6

Duality Theory and Sensitivity Analysis

One of the most important discoveries in the early development of linear programming was the concept of duality and its many important ramifications. This discovery revealed that every linear programming problem has associated with it another linear programming problem called the **dual.** The relationships between the dual problem and the original problem (called the **primal**) prove to be extremely useful in a variety of ways. For example, you soon will see that the shadow prices described in Sec. 4.7 actually are provided by the optimal solution for the dual problem. We shall describe many other valuable applications of duality theory in this chapter as well.

One of the key uses of duality theory lies in the interpretation and implementation of *sensitivity analysis.* As we already mentioned in Secs. 2.3, 3.3, and 4.7, sensitivity analysis is a very important part of almost every linear programming study. Because most of the parameter values used in the original model are just *estimates* of future conditions, the effect on the optimal solution if other conditions prevail instead needs to be investigated. Furthermore, certain parameter values (such as resource amounts) may represent *managerial decisions,* in which case the choice of the parameter values may be the main issue to be studied, which can be done through sensitivity analysis.

For greater clarity, the first three sections discuss duality theory under the assumption that the *primal* linear programming problem is in *our standard form* (but with no restriction that the b_i values need to be positive). Other forms are then discussed in Sec. 6.4. We begin the chapter by introducing the essence of duality theory and its applications. We then describe the economic interpretation of the dual problem (Sec. 6.2) and delve deeper into the relationships between the primal and dual problems (Sec. 6.3). Section 6.5 focuses on the role of duality theory in sensitivity analysis. The basic procedure for sensitivity analysis (which is based on the fundamental insight of Sec. 5.3) is summarized in Sec. 6.6 and illustrated in Sec. 6.7.

6.1 THE ESSENCE OF DUALITY THEORY

Given our standard form for the *primal problem* at the left (perhaps after conversion from another form), its *dual problem* has the form shown to the right.

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Primal Problem	*Dual Problem*
Maximize $Z = \sum_{j=1}^{n} c_j x_j$,	Minimize $W = \sum_{i=1}^{m} b_i y_i$,
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \le b_i$, for $i = 1, 2, \dots, m$	$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$, for $j = 1, 2, \dots, n$
and	and
$x_j \ge 0$, for $j = 1, 2, \dots, n$.	$y_i \ge 0$, for $i = 1, 2, \dots, m$.

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Thus, the dual problem uses exactly the same *parameters* as the primal problem, but in different locations. To highlight the comparison, now look at these same two problems in matrix notation (as introduced at the beginning of Sec. 5.2), where $\mathbf{c}$ and $\mathbf{y} = [y_1, y_2, \dots, y_m]$ are row vectors but $\mathbf{b}$ and $\mathbf{x}$ are column vectors.

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$,	Minimize $\mathbf{W} = \mathbf{yb}$,
subject to	subject to
$\mathbf{Ax} \le \mathbf{b}$	$\mathbf{yA} \ge \mathbf{c}$
and	and
$\mathbf{x} \ge \mathbf{0}$.	$\mathbf{y} \ge \mathbf{0}$.

To illustrate, the primal and dual problems for the Wyndor Glass Co. example of Sec. 3.1 are shown in Table 6.1 in both algebraic and matrix form.

The **primal-dual table** for linear programming (Table 6.2) also helps to highlight the correspondence between the two problems. It shows all the linear programming parameters (the a_{ij}, b_i, and c_j) and how they are used to construct the two problems. All the headings for the primal problem are horizontal, whereas the headings for the dual problem are read by turning the book sideways. For the primal problem, each *column* (except the Right Side column) gives the coefficients of a single variable in the respective constraints and then in the objective function, whereas each *row* (except the bottom one) gives the parameters for a single contraint. For the dual problem, each *row* (except the Right Side row) gives the coefficients of a single variable in the respective constraints and then in the objective function, whereas each *column* (except the rightmost one) gives the parameters for a single constraint. In addition, the Right Side column gives the right-hand sides for the primal problem and the objective function coefficients for the dual problem, whereas the bottom row gives the objective function coefficients for the primal problem and the right-hand sides for the dual problem.

TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

Primal Problem in Algebraic Form	*Dual Problem in Algebraic Form*
Maximize $Z = 3x_1 + 5x_2,$ subject to $x_1 \le 4$, $2x_2 \le 12$, $3x_1 + 2x_2 \le 18$ and $x_1 \ge 0, \quad x_2 \ge 0.$	Minimize $W = 4y_1 + 12y_2 + 18y_3,$ subject to $y_1 + 3y_3 \ge 3$, $2y_2 + 2y_3 \ge 5$ and $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$
Primal Problem in Matrix Form	*Dual Problem in Matrix Form*
Maximize $Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$ subject to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$	Minimize $W = [y_1, y_2, y_3]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$ subject to $[y_1, y_2, y_3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \ge [3, 5]$ and $[y_1, y_2, y_3] \ge [0, 0, 0].$

Consequently, (1) the parameters for a *constraint* in either problem are the coefficients of a *variable* in the other problem and (2) the coefficients for the *objective function* of either problem are the *right sides* for the other problem. Thus, there is a direct correspondence between these entities in the two problems, as summarized in Table 6.3. These correspondences are a key to some of the applications of duality theory, including sensitivity analysis.

Origin of the Dual Problem

Duality theory is based directly on the fundamental insight (particularly with regard to row 0) presented in Sec. 5.3. To see why, we continue to use the notation introduced in Table 5.10 for row 0 of the *final* tableau, except for replacing Z^* by W^* and dropping the asterisks from $\mathbf{z}^*$ and $\mathbf{y}^*$ when referring to *any* tableau. Thus, at *any* given iteration of the simplex method for the primal problem, the current numbers in row 0 are denoted as shown in the (partial) tableau given in Table 6.4. For the coefficients of $x_1, x_2, \ldots, x_n$, recall that $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ denotes the vector that the simplex method added to the vector of *initial* coefficients, $-\mathbf{c}$, in the process of reaching the current tableau. (Do not confuse $\mathbf{z}$ with the value of the objective function Z.) Similarly, since the *initial* coefficients of $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ in row 0 all are 0, $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ denotes the vector that the simplex method has added to these coefficients. Also recall [see Eq. (1) in the

TABLE 6.2 Primal-dual table for linear programming, illustrated by the Wyndor Glass Co. example

(a) General Case

			Primal Problem					
			Coefficient of:				Right Side	
			x_1	x_2	$\cdots$	x_n		
Dual Problem	Coefficient of:	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\leq b_1$	Coefficients for Objective Function (Minimize)
		y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\leq b_2$	
		$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	
		y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\leq b_m$	
		Right Side	VI	VI	$\cdots$	VI		
			c_1	c_2	$\cdots$	c_n		
			Coefficients for Objective Function (Maximize)					

(b) Wyndor Glass Co. Example

	x_1	x_2	
y_1	1	0	$\leq$ 4
y_2	0	2	$\leq$ 12
y_3	3	2	$\leq$ 18
	VI	VI	
	3	5	

"Mathematical Summary" subsection of Sec. 5.3] that the fundamental insight led to the following relationships between these quantities and the parameters of the original model:

$$W = \mathbf{yb} = \sum_{i=1}^{m} b_i y_i,$$

$$\mathbf{z} = \mathbf{yA}, \qquad \text{so} \qquad z_j = \sum_{i=1}^{m} a_{ij} y_i, \qquad \text{for } j = 1, 2, \ldots, n.$$

To illustrate these relationships with the Wyndor example, the first equation gives $W = 4y_1 + 12y_2 + 18y_3$, which is just the objective function for the dual problem shown

TABLE 6.3 Correspondence between entities in primal and dual problems

One Problem		Other Problem
Constraint i	$\longleftrightarrow$	Variable i
Objective function	$\longleftrightarrow$	Right sides

TABLE 6.4 Notation for entries in row 0 of a simplex tableau

Iteration	Basic Variable	Eq.	Coefficient of:									Right Side
			Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

in the upper right-hand box of Table 6.1. The second set of equations give $z_1 = y_1 + 3y_3$ and $z_2 = 2y_2 + 2y_3$, which are the left-hand sides of the functional constraints for this dual problem. Thus, by subtracting the right-hand sides of these $\geq$ constraints ($c_1 = 3$ and $c_2 = 5$), $(z_1 - c_1)$ and $(z_2 - c_2)$ can be interpreted as being the *surplus variables* for these functional constraints.

The remaining key is to express what the simplex method tries to accomplish (according to the optimality test) in terms of these symbols. Specifically, it seeks a set of basic variables, and the corresponding BF solution, such that *all* coefficients in row 0 are *nonnegative.* It then stops with this optimal solution. Using the notation in Table 6.4, this goal is expressed symbolically as follows:

Condition for Optimality:

$$z_j - c_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n,$$
$$y_i \geq 0 \qquad \text{for } i = 1, 2, \ldots, m.$$

After we substitute the preceding expression for z_j, the condition for optimality says that the simplex method can be interpreted as seeking values for $y_1, y_2, \ldots, y_m$ such that

$$W = \sum_{i=1}^{m} b_i y_i,$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

But, except for lacking an objective for W, this problem is precisely the *dual problem*! To complete the formulation, let us now explore what the missing objective should be.

Since W is just the current value of Z, and since the objective for the primal problem is to maximize Z, a natural first reaction is that W should be maximized also. However, this is not correct for the following rather subtle reason: The only *feasible* solutions for this new problem are those that satisfy the condition for *optimality* for the primal problem. Therefore, it is *only* the optimal solution for the primal problem that corresponds to a feasible solution for this new problem. As a consequence, the optimal value of Z in the primal problem is the *minimum* feasible value of W in the new problem, so W should be minimized. (The full justification for this conclusion is provided by the relationships we develop in Sec. 6.3.) Adding this objective of minimizing W gives the *complete* dual problem.

Consequently, the dual problem may be viewed as a restatement in linear programming terms of the *goal* of the simplex method, namely, to reach a solution for the primal problem that *satisfies the optimality test. Before* this goal has been reached, the corresponding **y** in row 0 (coefficients of slack variables) of the current tableau must be *infeasible* for the *dual problem.* However, *after* the goal is reached, the corresponding **y** must be an *optimal solution* (labeled **y***) for the *dual problem,* because it is a feasible solution that attains the minimum feasible value of W. This optimal solution $(y_1^*, y_2^*, \dots, y_m^*)$ provides for the primal problem the shadow prices that were described in Sec. 4.7. Furthermore, this optimal W is just the optimal value of Z, so the *optimal objective function values are equal* for the two problems. This fact also implies that $\mathbf{cx} \le \mathbf{yb}$ for any **x** and **y** that are *feasible* for the primal and dual problems, respectively.

