{5}$	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	1
x_3	(2)	0	$\frac{3}{5}$	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	3

(a) Use the fundamental insight presented in Sec. 5.3 to identify the value of (c_1, c_2, c_3) that was used.
(b) Use the fundamental insight presented in Sec. 5.3 to identify the value of b that was used.
(c) Calculate the value of Z^* in two ways, where one way uses your results from part (*a*) and the other way uses your result from part (*b*). Show your two methods for finding Z^*.

5.3-8. For iteration 2 of the example in Sec. 5.3, the following expression was shown:

$$\text{Final row } 0 = [-3, \quad -5 \quad 0, \quad 0, \quad 0 \quad 0] + [0, \quad \tfrac{3}{2}, \quad 1]\left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right].$$

Derive this expression by combining the algebraic operations (in matrix form) for iterations 1 and 2 that affect row 0.

5.3-9. Most of the description of the fundamental insight presented in Sec. 5.3 assumes that the problem is in our standard form. Now consider each of the following other forms, where the additional adjustments in the initialization step are those presented in Sec. 4.6, including the use of artificial variables and the Big M method where appropriate. Describe the resulting adjustments in the fundamental insight.
(a) Equality constraints
(b) Functional constraints in $\ge$ form
(c) Negative right-hand sides
(d) Variables allowed to be negative (with no lower bound)

5.3-10. Reconsider the model in Prob. 4.6-6. Use artificial variables and the Big M method to construct the complete first simplex tableau for the simplex method, and then identify the columns that will contains **S*** for applying the fundamental insight in the final tableau. Explain why these are the appropriate columns.

5.3-11. Consider the following problem.

Minimize $Z = 2x_1 + 3x_2 + 2x_3$,

subject to

$$x_1 + 4x_2 + 2x_3 \geq 8$$
$$3x_1 + 2x_2 + 2x_3 \geq 6$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4 and x_6 be the surplus variables for the first and second constraints, respectively. Let $\bar{x}_5$ and $\bar{x}_7$ be the corresponding artificial variables. After you make the adjustments described in Sec. 4.6 for this model form when using the Big *M* method, the initial simplex tableau ready to apply the simplex method is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	x_6	$\bar{x}_7$	Right Side
Z	(0)	−1	$-4M + 2$	$-6M + 3$	$-2M + 2$	M	0	M	0	$-14M$
$\bar{x}_5$	(1)	0	1	4	2	−1	1	0	0	8
$\bar{x}_7$	(2)	0	3	2	0	0	0	−1	1	6

After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	x_6	$\bar{x}_7$	Right Side
Z	(0)	−1					$M - 0.5$		$M - 0.5$	
x_2	(1)	0					0.3		−0.1	
x_1	(2)	0					−0.2		0.4	

(a) Based on the above tableaux, use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.

(b) Examine the mathematical logic presented in Sec. 5.3 to validate the fundamental insight (see the $\mathbf{T^*} = \mathbf{MT}$ and $\mathbf{t^*} = \mathbf{t} + \mathbf{vT}$ equations and the subsequent derivations of **M** and **v**). This logic assumes that the original model fits our standard form, whereas the current problem does not fit this form. Show how, with minor adjustments, this same logic applies to the current problem when **t** is row 0 and **T** is rows 1 and 2 in the initial simplex tableau given above. Derive **M** and **v** for this problem.

(c) When you apply the $\mathbf{t^*} = \mathbf{t} + \mathbf{vT}$ equation, another option is to use $\mathbf{t} = [2, 3, 2, 0, M, 0, M, 0]$, which is the *preliminary* row 0 before the algebraic elimination of the nonzero coefficients of the initial basic variables $\bar{x}_5$ and $\bar{x}_7$. Repeat part (*b*) for this equation with this new **t**. After you derive the new **v**, show that this equation yields the same final row 0 for this problem as the equation derived in part (*b*).

(d) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

5.3-12. Consider the following problem.

Maximize $Z = 2x_1 + 4x_2 + 3x_3$,

subject to

$$x_1 + 3x_2 + 2x_3 = 20$$
$$x_1 + 5x_2 \qquad \geq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let $\bar{x}_4$ be the artificial variable for the first constraint. Let x_5 and $\bar{x}_6$ be the surplus variable and artificial variable, respectively, for the second constraint.

You are now given the information that a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
Z	(0)	1				$M + 2$	0	M	
x_1	(1)	0				1	0	0	
x_5	(2)	0				1	1	−1	

(a) Extend the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.

(b) Identify the defining equations of the CPF solution corresponding to the optimal solution in the final simplex tableau.

5.3-13. Consider the following problem.

Maximize $Z = 3x_1 + 7x_2 + 2x_3$,

subject to

$$-2x_1 + 2x_2 + x_3 \leq 10$$
$$3x_1 + x_2 - x_3 \leq 20$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

You are given the fact that the basic variables in the optimal solution are x_1 and x_3.

(a) Introduce slack variables, and then use the given information to find the optimal solution directly by Gaussian elimination.

(b) Extend the work in part (*a*) to find the shadow prices.

(c) Use the given information to identify the defining equations of the optimal CPF solution, and then solve these equations to obtain the optimal solution.

(d) Construct the basis matrix **B** for the optimal BF solution, invert **B** manually, and then use this $\mathbf{B}^{-1}$ to solve for the optimal solution and the shadow prices $\mathbf{y}^*$. Then apply the optimality test for the revised simplex method to verify that this solution is optimal.

(e) Given $\mathbf{B}^{-1}$ and $\mathbf{y}^*$ from part (*d*), use the fundamental insight presented in Sec. 5.3 to construct the complete final simplex tableau.