To illustrate, the left-hand side of Table 6.5 shows row 0 for the respective iterations when the simplex method is applied to the Wyndor Glass Co. example. In each case, row 0 is partitioned into three parts: the coefficients of the decision variables (x_1, x_2), the coefficients of the slack variables (x_3, x_4, x_5), and the right-hand side (value of Z). Since the coefficients of the slack variables give the corresponding values of the dual variables (y_1, y_2, y_3), each row 0 identifies a corresponding solution for the dual problem, as shown in the y_1, y_2, and y_3 columns of Table 6.5. To interpret the next two columns, recall that $(z_1 - c_1)$ and $(z_2 - c_2)$ are the surplus variables for the functional constraints in the dual problem, so the full dual problem after augmenting with these surplus variables is

$$\text{Minimize} \qquad W = 4y_1 + 12y_2 + 18y_3,$$

subject to

$$\begin{aligned} y_1 \qquad + 3y_3 - (z_1 - c_1) &= 3 \\ 2y_2 + 2y_3 - (z_2 - c_2) &= 5 \end{aligned}$$

and

$$y_1 \ge 0, \qquad y_2 \ge 0, \qquad y_3 \ge 0.$$

Therefore, by using the numbers in the y_1, y_2, and y_3 columns, the values of these surplus variables can be calculated as

$$\begin{aligned} z_1 - c_1 &= y_1 + 3y_3 - 3, \\ z_2 - c_2 &= 2y_2 + 2y_3 - 5. \end{aligned}$$

TABLE 6.5 Row 0 and corresponding dual solution for each iteration for the Wyndor Glass Co. example

	Primal Problem	Dual Problem					
Iteration	**Row 0**	y_1	y_2	y_3	$z_1 - c_1$	$z_2 - c_2$	W
0	$[-3, \ -5 \ \vdots \ 0, \ 0, \ 0 \ \vdots \ 0]$	0	0	0	-3	-5	0
1	$[-3, \ 0 \ \vdots \ 0, \ \frac{5}{2}, \ 0 \ \vdots \ 30]$	0	$\frac{5}{2}$	0	-3	0	30
2	$[0, \ 0 \ \vdots \ 0, \ \frac{3}{2}, \ 1 \ \vdots \ 36]$	0	$\frac{3}{2}$	1	0	0	36

Thus, a negative value for either surplus variable indicates that the corresponding constraint is violated. Also included in the rightmost column of the table is the calculated value of the dual objective function $W = 4y_1 + 12y_2 + 18y_3$.

As displayed in Table 6.4, *all* these quantities to the right of row 0 in Table 6.5 already are identified by row 0 without requiring any new calculations. In particular, note in Table 6.5 how *each* number obtained for the dual problem already appears in row 0 in the spot indicated by Table 6.4.

For the initial row 0, Table 6.5 shows that the corresponding dual solution $(y_1, y_2, y_3) = (0, 0, 0)$ is infeasible because both surplus variables are negative. The first iteration succeeds in eliminating one of these negative values, but not the other. After two iterations, the optimality test is satisfied for the primal problem because all the dual variables and surplus variables are nonnegative. This dual solution $(y_1^*, y_2^*, y_3^*) = (0, \frac{3}{2}, 1)$ is optimal (as could be verified by applying the simplex method directly to the dual problem), so the optimal value of Z and W is $Z^* = 36 = W^*$.

Summary of Primal-Dual Relationships

Now let us summarize the newly discovered key relationships between the primal and dual problems.

Weak duality property: If $\mathbf{x}$ is a feasible solution for the primal problem and $\mathbf{y}$ is a feasible solution for the dual problem, then

$$\mathbf{cx} \leq \mathbf{yb}.$$

For example, for the Wyndor Glass Co. problem, one feasible solution is $x_1 = 3$, $x_2 = 3$, which yields $Z = \mathbf{cx} = 24$, and one feasible solution for the dual problem is $y_1 = 1$, $y_2 = 1$, $y_3 = 2$, which yields a larger objective function value $W = \mathbf{yb} = 52$. These are just sample feasible solutions for the two problems. For *any* such pair of feasible solutions, this inequality must hold because the *maximum* feasible value of $Z = \mathbf{cx}$ (36) *equals* the *minimum* feasible value of the dual objective function $W = \mathbf{yb}$, which is our next property.

Strong duality property: If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

Thus, these two properties imply that $\mathbf{cx} < \mathbf{yb}$ for feasible solutions if one or both of them are *not optimal* for their respective problems, whereas equality holds when both are optimal.

The *weak duality property* describes the relationship between any pair of solutions for the primal and dual problems where *both* solutions are *feasible* for their respective problems. At each iteration, the simplex method finds a specific pair of solutions for the two problems, where the primal solution is feasible but the dual solution is *not feasible* (except at the final iteration). Our next property describes this situation and the relationship between this pair of solutions.

Complementary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution $\mathbf{x}$ for the primal problem and a **complementary solution** $\mathbf{y}$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx} = \mathbf{yb}.$$

If $\mathbf{x}$ is *not optimal* for the primal problem, then $\mathbf{y}$ is *not feasible* for the dual problem.

To illustrate, after one iteration for the Wyndor Glass Co. problem, $x_1 = 0$, $x_2 = 6$, and $y_1 = 0$, $y_2 = \frac{5}{2}$, $y_3 = 0$, with $\mathbf{cx} = 30 = \mathbf{yb}$. This $\mathbf{x}$ is feasible for the primal problem, but this $\mathbf{y}$ is not feasible for the dual problem (since it violates the constraint, $y_1 + 3y_3 \geq 3$).

The complementary solutions property also holds at the final iteration of the simplex method, where an optimal solution is found for the primal problem. However, more can be said about the complementary solution $\mathbf{y}$ in this case, as presented in the next property.

Complementary optimal solutions property: At the final iteration, the simplex method simultaneously identifies an optimal solution $\mathbf{x}^*$ for the primal problem and a **complementary optimal solution** $\mathbf{y}^*$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

The y_i^* are the shadow prices for the primal problem.

For the example, the final iteration yields $x_1^* = 2$, $x_2^* = 6$, and $y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$, with $\mathbf{cx}^* = 36 = \mathbf{y}^*\mathbf{b}$.

We shall take a closer look at some of these properties in Sec. 6.3. There you will see that the complementary solutions property can be extended considerably further. In particular, after slack and surplus variables are introduced to augment the respective problems, every *basic* solution in the primal problem has a complementary *basic* solution in the dual problem. We already have noted that the simplex method identifies the values of the surplus variables for the dual problem as $z_j - c_j$ in Table 6.4. This result then leads to an additional *complementary slackness property* that relates the basic variables in one problem to the nonbasic variables in the other (Tables 6.7 and 6.8), but more about that later.

In Sec. 6.4, after describing how to construct the dual problem when the primal problem is *not* in our standard form, we discuss another very useful property, which is summarized as follows:

Symmetry property: For *any* primal problem and its dual problem, all relationships between them must be *symmetric* because the dual of this dual problem is this primal problem.

Therefore, all the preceding properties hold regardless of which of the two problems is labeled as the primal problem. (The direction of the inequality for the weak duality property does require that the primal problem be expressed or reexpressed in maximization form and the dual problem in minimization form.) Consequently, the simplex method can be applied to either problem, and it simultaneously will identify complementary solutions (ultimately a complementary optimal solution) for the other problem.

So far, we have focused on the relationships between *feasible* or *optimal* solutions in the primal problem and corresponding solutions in the dual problem. However, it is possible that the primal (or dual) problem either has *no feasible solutions* or has feasible solutions but *no optimal solution* (because the objective function is unbounded). Our final property summarizes the primal-dual relationships under all these possibilities.

Duality theorem: The following are the only possible relationships between the primal and dual problems.

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions.*
3. If one problem has *no feasible solutions,* then the other problem has either *no feasible solutions* or an *unbounded* objective function.

Applications

As we have just implied, one important application of duality theory is that the *dual* problem can be solved directly by the simplex method in order to identify an optimal solution for the primal problem. We discussed in Sec. 4.8 that the number of functional constraints affects the computational effort of the simplex method far more than the number of variables does. If $m > n$, so that the dual problem has fewer functional constraints (n) than the primal problem (m), then applying the simplex method directly to the dual problem instead of the primal problem probably will achieve a substantial reduction in computational effort.

The *weak* and *strong duality properties* describe key relationships between the primal and dual problems. One useful application is for evaluating a proposed solution for the primal problem. For example, suppose that $\mathbf{x}$ is a feasible solution that has been proposed for implementation and that a feasible solution $\mathbf{y}$ has been found by inspection for the dual problem such that $\mathbf{cx} = \mathbf{yb}$. In this case, $\mathbf{x}$ must be *optimal* without the simplex method even being applied! Even if $\mathbf{cx} < \mathbf{yb}$, then $\mathbf{yb}$ still provides an upper bound on the optimal value of Z, so if $\mathbf{yb} - \mathbf{cx}$ is small, intangible factors favoring $\mathbf{x}$ may lead to its selection without further ado.

One of the key applications of the complementary solutions property is its use in the dual simplex method presented in Sec. 7.1. This algorithm operates on the primal problem exactly as if the simplex method were being applied simultaneously to the dual problem, which can be done because of this property. Because the roles of row 0 and the right side in the simplex tableau have been reversed, the dual simplex method requires that row 0 *begin and remain nonnegative* while the right side *begins* with some *negative* values (subsequent iterations strive to reach a nonnegative right side). Consequently, this algorithm occasionally is used because it is more convenient to set up the initial tableau in this form than in the form required by the simplex method. Furthermore, it frequently is used for reoptimization (discussed in Sec. 4.7), because changes in the original model lead to the revised final tableau fitting this form. This situation is common for certain types of sensitivity analysis, as you will see later in the chapter.

In general terms, duality theory plays a central role in sensitivity analysis. This role is the topic of Sec. 6.5.

Another important application is its use in the economic interpretation of the dual problem and the resulting insights for analyzing the primal problem. You already have seen one example when we discussed shadow prices in Sec. 4.7. The next section describes how this interpretation extends to the entire dual problem and then to the simplex method.

6.2 ECONOMIC INTERPRETATION OF DUALITY

The economic interpretation of duality is based directly upon the typical interpretation for the primal problem (linear programming problem in our standard form) presented in Sec. 3.2. To refresh your memory, we have summarized this interpretation of the primal problem in Table 6.6.

Interpretation of the Dual Problem

To see how this interpretation of the primal problem leads to an economic interpretation for the dual problem,[1] note in Table 6.4 that W is the value of Z (total profit) at the current iteration. Because

$$W = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m,$$

each $b_i y_i$ can thereby be interpreted as the current *contribution to profit* by having b_i units of resource i available for the primal problem. Thus,

> The dual variable y_i is interpreted as the contribution to profit per unit of resource i ($i = 1, 2, \dots, m$), when the current set of basic variables is used to obtain the primal solution.

In other words, the y_i values (or y_i^* values in the optimal solution) are just the shadow prices discussed in Sec. 4.7.

For example, when iteration 2 of the simplex method finds the optimal solution for the Wyndor problem, it also finds the optimal values of the dual variables (as shown in the bottom row of Table 6.5) to be $y_1^* = 0$, $y_2^* = \frac{3}{2}$, and $y_3^* = 1$. These are precisely the shadow prices found in Sec. 4.7 for this problem through graphical analysis. Recall that the resources for the Wyndor problem are the production capacities of the three plants being made available to the two new products under consideration, so that b_i is the number of hours of production time per week being made available in Plant i for these new products, where $i = 1, 2, 3$. As discussed in Sec. 4.7, the shadow prices indicate that individually increasing any b_i by 1 would increase the optimal value of the objective function (total weekly profit in units of thousands of dollars) by y_i^*. Thus, y_i^* can be interpreted as the contribution to profit per unit of resource i when using the optimal solution.

[1]Actually, several slightly different interpretations have been proposed. The one presented here seems to us to be the most useful because it also directly interprets what the simplex method does in the primal problem.

TABLE 6.6 Economic interpretation of the primal problem

Quantity	Interpretation
x_j	Level of activity j ($j = 1, 2, \dots, n$)
c_j	Unit profit from activity j
Z	Total profit from all activities
b_i	Amount of resource i available ($i = 1, 2, \dots, m$)
a_{ij}	Amount of resource i consumed by each unit of activity j

This interpretation of the dual variables leads to our interpretation of the overall dual problem. Specifically, since each unit of activity j in the primal problem consumes a_{ij} units of resource i,

> $\Sigma_{i=1}^{m} a_{ij}y_i$ is interpreted as the current contribution to profit of the mix of resources that would be consumed if 1 unit of activity j were used ($j = 1, 2, \ldots, n$).

For the Wyndor problem, 1 unit of activity j corresponds to producing 1 batch of product j per week, where $j = 1, 2$. The mix of resources consumed by producing 1 batch of product 1 is 1 hour of production time in Plant 1 and 3 hours in Plant 3. The corresponding mix per batch of product 2 is 2 hours each in Plants 2 and 3. Thus, $y_1 + 3y_3$ and $2y_2 + 2y_3$ are interpreted as the current contributions to profit (in thousands of dollars per week) of these respective mixes of resources per batch produced per week of the respective products.

For each activity j, this same mix of resources (and more) probably can be used in other ways as well, but no alternative use should be considered if it is less profitable than 1 unit of activity j. Since c_j is interpreted as the unit profit from activity j, each functional constraint in the dual problem is interpreted as follows:

> $\Sigma_{i=1}^{m} a_{ij}y_i \geq c_j$ says that the actual contribution to profit of the above mix of resources must be at least as much as if they were used by 1 unit of activity j; otherwise, we would not be making the best possible use of these resources.

For the Wyndor problem, the unit profits (in thousands of dollars per week) are $c_1 = 3$ and $c_2 = 5$, so the dual functional constraints with this interpretation are $y_1 + y_3 \geq 3$ and $2y_2 + 2y_3 \geq 5$. Similarly, the interpretation of the nonnegativity constraints is the following:

> $y_i \geq 0$ says that the contribution to profit of resource i ($i = 1, 2, \ldots, m$) must be nonnegative: otherwise, it would be better not to use this resource at all.

The objective

$$\text{Minimize} \qquad W = \sum_{i=1}^{m} b_i y_i$$

can be viewed as minimizing the total implicit value of the resources consumed by the activities. For the Wyndor problem, the total implicit value (in thousands of dollars per week) of the resources consumed by the two products is $W = 4y_1 + 12y_2 + 18y_3$.

This interpretation can be sharpened somewhat by differentiating between basic and nonbasic variables in the primal problem for any given BF solution $(x_1, x_2, \ldots, x_{n+m})$. Recall that the *basic* variables (the only variables whose values can be nonzero) *always* have a coefficient of *zero* in row 0. Therefore, referring again to Table 6.4 and the accompanying equation for z_j, we see that

$$\sum_{i=1}^{m} a_{ij}y_i = c_j, \qquad \text{if } x_j > 0 \qquad (j = 1, 2, \ldots, n),$$

$$y_i = 0, \qquad \text{if } x_{n+i} > 0 \qquad (i = 1, 2, \ldots, m).$$

(This is one version of the complementary slackness property discussed in the next section.) The economic interpretation of the first statement is that whenever an activity j operates at a strictly positive level ($x_j > 0$), the marginal value of the resources it consumes *must equal* (as opposed to exceeding) the unit profit from this activity. The second statement implies that the marginal value of resource i is *zero* ($y_i = 0$) whenever the supply of this resource is not exhausted by the activities ($x_{n+i} > 0$). In economic terminology, such a resource is a "free good"; the price of goods that are oversupplied must drop to zero by the law of supply and demand. This fact is what justifies interpreting the objective for the dual problem as minimizing the total implicit value of the resources *consumed,* rather than the resources *allocated.*

To illustrate these two statements, consider the optimal BF solution (2, 6, 2, 0, 0) for the Wyndor problem. The basic variables are x_1, x_2, and x_3, so their coefficients in row 0 are zero, as shown in the bottom row of Table 6.5. This bottom row also gives the corresponding dual solution: $y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$, with surplus variables $(z_1^* - c_1) = 0$ and $(z_2^* - c_2) = 0$. Since $x_1 > 0$ and $x_2 > 0$, both these surplus variables and direct calculations indicate that $y_1^* + 3y_3^* = c_1 = 3$ and $2y_2^* + 2y_3^* = c_2 = 5$. Therefore, the value of the resources consumed per batch of the respective products produced does indeed equal the respective unit profits. The slack variable for the constraint on the amount of Plant 1 capacity used is $x_3 > 0$, so the marginal value of adding any Plant 1 capacity would be zero ($y_1^* = 0$).

Interpretation of the Simplex Method

The interpretation of the dual problem also provides an economic interpretation of what the simplex method does in the primal problem. The *goal* of the simplex method is to find how to use the available resources in the most profitable feasible way. To attain this goal, we must reach a BF solution that satisfies all the *requirements* on profitable use of the resources (the constraints of the dual problem). These requirements comprise the *condition for optimality* for the algorithm. For any given BF solution, the requirements (dual constraints) associated with the basic variables are automatically satisfied (with equality). However, those associated with nonbasic variables may or may not be satisfied.

In particular, if an original variable x_j is nonbasic so that activity j is not used, then the current contribution to profit of the resources that would be required to undertake each unit of activity j

$$\sum_{i=1}^{m} a_{ij} y_i$$

may be smaller than, larger than, or equal to the unit profit c_j obtainable from the activity. If it is smaller, so that $z_j - c_j < 0$ in row 0 of the simplex tableau, then these resources can be used more profitably by initiating this activity. If it is larger ($z_j - c_j > 0$), then these resources already are being assigned elsewhere in a more profitable way, so they should not be diverted to activity j. If $z_j - c_j = 0$, there would be no change in profitability by initiating activity j.

Similarly, if a slack variable x_{n+i} is nonbasic so that the total allocation b_i of resource i is being used, then y_i is the current contribution to profit of this resource on a marginal basis. Hence, if $y_i < 0$, profit can be increased by cutting back on the use of this resource (i.e., increasing x_{n+i}). If $y_i > 0$, it is worthwhile to continue fully using this resource, whereas this decision does not affect profitability if $y_i = 0$.

Therefore, what the simplex method does is to examine all the nonbasic variables in the current BF solution to see which ones can provide a *more profitable use of the resources* by being increased. If *none* can, so that no feasible shifts or reductions in the current proposed use of the resources can increase profit, then the current solution must be optimal. If one or more can, the simplex method selects the variable that, if increased by 1, would *improve the profitability* of the use of the resources the most. It then actually increases this variable (the entering basic variable) as much as it can until the marginal values of the resources change. This increase results in a new BF solution with a new row 0 (dual solution), and the whole process is repeated.

The economic interpretation of the dual problem considerably expands our ability to analyze the primal problem. However, you already have seen in Sec. 6.1 that this interpretation is just one ramification of the relationships between the two problems. In the next section, we delve into these relationships more deeply.

6.3 PRIMAL-DUAL RELATIONSHIPS

Because the dual problem is a linear programming problem, it also has corner-point solutions. Furthermore, by using the augmented form of the problem, we can express these corner-point solutions as basic solutions. Because the functional constraints have the $\geq$ form, this augmented form is obtained by *subtracting* the surplus (rather than adding the slack) from the left-hand side of each constraint j ($j = 1, 2, \ldots, n$).[1] This surplus is

$$z_j - c_j = \sum_{i=1}^{m} a_{ij} y_i - c_j, \qquad \text{for } j = 1, 2, \ldots, n.$$

Thus, $z_j - c_j$ plays the role of the *surplus variable* for constraint j (or its slack variable if the constraint is multiplied through by -1). Therefore, augmenting each corner-point solution $(y_1, y_2, \ldots, y_m)$ yields a basic solution $(y_1, y_2, \ldots, y_m, z_1 - c_1, z_2 - c_2, \ldots, z_n - c_n)$ by using this expression for $z_j - c_j$. Since the augmented form of the dual problem has n functional constraints and $n + m$ variables, each basic solution has n basic variables and m nonbasic variables. (Note how m and n reverse their previous roles here because, as Table 6.3 indicates, dual constraints correspond to primal variables and dual variables correspond to primal constraints.)

Complementary Basic Solutions

One of the important relationships between the primal and dual problems is a direct correspondence between their basic solutions. The key to this correspondence is row 0 of the simplex tableau for the primal basic solution, such as shown in Table 6.4 or 6.5. Such a row 0 can be obtained for *any* primal basic solution, feasible or not, by using the formulas given in the bottom part of Table 5.8.

Note again in Tables 6.4 and 6.5 how a complete solution for the dual problem (including the surplus variables) can be read directly from row 0. Thus, because of its coefficient in

[1] You might wonder why we do not also introduce *artificial variables* into these constraints as discussed in Sec. 4.6. The reason is that these variables have no purpose other than to change the feasible region temporarily as a convenience in starting the simplex method. We are not interested now in applying the simplex method to the dual problem, and we do not want to change its feasible region.

TABLE 6.7 Association between variables in primal and dual problems

	Primal Variable	Associated Dual Variable
Any problem	(Decision variable) x_j (Slack variable) x_{n+i}	$z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$ y_i (decision variable) $i = 1, 2, \ldots, m$
Wyndor problem	Decision variables: x_1 x_2 Slack variables: x_3 x_4 x_5	$z_1 - c_1$ (surplus variables) $z_2 - c_2$ y_1 (decision variables) y_2 y_3

row 0, each variable in the primal problem has an associated variable in the dual problem, as summarized in Table 6.7, first for any problem and then for the Wyndor problem.

A key insight here is that the dual solution read from row 0 must also be a basic solution! The reason is that the m basic variables for the primal problem are required to have a coefficient of zero in row 0, which thereby requires the m associated dual variables to be zero, i.e., nonbasic variables for the dual problem. The values of the remaining n (basic) variables then will be the simultaneous solution to the system of equations given at the beginning of this section. In matrix form, this system of equations is $\mathbf{z} - \mathbf{c} = \mathbf{yA} - \mathbf{c}$, and the fundamental insight of Sec. 5.3 actually identifies its solution for $\mathbf{z} - \mathbf{c}$ and $\mathbf{y}$ as being the corresponding entries in row 0.

Because of the symmetry property quoted in Sec. 6.1 (and the direct association between variables shown in Table 6.7), the correspondence between basic solutions in the primal and dual problems is a symmetric one. Furthermore, a pair of complementary basic solutions has the same objective function value, shown as W in Table 6.4.

Let us now summarize our conclusions about the correspondence between primal and dual basic solutions, where the first property extends the complementary solutions property of Sec. 6.1 to the augmented forms of the two problems and then to any basic solution (feasible or not) in the primal problem.

> **Complementary basic solutions property:** Each *basic* solution in the *primal problem* has a **complementary basic solution** in the *dual problem,* where their respective objective function values (Z and W) are equal. Given row 0 of the simplex tableau for the primal basic solution, the complementary dual basic solution ($\mathbf{y}, \mathbf{z} - \mathbf{c}$) is found as shown in Table 6.4.

The next property shows how to identify the basic and nonbasic variables in this complementary basic solution.

> **Complementary slackness property:** Given the association between variables in Table 6.7, the variables in the primal basic solution and the complementary dual basic solution satisfy the **complementary slackness** relationship shown in Table 6.8. Furthermore, this relationship is a symmetric one, so that these two basic solutions are complementary to each other.

The reason for using the name *complementary slackness* for this latter property is that it says (in part) that for each pair of associated variables, if one of them has *slack* in its

TABLE 6.8 Complementary slackness relationship for complementary basic solutions

Primal Variable	Associated Dual Variable	
Basic	Nonbasic	(m variables)
Nonbasic	Basic	(n variables)

nonnegativity constraint (a basic variable > 0), then the other one must have *no slack* (a nonbasic variable = 0). We mentioned in Sec. 6.2 that this property has a useful economic interpretation for linear programming problems.

Example. To illustrate these two properties, again consider the Wyndor Glass Co. problem of Sec. 3.1. All eight of its basic solutions (five feasible and three infeasible) are shown in Table 6.9. Thus, its dual problem (see Table 6.1) also must have eight basic solutions, each complementary to one of these primal solutions, as shown in Table 6.9.

The three BF solutions obtained by the simplex method for the primal problem are the first, fifth, and sixth primal solutions shown in Table 6.9. You already saw in Table 6.5 how the complementary basic solutions for the dual problem can be read directly from row 0, starting with the coefficients of the slack variables and then the original variables. The other dual basic solutions also could be identified in this way by constructing row 0 for each of the other primal basic solutions, using the formulas given in the bottom part of Table 5.8.

Alternatively, for each primal basic solution, the complementary slackness property can be used to identify the basic and nonbasic variables for the complementary dual basic solution, so that the system of equations given at the beginning of the section can be

TABLE 6.9 Complementary basic solutions for the Wyndor Glass Co. example

	Primal Problem			Dual Problem	
No.	Basic Solution	Feasible?	$Z = W$	Feasible?	Basic Solution
1	(0, 0, 4, 12, 18)	Yes	0	No	$(0, 0, 0, -3, -5)$
2	(4, 0, 0, 12, 6)	Yes	12	No	$(3, 0, 0, 0, -5)$
3	$(6, 0, -2, 12, 0)$	No	18	No	$(0, 0, 1, 0, -3)$
4	(4, 3, 0, 6, 0)	Yes	27	No	$\left(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0\right)$
5	(0, 6, 4, 0, 6)	Yes	30	No	$\left(0, \frac{5}{2}, 0, -3, 0\right)$
6	(2, 6, 2, 0, 0)	Yes	36	Yes	$\left(0, \frac{3}{2}, 1, 0, 0\right)$
7	$(4, 6, 0, 0, -6)$	No	42	Yes	$\left(3, \frac{5}{2}, 0, 0, 0\right)$
8	$(0, 9, 4, -6, 0)$	No	45	Yes	$\left(0, 0, \frac{5}{2}, \frac{9}{2}, 0\right)$

solved directly to obtain this complementary solution. For example, consider the next-to-last primal basic solution in Table 6.9, (4, 6, 0, 0, −6). Note that x_1, x_2, and x_5 are *basic variables,* since these variables are not equal to 0. Table 6.7 indicates that the associated dual variables are $(z_1 - c_1)$, $(z_2 - c_2)$, and y_3. Table 6.8 specifies that these associated dual variables are *nonbasic variables* in the complementary basic solution, so

$$z_1 - c_1 = 0, \qquad z_2 - c_2 = 0, \qquad y_3 = 0.$$

Consequently, the augmented form of the functional constraints in the dual problem,

$$\begin{aligned} y_1 \quad\; &+ 3y_3 - (z_1 - c_1) = 3 \\ 2y_2 &+ 2y_3 - (z_2 - c_2) = 5, \end{aligned}$$

reduce to

$$\begin{aligned} y_1 \quad\; &+ 0 - 0 = 3 \\ 2y_2 &+ 0 - 0 = 5, \end{aligned}$$

so that $y_1 = 3$ and $y_2 = \frac{5}{2}$. Combining these values with the values of 0 for the nonbasic variables gives the basic solution $(3, \frac{5}{2}, 0, 0, 0)$, shown in the rightmost column and next-to-last row of Table 6.9. Note that this dual solution is feasible for the dual problem because all five variables satisfy the nonnegativity constraints.

Finally, notice that Table 6.9 demonstrates that $(0, \frac{3}{2}, 1, 0, 0)$ is the optimal solution for the dual problem, because it is the basic *feasible* solution with minimal W (36).

Relationships between Complementary Basic Solutions

We now turn our attention to the relationships between complementary basic solutions, beginning with their *feasibility* relationships. The middle columns in Table 6.9 provide some valuable clues. For the pairs of complementary solutions, notice how the yes or no answers on feasibility also satisfy a complementary relationship in most cases. In particular, with one exception, whenever one solution is feasible, the other is not. (It also is possible for *neither* solution to be feasible, as happened with the third pair.) The one exception is the sixth pair, where the primal solution is known to be optimal. The explanation is suggested by the $Z = W$ column. Because the sixth dual solution also is optimal (by the complementary optimal solutions property), with $W = 36$, the first five dual solutions *cannot be feasible* because $W < 36$ (remember that the dual problem objective is to *minimize* W). By the same token, the last two primal solutions cannot be feasible because $Z > 36$.

This explanation is further supported by the strong duality property that optimal primal and dual solutions have $Z = W$.

Next, let us state the *extension* of the complementary optimal solutions property of Sec. 6.1 for the augmented forms of the two problems.

Complementary optimal basic solutions property: Each *optimal* basic solution in the *primal problem* has a **complementary optimal basic solution** in the dual problem, where their respective objective function values (Z and W) are equal. Given row 0 of the simplex tableau for the optimal primal solution, the complementary optimal dual solution $(\mathbf{y}^*, \mathbf{z}^* - \mathbf{c})$ is found as shown in Table 6.4.

TABLE 6.10 Classification of basic solutions

		Satisfies Condition for Optimality? Yes	No
Feasible?	Yes	Optimal	Suboptimal
	No	Superoptimal	Neither feasible nor superoptimal

To review the reasoning behind this property, note that the dual solution ($\mathbf{y}^*$, $\mathbf{z}^* - \mathbf{c}$) must be feasible for the dual problem because the condition for optimality for the primal problem requires that *all* these dual variables (including surplus variables) be *nonnegative.* Since this solution is *feasible,* it must be *optimal* for the dual problem by the weak duality property (since $W = Z$, so $\mathbf{y}^*\mathbf{b} = \mathbf{c}\mathbf{x}^*$ where $\mathbf{x}^*$ is optimal for the primal problem).

Basic solutions can be classified according to whether they satisfy each of two conditions. One is the *condition for feasibility,* namely, whether *all* the variables (including slack variables) in the augmented solution are *nonnegative.* The other is the *condition for optimality,* namely, whether *all* the coefficients in row 0 (i.e., all the variables in the complementary basic solution) are *nonnegative.* Our names for the different types of basic solutions are summarized in Table 6.10. For example, in Table 6.9, primal basic solutions 1, 2, 4, and 5 are suboptimal, 6 is optimal, 7 and 8 are superoptimal, and 3 is neither feasible nor superoptimal.

Given these definitions, the general relationships between complementary basic solutions are summarized in Table 6.11. The resulting range of possible (common) values for the objective functions ($Z = W$) for the first three pairs given in Table 6.11 (the last pair can have any value) is shown in Fig. 6.1. Thus, while the simplex method is dealing directly with suboptimal basic solutions and working toward optimality in the primal problem, it is simultaneously dealing indirectly with complementary superoptimal solutions and working toward feasibility in the dual problem. Conversely, it sometimes is more convenient (or necessary) to work directly with superoptimal basic solutions and to move toward feasibility in the primal problem, which is the purpose of the dual simplex method described in Sec. 7.1.

The third and fourth columns of Table 6.11 introduce two other common terms that are used to describe a pair of complementary basic solutions. The two solutions are said to be **primal feasible** if the primal basic solution is feasible, whereas they are called **dual feasible** if the complementary dual basic solution is feasible for the dual problem. Using

TABLE 6.11 Relationships between complementary basic solutions

Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions: Primal Feasible?	Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No

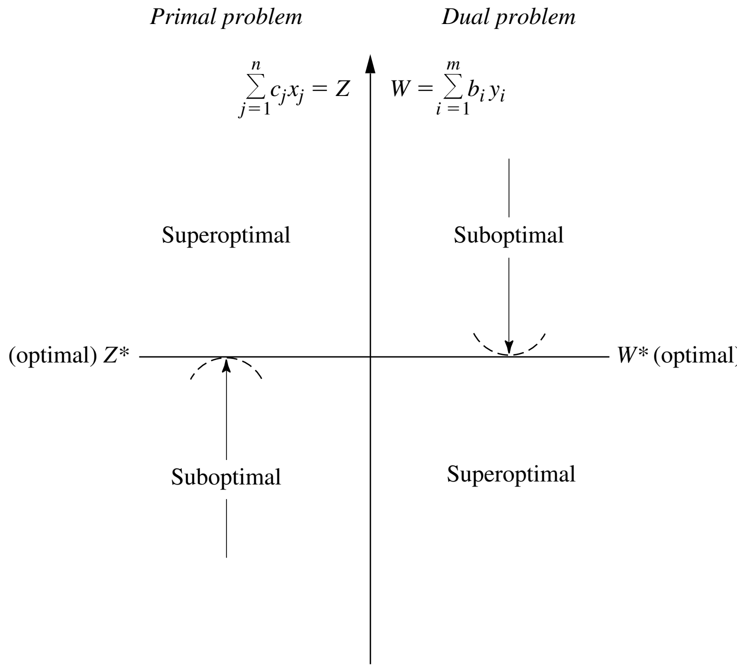

FIGURE 6.1
Range of possible values of $Z = W$ for certain types of complementary basic solutions.

this terminology, the simplex method deals with primal feasible solutions and strives toward achieving dual feasibility as well. When this is achieved, the two complementary basic solutions are optimal for their respective problems.

These relationships prove very useful, particularly in sensitivity analysis, as you will see later in the chapter.

6.4 ADAPTING TO OTHER PRIMAL FORMS

Thus far it has been assumed that the model for the primal problem is in our standard form. However, we indicated at the beginning of the chapter that any linear programming problem, whether in our standard form or not, possesses a dual problem. Therefore, this section focuses on how the dual problem changes for other primal forms.

Each nonstandard form was discussed in Sec. 4.6, and we pointed out how it is possible to convert each one to an equivalent standard form if so desired. These conversions are summarized in Table 6.12. Hence, you always have the option of converting any model to our standard form and *then* constructing its dual problem in the usual way. To illustrate, we do this for our standard dual problem (it must have a dual also) in Table 6.13. Note that what we end up with is just our standard primal problem! Since any pair of primal and dual problems can be converted to these forms, this fact implies that the dual of the dual problem always is the primal problem. Therefore, for any primal problem and its dual problem, all relationships between them must be symmetric. This is just the symmetry property already stated in Sec. 6.1 (without proof), but now Table 6.13 demonstrates why it holds.

TABLE 6.12 Conversions to standard form for linear programming models

Nonstandard Form	Equivalent Standard Form
Minimize Z	Maximize $(-Z)$
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ and $-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$
x_j unconstrained in sign	$x_j^+ - x_j^-$, $x_j^+ \geq 0$, $x_j^- \geq 0$

One consequence of the symmetry property is that all the statements made earlier in the chapter about the relationships of the dual problem to the primal problem also hold in reverse.

Another consequence is that it is immaterial which problem is called the primal and which is called the dual. In practice, you might see a linear programming problem fitting our standard form being referred to as the dual problem. The convention is that the model formulated to fit the actual problem is called the primal problem, regardless of its form.

Our illustration of how to construct the dual problem for a nonstandard primal problem did not involve either equality constraints or variables unconstrained in sign. Actually, for these two forms, a shortcut is available. It is possible to show (see Probs. 6.4-7 and 6.4-2*a*) that an *equality constraint* in the primal problem should be treated just like a $\leq$ constraint in

TABLE 6.13 Constructing the dual of the dual problem

Dual Problem		*Converted to Standard Form*
Minimize $W = \mathbf{yb}$, subject to $\mathbf{yA} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.	$\longrightarrow$	Maximize $(-W) = -\mathbf{yb}$, subject to $-\mathbf{yA} \leq -\mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.
		$\downarrow$
Converted to Standard Form		*Its Dual Problem*
Maximize $Z = \mathbf{cx}$, subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.	$\longleftarrow$	Minimize $(-Z) = -\mathbf{cx}$, subject to $-\mathbf{Ax} \geq -\mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

constructing the dual problem except that the nonnegativity constraint for the corresponding dual variable should be *deleted* (i.e., this variable is unconstrained in sign). By the symmetry property, deleting a nonnegativity constraint in the primal problem affects the dual problem only by changing the corresponding inequality constraint to an equality constraint.

Another shortcut involves functional constraints in $\geq$ form for a maximization problem. The straightforward (but longer) approach would begin by converting each such constraint to $\leq$ form

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \longrightarrow -\sum_{j=1}^{n} a_{ij}x_j \leq -b_i.$$

Constructing the dual problem in the usual way then gives $-a_{ij}$ as the coefficient of y_i in functional constraint j (which has $\geq$ form) and a coefficient of $-b_i$ in the objective function (which is to be minimized), where y_i also has a nonnegativity constraint $y_i \geq 0$. Now suppose we define a new variable $y_i' = -y_i$. The changes caused by expressing the dual problem in terms of y_i' instead of y_i are that (1) the coefficients of the variable become a_{ij} for functional constraint j and b_i for the objective function and (2) the constraint on the variable becomes $y_i' \leq 0$ (a *nonpositivity constraint*). The shortcut is to use y_i' instead of y_i as a dual variable so that the parameters in the original constraint (a_{ij} and b_i) immediately become the coefficients of this variable in the dual problem.

Here is a useful mnemonic device for remembering what the forms of dual constraints should be. With a maximization problem, it might seem *sensible* for a functional constraint to be in $\leq$ form, slightly *odd* to be in $=$ form, and somewhat *bizarre* to be in $\geq$ form. Similarly, for a minimization problem, it might seem *sensible* to be in $\geq$ form, slightly *odd* to be in $=$ form, and somewhat *bizarre* to be in $\leq$ form. For the constraint on an individual variable in either kind of problem, it might seem *sensible* to have a nonnegativity constraint, somewhat *odd* to have no constraint (so the variable is unconstrained in sign), and quite *bizarre* for the variable to be restricted to be *less* than or equal to zero. Now recall the correspondence between entities in the primal and dual problems indicated in Table 6.3; namely, functional constraint i in one problem corresponds to variable i in the other problem, and vice versa. The *sensible-odd-bizarre method,* or **SOB method** for short, says that the form of a functional constraint or the constraint on a variable in the dual problem should be sensible, odd, or bizarre, depending on whether the form for the corresponding entity in the primal problem is sensible, odd, or bizarre. Here is a summary.

The SOB Method for Determining the Form of Constraints in the Dual.[1]

1. Formulate the primal problem in either maximization form or minimization form, and then the dual problem automatically will be in the other form.
2. Label the different forms of functional constraints and of constraints on individual variables in the primal problem as being sensible, odd, or bizarre according to Table 6.14.

[1]This particular mnemonic device (and a related one) for remembering what the forms of dual constraints should be has been suggested by Arthur T. Benjamin, a mathematics professor at Harvey Mudd College. An interesting and wonderfully bizarre fact about Professor Benjamin himself is that he is one of the world's great human calculators who can perform such feats as quickly multiplying six-digit numbers in his head.

TABLE 6.14 Corresponding primal-dual forms

Label	Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
	Maximize Z (or W)		Minimize W (or Z)
	Constraint i:		Variable y_i (or x_i):
Sensible	$\leq$ form	$\longleftrightarrow$	$y_i \geq 0$
Odd	$=$ form	$\longleftrightarrow$	Unconstrained
Bizarre	$\geq$ form	$\longleftrightarrow$	$y_i' \leq 0$
	Variable x_j (or y_j):		Constraint j:
Sensible	$x_j \geq 0$	$\longleftrightarrow$	$\geq$ form
Odd	Unconstrained	$\longleftrightarrow$	$=$ form
Bizarre	$x_j' \leq 0$	$\longleftrightarrow$	$\leq$ form

The labeling of the functional constraints depends on whether the problem is a *maximization* problem (use the second column) or a *minimization* problem (use the third column).

3. For each constraint on an *individual variable* in the dual problem, use the form that has the same label as for the functional constraint in the primal problem that corresponds to this dual variable (as indicated by Table 6.3).
4. For each *functional constraint* in the dual problem, use the form that has the same label as for the constraint on the corresponding individual variable in the primal problem (as indicated by Table 6.3).

The arrows between the second and third columns of Table 6.14 spell out the correspondence between the forms of constraints in the primal and dual. Note that the correspondence always is between a functional constraint in one problem and a constraint on an individual variable in the other problem. Since the primal problem can be either a maximization or minimization problem, where the dual then will be of the opposite type, the second column of the table gives the form for whichever is the maximization problem and the third column gives the form for the other problem (a minimization problem).

To illustrate, consider the radiation therapy example presented in Sec. 3.4. (Its model is shown on p. 46.) To show the conversion in both directions in Table 6.14, we begin with the maximization form of this model as the primal problem, before using the (original) minimization form.

The primal problem in maximization form is shown on the left side of Table 6.15. By using the second column of Table 6.14 to represent this problem, the arrows in this table indicate the form of the dual problem in the third column. These same arrows are used in Table 6.15 to show the resulting dual problem. (Because of these arrows, we have placed the functional constraints last in the dual problem rather than in their usual top position.) Beside each constraint in both problems, we have inserted (in parentheses) an S, O, or B to label the form as sensible, odd, or bizarre. As prescribed by the SOB method, the label for each dual constraint always is the same as for the corresponding primal constraint.

TABLE 6.15 **One primal-dual form for the radiation therapy example**

Primal Problem		Dual Problem	
Maximize $-Z = -0.4x_1 - 0.5x_2$,		Minimize $W = 2.7y_1 + 6y_2 + 6y_3'$,	
subject to		subject to	
(S)	$0.3x_1 + 0.1x_2 \le 2.7$ ⟷	$y_1 \ge 0$	(S)
(O)	$0.5x_1 + 0.5x_2 = 6$ ⟷	y_2 unconstrained in sign	(O)
(B)	$0.6x_1 + 0.4x_2 \ge 6$ ⟷	$y_3' \le 0$	(B)
and		and	
(S)	$x_1 \ge 0$ ⟷	$0.3y_1 + 0.5y_2 + 0.6y_3' \ge -0.4$	(S)
(S)	$x_2 \ge 0$ ⟷	$0.1y_1 + 0.5y_2 + 0.4y_3' \ge -0.5$	(S)

However, there was no need (other than for illustrative purposes) to convert the primal problem to maximization form. Using the original minimization form, the equivalent primal problem is shown on the left side of Table 6.16. Now we use the *third column* of Table 6.14 to represent this primal problem, where the arrows indicate the form of the dual problem in the *second column.* These same arrows in Table 6.16 show the resulting dual problem on the right side. Again, the labels on the constraints show the application of the SOB method.

Just as the primal problems in Tables 6.15 and 6.16 are equivalent, the two dual problems also are completely equivalent. The key to recognizing this equivalency lies in the fact that the variables in each version of the dual problem are the negative of those in the other version ($y_1' = -y_1$, $y_2' = -y_2$, $y_3 = -y_3'$). Therefore, for each version, if the variables in the other version are used instead, and if both the objective function and the constraints are multiplied through by -1, then the other version is obtained. (Problem 6.4-5 asks you to verify this.)

If the simplex method is to be applied to either a primal or a dual problem that has any variables constrained to be *nonpositive* (for example, $y_3' \le 0$ in the dual problem of Table 6.15), this variable may be replaced by its *nonnegative* counterpart (for example, $y_3 = -y_3'$).

TABLE 6.16 **The other primal-dual form for the radiation therapy example**

Primal Problem		Dual Problem	
Minimize $Z = 0.4x_1 + 0.5x_2$,		Maximize $W = 2.7y_1' + 6y_2' + 6y_3$,	
subject to		subject to	
(B)	$0.3x_1 + 0.1x_2 \le 2.7$ ⟷	$y_1' \le 0$	(B)
(O)	$0.5x_1 + 0.5x_2 = 6$ ⟷	y_2' unconstrained in sign	(O)
(S)	$0.6x_1 + 0.4x_2 \ge 6$ ⟷	$y_3 \ge 0$	(S)
and		and	
(S)	$x_1 \ge 0$ ⟷	$0.3y_1' + 0.5y_2' + 0.6y_3 \le 0.4$	(S)
(S)	$x_2 \ge 0$ ⟷	$0.1y_1' + 0.5y_2' + 0.4y_3 \le 0.6$	(S)

When artificial variables are used to help the simplex method solve a primal problem, the duality interpretation of row 0 of the simplex tableau is the following: Since artificial variables play the role of slack variables, their coefficients in row 0 now provide the values of the corresponding dual variables in the complementary basic solution for the dual problem. Since artificial variables are used to replace the real problem with a more convenient artificial problem, this dual problem actually is the dual of the artificial problem. However, after all the artificial variables become nonbasic, we are back to the real primal and dual problems. With the two-phase method, the artificial variables would need to be retained in phase 2 in order to read off the complete dual solution from row 0. With the Big M method, since M has been added initially to the coefficient of each artificial variable in row 0, the current value of each corresponding dual variable is the current coefficient of this artificial variable *minus* M.

For example, look at row 0 in the final simplex tableau for the radiation therapy example, given at the bottom of Table 4.12 on p. 142. After M is subtracted from the coefficients of the artificial variables $\bar{x}_4$ and $\bar{x}_6$, the optimal solution for the corresponding dual problem given in Table 6.15 is read from the coefficients of x_3, $\bar{x}_4$, and $\bar{x}_6$ as $(y_1, y_2, y_3') = (0.5, -1.1, 0)$. As usual, the surplus variables for the two functional constraints are read from the coefficients of x_1 and x_2 as $z_1 - c_1 = 0$ and $z_2 - c_2 = 0$.

6.5 THE ROLE OF DUALITY THEORY IN SENSITIVITY ANALYSIS

As described further in the next two sections, sensitivity analysis basically involves investigating the effect on the optimal solution of making changes in the values of the model parameters a_{ij}, b_i, and c_j. However, changing parameter values in the primal problem also changes the corresponding values in the dual problem. Therefore, you have your choice of which problem to use to investigate each change. Because of the primal-dual relationships presented in Secs. 6.1 and 6.3 (especially the complementary basic solutions property), it is easy to move back and forth between the two problems as desired. In some cases, it is more convenient to analyze the dual problem directly in order to determine the complementary effect on the primal problem. We begin by considering two such cases.

Changes in the Coefficients of a Nonbasic Variable

Suppose that the changes made in the original model occur in the coefficients of a variable that was nonbasic in the original optimal solution. What is the effect of these changes on this solution? Is it still feasible? Is it still optimal?

Because the variable involved is nonbasic (value of zero), changing its coefficients cannot affect the feasibility of the solution. Therefore, the open question in this case is whether it is still optimal. As Tables 6.10 and 6.11 indicate, an equivalent question is whether the complementary basic solution for the dual problem is still feasible after these changes are made. Since these changes affect the dual problem by changing only one constraint, this question can be answered simply by checking whether this complementary basic solution still satisfies this revised constraint.

We shall illustrate this case in the corresponding subsection of Sec. 6.7 after developing a relevant example.

Introduction of a New Variable

As indicated in Table 6.6, the decision variables in the model typically represent the levels of the various activities under consideration. In some situations, these activities were selected from a larger group of *possible* activities, where the remaining activities were not included in the original model because they seemed less attractive. Or perhaps these other activities did not come to light until after the original model was formulated and solved. Either way, the key question is whether any of these previously unconsidered activities are sufficiently worthwhile to warrant initiation. In other words, would adding any of these activities to the model change the original optimal solution?

Adding another activity amounts to introducing a new variable, with the appropriate coefficients in the functional constraints and objective function, into the model. The only resulting change in the dual problem is to add a *new constraint* (see Table 6.3).

After these changes are made, would the original optimal solution, along with the new variable equal to zero (nonbasic), still be optimal for the primal problem? As for the preceding case, an equivalent question is whether the complementary basic solution for the dual problem is still feasible. And, as before, this question can be answered simply by checking whether this complementary basic solution satisfies one constraint, which in this case is the new constraint for the dual problem.

To illustrate, suppose for the Wyndor Glass Co. problem of Sec. 3.1 that a possible third new product now is being considered for inclusion in the product line. Letting x_{new} represent the production rate for this product, we show the resulting revised model as follows:

$$\text{Maximize} \quad Z = 3x_1 + 5x_2 + 4x_{\text{new}},$$

subject to

$$\begin{aligned} x_1 \phantom{{}+2x_2} + 2x_{\text{new}} &\le 4 \\ 2x_2 + 3x_{\text{new}} &\le 12 \\ 3x_1 + 2x_2 + x_{\text{new}} &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_{\text{new}} \ge 0.$$

After we introduced slack variables, the original optimal solution for this problem without x_{new} (given by Table 4.8) was $(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$. Is this solution, along with $x_{\text{new}} = 0$, still optimal?

To answer this question, we need to check the complementary basic solution for the dual problem. As indicated by the *complementary optimal basic solutions property* in Sec. 6.3, this solution is given in row 0 of the *final* simplex tableau for the primal problem, using the locations shown in Table 6.4 and illustrated in Table 6.5. Therefore, as given in both the bottom row of Table 6.5 and the sixth row of Table 6.9, the solution is

$$(y_1, y_2, y_3, z_1 - c_1, z_2 - c_2) = \left(0, \frac{3}{2}, 1, 0, 0\right).$$

(Alternatively, this complementary basic solution can be derived in the way that was illustrated in Sec. 6.3 for the complementary basic solution in the next-to-last row of Table 6.9.)

Since this solution was optimal for the original dual problem, it certainly satisfies the original dual constraints shown in Table 6.1. But does it satisfy this new dual constraint?

$$2y_1 + 3y_2 + y_3 \geq 4$$

Plugging in this solution, we see that

$$2(0) + 3\left(\frac{3}{2}\right) + (1) \geq 4$$

is satisfied, so this dual solution is still feasible (and thus still optimal). Consequently, the original primal solution (2, 6, 2, 0, 0), along with $x_{\text{new}} = 0$, is still optimal, so this third possible new product should *not* be added to the product line.

This approach also makes it very easy to conduct sensitivity analysis on the coefficients of the new variable added to the primal problem. By simply checking the new dual constraint, you can immediately see how far any of these parameter values can be changed before they affect the feasibility of the dual solution and so the optimality of the primal solution.

Other Applications

Already we have discussed two other key applications of duality theory to sensitivity analysis, namely, shadow prices and the dual simplex method. As described in Secs. 4.7 and 6.2, the optimal dual solution $(y_1^*, y_2^*, \ldots, y_m^*)$ provides the shadow prices for the respective resources that indicate how Z would change if (small) changes were made in the b_i (the resource amounts). The resulting analysis will be illustrated in some detail in Sec. 6.7.

In more general terms, the economic interpretation of the dual problem and of the simplex method presented in Sec. 6.2 provides some useful insights for sensitivity analysis.

When we investigate the effect of changing the b_i or the a_{ij} values (for basic variables), the original optimal solution may become a *superoptimal* basic solution (as defined in Table 6.10) instead. If we then want to *reoptimize* to identify the new optimal solution, the dual simplex method (discussed at the end of Secs. 6.1 and 6.3) should be applied, starting from this basic solution.

We mentioned in Sec. 6.1 that sometimes it is more efficient to solve the dual problem directly by the simplex method in order to identify an optimal solution for the primal problem. When the solution has been found in this way, sensitivity analysis for the primal problem then is conducted by applying the procedure described in the next two sections directly to the dual problem and then inferring the complementary effects on the primal problem (e.g., see Table 6.11). This approach to sensitivity analysis is relatively straightforward because of the close primal-dual relationships described in Secs. 6.1 and 6.3. (See Prob. 6.6-3.)

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6.6 THE ESSENCE OF SENSITIVITY ANALYSIS

The work of the operations research team usually is not even nearly done when the simplex method has been successfully applied to identify an optimal solution for the model. As we pointed out at the end of Sec. 3.3, one assumption of linear programming is that all the parameters of the model (a_{ij}, b_i, and c_j) are *known constants*. Actually, the parameter values used in the model normally are just *estimates* based on a *prediction of future conditions*. The data obtained to develop these estimates often are rather crude or nonexistent, so that the parameters in the original formulation may represent little more than quick rules of thumb provided by harassed line personnel. The data may even represent deliberate overestimates or underestimates to protect the interests of the estimators.

Thus, the successful manager and operations research staff will maintain a healthy skepticism about the original numbers coming out of the computer and will view them in many cases as only a starting point for further analysis of the problem. An "optimal" solution is optimal only with respect to the specific model being used to represent the real problem, and such a solution becomes a reliable guide for action only after it has been verified as performing well for other reasonable representations of the problem. Furthermore, the model parameters (particularly b_i) sometimes are set as a result of managerial policy decisions (e.g., the amount of certain resources to be made available to the activities), and these decisions should be reviewed after their potential consequences are recognized.

For these reasons it is important to perform **sensitivity analysis** to investigate the effect on the optimal solution provided by the simplex method if the parameters take on other possible values. Usually there will be some parameters that can be assigned any reasonable value without the optimality of this solution being affected. However, there may also be parameters with likely alternative values that would yield a new optimal solution. This situation is particularly serious if the original solution would then have a substantially inferior value of the objective function, or perhaps even be infeasible!

Therefore, one main purpose of sensitivity analysis is to identify the **sensitive parameters** (i.e., the parameters whose values cannot be changed without changing the optimal solution). For certain parameters that are not categorized as sensitive, it is also very helpful to determine the *range of values* of the parameter over which the optimal solution will remain unchanged. (We call this range of values the *allowable range to stay optimal.*) In some cases, changing a parameter value can affect the *feasibility* of the optimal BF solution. For such parameters, it is useful to determine the range of values over which the optimal BF solution (with adjusted values for the basic variables) will remain feasible. (We call this range of values the *allowable range to stay feasible.*) In the next section, we will describe the specific procedures for obtaining this kind of information.

Such information is invaluable in two ways. First, it identifies the more important parameters, so that special care can be taken to estimate them closely and to select a solution that performs well for most of their likely values. Second, it identifies the parameters that will need to be monitored particularly closely as the study is implemented. If it is discovered that the true value of a parameter lies outside its allowable range, this immediately signals a need to change the solution.

For small problems, it would be straightforward to check the effect of a variety of changes in parameter values simply by reapplying the simplex method each time to see if the optimal solution changes. This is particularly convenient when using a spreadsheet formulation. Once the Solver has been set up to obtain an optimal solution, all you have to do is make any desired change on the spreadsheet and then click on the Solve button again.

However, for larger problems of the size typically encountered in practice, sensitivity analysis would require an exorbitant computational effort if it were necessary to reapply the simplex method from the beginning to investigate each new change in a parameter value. Fortunately, the fundamental insight discussed in Sec. 5.3 virtually eliminates computational effort. The basic idea is that the fundamental insight *immediately* reveals just how any changes in the original model would change the numbers in the final simplex tableau (assuming that the *same* sequence of algebraic operations originally performed by the simplex method were to be *duplicated*). Therefore, after making a few simple calculations to revise this tableau, we can check easily whether the original optimal BF solution is now nonoptimal (or infeasible). If so, this solution would be used as the initial basic solution to restart the simplex method (or dual simplex method) to find the new optimal solution, if desired. If the changes in the model are not major, only a very few iterations should be required to reach the new optimal solution from this "advanced" initial basic solution.

To describe this procedure more specifically, consider the following situation. The simplex method already has been used to obtain an optimal solution for a linear programming model with specified values for the b_i, c_j, and a_{ij} parameters. To initiate sensitivity analysis, at least one of the parameters is changed. After the changes are made, let $\bar{b}_i$, $\bar{c}_j$, and $\bar{a}_{ij}$ denote the values of the various parameters. Thus, in matrix notation,

$$\mathbf{b} \rightarrow \bar{\mathbf{b}}, \qquad \mathbf{c} \rightarrow \bar{\mathbf{c}}, \qquad \mathbf{A} \rightarrow \bar{\mathbf{A}},$$

for the revised model.

The first step is to revise the final simplex tableau to reflect these changes. Continuing to use the notation presented in Table 5.10, as well as the accompanying formulas for the fundamental insight [(1) $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T}$ and (2) $\mathbf{T}^* = \mathbf{S}^*\mathbf{T}$], we see that the revised final tableau is calculated from $\mathbf{y}^*$ and $\mathbf{S}^*$ (which have not changed) and the new initial tableau, as shown in Table 6.17.

Example (Variation 1 of the Wyndor Model). To illustrate, suppose that the first revision in the model for the Wyndor Glass Co. problem of Sec. 3.1 is the one shown in Table 6.18.

Thus, the changes from the original model are $c_1 = 3 \rightarrow 4$, $a_{31} = 3 \rightarrow 2$, and $b_2 = 12 \rightarrow 24$. Figure 6.2 shows the graphical effect of these changes. For the original model, the simplex method already has identified the optimal CPF solution as (2, 6), lying at the intersection of the two constraint boundaries, shown as dashed lines $2x_2 = 12$ and $3x_1 + 2x_2 = 18$. Now the revision of the model has shifted both of these constraint boundaries as shown by the dark lines $2x_2 = 24$ and $2x_1 + 2x_2 = 18$. Consequently, the previous

TABLE 6.17 **Revised final simplex tableau resulting from changes in original model**

		Coefficient of:			
	Eq.	**Z**	**Original Variables**	**Slack Variables**	**Right Side**
New initial tableau	(0)	1	$-\bar{\mathbf{c}}$	$\mathbf{0}$	0
	(1, 2, . . . , m)	$\mathbf{0}$	$\bar{\mathbf{A}}$	$\mathbf{I}$	$\bar{\mathbf{b}}$
Revised final tableau	(0)	1	$\mathbf{z}^* - \bar{\mathbf{c}} = \mathbf{y}^*\bar{\mathbf{A}} - \bar{\mathbf{c}}$	$\mathbf{y}^*$	$Z^* = \mathbf{y}^*\bar{\mathbf{b}}$
	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{A}^* = \mathbf{S}^*\bar{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$

TABLE 6.18 **The original model and the first revised model (variation 1) for conducting sensitivity analysis on the Wyndor Glass Co. model**

Original Model	*Revised Model*
Maximize $Z = [3, 5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,	Maximize $Z = [4, 5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,
subject to	subject to
$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$
and	and
$\mathbf{x} \ge \mathbf{0}$.	$\mathbf{x} \ge \mathbf{0}$.

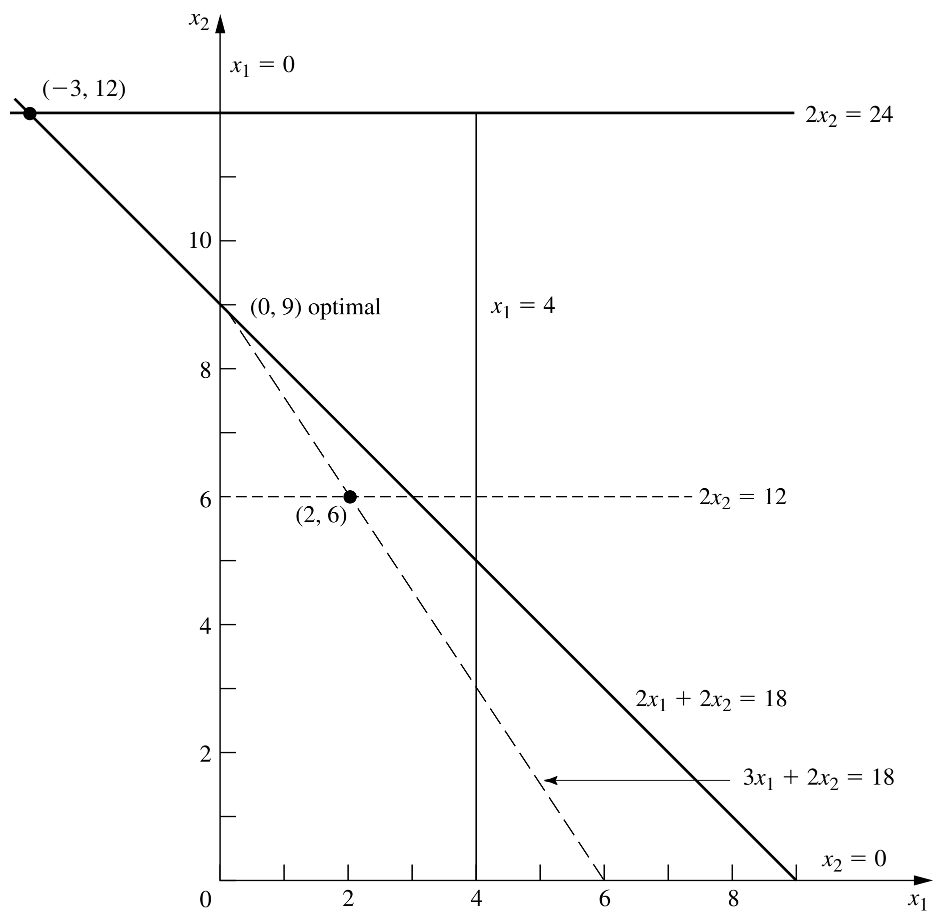


FIGURE 6.2
Shift of the final corner-point solution from (2, 6) to (−3, 12) for Variation 1 of the Wyndor Glass Co. model where $c_1 = 3 \rightarrow 4$, $a_{31} = 3 \rightarrow 2$, and $b_2 = 12 \rightarrow 24$.

CPF solution (2, 6) now shifts to the new intersection (−3, 12), which is a corner-point *infeasible* solution for the revised model. The procedure described in the preceding paragraphs finds this shift *algebraically* (in augmented form). Furthermore, it does so in a manner that is very efficient even for huge problems where graphical analysis is impossible.

To carry out this procedure, we begin by displaying the parameters of the revised model in matrix form:

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}.$$

The resulting new initial simplex tableau is shown at the top of Table 6.19. Below this tableau is the original final tableau (as first given in Table 4.8). We have drawn dark boxes around the portions of this final tableau that the changes in the model definitely *do not change,* namely, the coefficients of the slack variables in both row 0 ($\mathbf{y}^*$) and the rest of the rows ($\mathbf{S}^*$). Thus,

$$\mathbf{y}^* = [0, \tfrac{3}{2}, 1], \qquad \mathbf{S}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

TABLE 6.19 Obtaining the revised final simplex tableau for Variation 1 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
New initial tableau	Z	(0)	1	−4	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	24
	x_5	(3)	0	2	2	0	0	1	18
Final tableau for original model	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
Revised final tableau	Z	(0)	1	−2	0	0	$\frac{3}{2}$	1	54
	x_3	(1)	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	−2

These coefficients of the slack variables necessarily are unchanged with the same algebraic operations originally performed by the simplex method because the coefficients of these same variables in the initial tableau are unchanged.

However, because other portions of the initial tableau have changed, there will be changes in the rest of the final tableau as well. Using the formulas in Table 6.17, we calculate the revised numbers in the rest of the final tableau as follows:

$$\mathbf{z}^* - \bar{\mathbf{c}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - [4, 5] = [-2, 0], \qquad Z^* = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54,$$

$$\mathbf{A}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix},$$

$$\mathbf{b}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}.$$

The resulting revised final tableau is shown at the bottom of Table 6.19.

Actually, we can substantially streamline these calculations for obtaining the revised final tableau. Because none of the coefficients of x_2 changed in the original model (tableau), none of them can change in the final tableau, so we can delete their calculation. Several other original parameters (a_{11}, a_{21}, b_1, b_3) also were not changed, so another shortcut is to calculate only the *incremental changes* in the final tableau in terms of the incremental changes in the initial tableau, ignoring those terms in the vector or matrix multiplication that involve zero change in the initial tableau. In particular, the only incremental changes in the initial tableau are $\Delta c_1 = 1$, $\Delta a_{31} = -1$, and $\Delta b_2 = 12$, so these are the only terms that need be considered. This streamlined approach is shown below, where a zero or dash appears in each spot where no calculation is needed.

$$\Delta(\mathbf{z}^* - \mathbf{c}) = \mathbf{y}^*\,\Delta\mathbf{A} - \Delta\mathbf{c} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 0 & — \\ 0 & — \\ -1 & — \end{bmatrix} - [1, —] = [-2, —].$$

$$\Delta Z^* = \mathbf{y}^*\,\Delta\mathbf{b} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = 18.$$

$$\Delta\mathbf{A}^* = \mathbf{S}^*\,\Delta\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & — \\ 0 & — \\ -1 & — \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & — \\ 0 & — \\ -\frac{1}{3} & — \end{bmatrix}.$$

$$\Delta\mathbf{b}^* = \mathbf{S}^*\,\Delta\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \end{bmatrix}.$$

Adding these increments to the original quantities in the final tableau (middle of Table 6.19) then yields the revised final tableau (bottom of Table 6.19).

This *incremental analysis* also provides a useful general insight, namely, that changes in the final tableau must be *proportional* to each change in the initial tableau. We illustrate in the next section how this property enables us to use linear interpolation or extrapolation to determine the range of values for a given parameter over which the final basic solution remains both feasible and optimal.

After obtaining the revised final simplex tableau, we next convert the tableau to proper form from Gaussian elimination (as needed). In particular, the basic variable for row i must have a coefficient of 1 in that row and a coefficient of 0 in every other row (including row 0) for the tableau to be in the proper form for identifying and evaluating the current basic solution. Therefore, if the changes have violated this requirement (which can occur only if the original constraint coefficients of a basic variable have been changed), further changes must be made to restore this form. This restoration is done by using Gaussian elimination, i.e., by successively applying step 3 of an iteration for the simplex method (see Chap. 4) as if each violating basic variable were an entering basic variable. Note that these algebraic operations may also cause further changes in the *right side* column, so that the current basic solution can be read from this column only when the proper form from Gaussian elimination has been fully restored.

For the example, the revised final simplex tableau shown in the top half of Table 6.20 is not in proper form from Gaussian elimination because of the column for the basic variable x_1. Specifically, the coefficient of x_1 in *its* row (row 3) is $\frac{2}{3}$ instead of 1, and it has nonzero coefficients (-2 and $\frac{1}{3}$) in rows 0 and 1. To restore proper form, row 3 is multiplied by $\frac{3}{2}$; then 2 times this new row 3 is added to row 0 and $\frac{1}{3}$ times new row 3 is subtracted from row 1. This yields the proper form from Gaussian elimination shown in

TABLE 6.20 Converting the revised final simplex tableau to proper form from Gaussian elimination for Variation 1 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
Revised final tableau	Z	(0)	1	-2	0	0	$\frac{3}{2}$	1	54
	x_3	(1)	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
Converted to proper form	Z	(0)	1	0	0	0	$\frac{1}{2}$	2	48
	x_3	(1)	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	7
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	-3

the bottom half of Table 6.20, which now can be used to identify the new values for the current (previously optimal) basic solution:

$$(x_1, x_2, x_3, x_4, x_5) = (-3, 12, 7, 0, 0).$$

Because x_1 is negative, this basic solution no longer is feasible. However, it is *superoptimal* (as defined in Table 6.10), and so *dual feasible*, because *all* the coefficients in row 0 still are *nonnegative.* Therefore, the dual simplex method can be used to reoptimize (if desired), by starting from this basic solution. (The sensitivity analysis routine in the OR Courseware includes this option.) Referring to Fig. 6.2 (and ignoring slack variables), the dual simplex method uses just one iteration to move from the corner-point solution $(-3, 12)$ to the optimal CPF solution (0, 9). (It is often useful in sensitivity analysis to identify the solutions that are optimal for some set of likely values of the model parameters and then to determine which of these solutions most *consistently* performs well for the various likely parameter values.)

If the basic solution $(-3, 12, 7, 0, 0)$ had been *neither* primal feasible nor dual feasible (i.e., if the tableau had negative entries in *both* the *right side* column and row 0), artificial variables could have been introduced to convert the tableau to the proper form for an initial simplex tableau.[1]

The General Procedure. When one is testing to see how *sensitive* the original optimal solution is to the various parameters of the model, the common approach is to check each parameter (or at least c_j and b_i) individually. In addition to finding allowable ranges as described in the next section, this check might include changing the value of the parameter from its initial estimate to other possibilities in the *range of likely values* (including the endpoints of this range). Then some combinations of simultaneous changes of parameter values (such as changing an entire functional constraint) may be investigated. *Each* time one (or more) of the parameters is changed, the procedure described and illustrated here would be applied. Let us now summarize this procedure.

Summary of Procedure for Sensitivity Analysis

1. *Revision of model:* Make the desired change or changes in the model to be investigated next.
2. *Revision of final tableau:* Use the fundamental insight (as summarized by the formulas on the bottom of Table 6.17) to determine the resulting changes in the final simplex tableau. (See Table 6.19 for an illustration.)
3. *Conversion to proper form from Gaussian elimination:* Convert this tableau to the proper form for identifying and evaluating the current basic solution by applying (as necessary) Gaussian elimination. (See Table 6.20 for an illustration.)
4. *Feasibility test:* Test this solution for feasibility by checking whether all its basic variable values in the right-side column of the tableau still are nonnegative.
5. *Optimality test:* Test this solution for optimality (if feasible) by checking whether all its nonbasic variable coefficients in row 0 of the tableau still are nonnegative.
6. *Reoptimization:* If this solution fails either test, the new optimal solution can be obtained (if desired) by using the current tableau as the initial simplex tableau (and making any necessary conversions) for the simplex method or dual simplex method.

[1]There also exists a primal-dual algorithm that can be directly applied to such a simplex tableau without any conversion.

The interactive routine entitled *sensitivity analysis* in the OR Courseware will enable you to efficiently practice applying this procedure. In addition, a demonstration in OR Tutor (also entitled *sensitivity analysis*) provides you with another example.

In the next section, we shall discuss and illustrate the application of this procedure to each of the major categories of revisions in the original model. This discussion will involve, in part, expanding upon the example introduced in this section for investigating changes in the Wyndor Glass Co. model. In fact, we shall begin by *individually* checking each of the preceding changes. At the same time, we shall integrate some of the applications of duality theory to sensitivity analysis discussed in Sec. 6.5.

6.7 APPLYING SENSITIVITY ANALYSIS

Sensitivity analysis often begins with the investigation of changes in the values of b_i, the amount of resource i ($i = 1, 2, \ldots, m$) being made available for the activities under consideration. The reason is that there generally is more flexibility in setting and adjusting these values than there is for the other parameters of the model. As already discussed in Secs. 4.7 and 6.2, the economic interpretation of the dual variables (the y_i) as shadow prices is extremely useful for deciding which changes should be considered.

Case 1—Changes in b_i

Suppose that the only changes in the current model are that one or more of the b_i parameters ($i = 1, 2, \ldots, m$) has been changed. In this case, the *only* resulting changes in the final simplex tableau are in the *right-side* column. Consequently, the tableau still will be in proper form from Gaussian elimination and all the nonbasic variable coefficients in row 0 still will be nonnegative. Therefore, both the *conversion to proper form from Gaussian elimination* and the *optimality test* steps of the general procedure can be skipped. After revising the right-side column of the tableau, the only question will be whether all the basic variable values in this column still are nonnegative (the feasibility test).

As shown in Table 6.17, when the vector of the b_i values is changed from $\mathbf{b}$ to $\overline{\mathbf{b}}$, the formulas for calculating the new *right-side* column in the final tableau are

Right side of final row 0: $Z^* = \mathbf{y}^*\overline{\mathbf{b}}$,
Right side of final rows 1, 2, . . . , m: $\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}}$.

(See the bottom of Table 6.17 for the location of the unchanged vector $\mathbf{y}^*$ and matrix $\mathbf{S}^*$ in the final tableau.)

Example (Variation 2 of the Wyndor Model). Sensitivity analysis is begun for the original Wyndor Glass Co. problem of Sec. 3.1 by examining the optimal values of the y_i dual variables ($y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$). These shadow prices give the marginal value of each resource i for the activities (two new products) under consideration, where marginal value is expressed in the units of Z (thousands of dollars of profit per week). As discussed in Sec. 4.7 (see Fig. 4.8), the total profit from these activities can be increased \$1,500 per week (y_2^* times \$1,000 per week) for each additional unit of resource 2 (hour of production time per week in Plant 2) that is made available. This increase in profit holds for relatively small changes that do not affect the feasibility of the current basic solution (and so do not affect the y_i^* values).

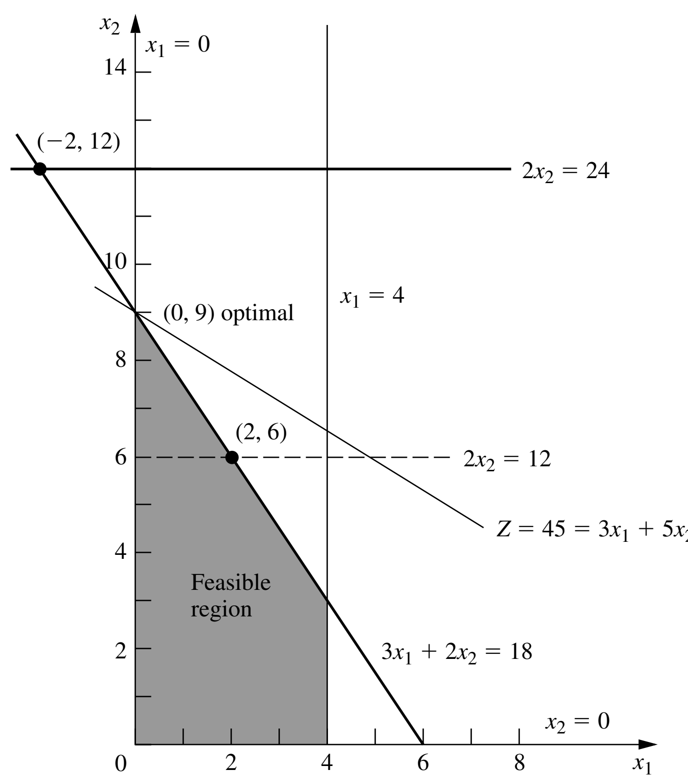
Consequently, the OR team has investigated the marginal profitability from the other current uses of this resource to determine if any are less than \$1,500 per week. This investigation reveals that one old product is far less profitable. The production rate for this product already has been reduced to the minimum amount that would justify its marketing expenses. However, it can be discontinued altogether, which would provide an additional 12 units of resource 2 for the new products. Thus, the next step is to determine the profit that could be obtained from the new products if this shift were made. This shift changes b_2 from 12 to 24 in the linear programming model. Figure 6.3 shows the graphical effect of this change, including the shift in the final corner-point solution from (2, 6) to (−2, 12). (Note that this figure differs from Fig. 6.2, which depicts Variation 1 of the Wyndor model, because the constraint $3x_1 + 2x_2 \leq 18$ has not been changed here.)

Thus, for Variation 2 of the Wyndor model, the only revision in the original model is the following change in the vector of the b_i values:

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}.$$

so only b_2 has a new value.

FIGURE 6.3
Feasible region for Variation 2 of the Wyndor Glass Co. model where $b_2 = 12 \rightarrow 24$.



Analysis of Variation 2. When the fundamental insight (Table 6.17) is applied, the effect of this change in b_2 on the original final simplex tableau (middle of Table 6.19) is that the entries in the right-side column change to the following values:

$$Z^* = \mathbf{y}^*\overline{\mathbf{b}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54,$$

$$\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}, \qquad \text{so} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}.$$

Equivalently, because the only change in the original model is $\Delta b_2 = 24 - 12 = 12$, incremental analysis can be used to calculate these same values more quickly. Incremental analysis involves calculating just the *increments* in the tableau values caused by the change (or changes) in the original model, and then adding these increments to the original values. In this case, the increments in Z^* and $\mathbf{b}^*$ are

$$\Delta Z^* = \mathbf{y}^*\Delta\mathbf{b} = \mathbf{y}^*\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{y}^*\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix},$$

$$\Delta\mathbf{b}^* = \mathbf{S}^*\,\Delta\mathbf{b} = \mathbf{S}^*\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{S}^*\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}.$$

Therefore, using the second component of $\mathbf{y}^*$ and the second column of $\mathbf{S}^*$, the only calculations needed are

$$\Delta Z^* = \frac{3}{2}(12) = 18, \qquad \text{so } Z^* = 36 + 18 = 54,$$

$$\Delta b_1^* = \frac{1}{3}(12) = 4, \qquad \text{so } b_1^* = 2 + 4 = 6,$$

$$\Delta b_2^* = \frac{1}{2}(12) = 6, \qquad \text{so } b_2^* = 6 + 6 = 12,$$

$$\Delta b_3^* = -\frac{1}{3}(12) = -4, \qquad \text{so } b_3^* = 2 - 4 = -2,$$

where the original values of these quantities are obtained from the right-side column in the original final tableau (middle of Table 6.19). The resulting revised final tableau corresponds completely to this original final tableau except for replacing the right-side column with these new values.

Therefore, the current (previously optimal) basic solution has become

$$(x_1, x_2, x_3, x_4, x_5) = (-2, 12, 6, 0, 0),$$

which fails the feasibility test because of the negative value. The dual simplex method now can be applied, starting with this revised simplex tableau, to find the new optimal so-

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$c_1 = 3,$	$c_2 = 5$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = 2,$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = 2,$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	-3	0	0	1	-1	6

lution. This method leads in just one iteration to the new final simplex tableau shown in Table 6.21. (Alternatively, the simplex method could be applied from the beginning, which also would lead to this final tableau in just one iteration in this case.) This tableau indicates that the new optimal solution is

$$(x_1, x_2, x_3, x_4, x_5) = (0, 9, 4, 6, 0),$$

with $Z = 45$, thereby providing an increase in profit from the new products of 9 units (\$9,000 per week) over the previous $Z = 36$. The fact that $x_4 = 6$ indicates that 6 of the 12 additional units of resource 2 are unused by this solution.

Based on the results with $b_2 = 24$, the relatively unprofitable old product will be discontinued and the unused 6 units of resource 2 will be saved for some future use. Since y_3^* still is positive, a similar study is made of the possibility of changing the allocation of resource 3, but the resulting decision is to retain the current allocation. Therefore, the current linear programming model at this point (Variation 2) has the parameter values and optimal solution shown in Table 6.21. This model will be used as the starting point for investigating other types of changes in the model later in this section. However, before turning to these other cases, let us take a broader look at the current case.

The Allowable Range to Stay Feasible. Although $\Delta b_2 = 12$ proved to be too large an increase in b_2 to retain feasibility (and so optimality) with the basic solution where x_1, x_2, and x_3 are the basic variables (middle of Table 6.19), the above incremental analysis shows immediately just how large an increase is feasible. In particular, note that

$$b_1^* = 2 + \frac{1}{3}\,\Delta b_2,$$

$$b_2^* = 6 + \frac{1}{2}\,\Delta b_2,$$

$$b_3^* = 2 - \frac{1}{3}\,\Delta b_2,$$

where these three quantities are the values of x_3, x_2, and x_1, respectively, for this basic solution. The solution remains feasible, and so optimal, as long as all three quantities remain nonnegative.

$$\begin{aligned}
2 + \frac{1}{3}\,\Delta b_2 \geq 0 &\Rightarrow \frac{1}{3}\,\Delta b_2 \geq -2 \Rightarrow \Delta b_2 \geq \ \ -6,\\
6 + \frac{1}{2}\,\Delta b_2 \geq 0 &\Rightarrow \frac{1}{2}\,\Delta b_2 \geq -6 \Rightarrow \Delta b_2 \geq -12,\\
2 - \frac{1}{3}\,\Delta b_2 \geq 0 &\Rightarrow 2 \geq \frac{1}{3}\,\Delta b_2 \quad \Rightarrow \Delta b_2 \leq \quad 6.
\end{aligned}$$

Therefore, since $b_2 = 12 + \Delta b_2$, the solution remains feasible only if

$$-6 \leq \Delta b_2 \leq 6, \qquad \text{that is,} \qquad 6 \leq b_2 \leq 18.$$

(Verify this graphically in Fig. 6.3.) As introduced in Sec. 4.7, this range of values for b_2 is referred to as its *allowable range to stay feasible.*

For any b_i, recall from Sec. 4.7 that its **allowable range to stay feasible** is the range of values over which the current optimal BF solution[1] (with adjusted values for the basic variables) remains feasible. Thus, the shadow price for b_i remains valid for evaluating the effect on Z of changing b_i only as long as b_i remains within this allowable range. (It is assumed that the change in this one b_i value is the only change in the model.) The adjusted values for the basic variables are obtained from the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$. The calculation of the allowable range to stay feasible then is based on finding the range of values of b_i such that $\mathbf{b}^* \geq \mathbf{0}$.

Many linear programming software packages use this same technique for automatically generating the allowable range to stay feasible for each b_i. (A similar technique, discussed under Cases 2*a* and 3, also is used to generate an *allowable range to stay optimal* for each c_j.) In Chap. 4, we showed the corresponding output for the Excel Solver and LINDO in Figs. 4.10 and 4.13, respectively. Table 6.22 summarizes this same output with respect to the b_i for the original Wyndor Glass Co. model. For example, both the *allowable increase* and *allowable decrease* for b_2 are 6, that is, $-6 \leq \Delta b_2 \leq 6$. The above analysis shows how these quantities were calculated.

[1]When there is more than one optimal BF solution for the current model (before changing b_i), we are referring here to the one obtained by the simplex method.

TABLE 6.22 Typical software output for sensitivity analysis of the right-hand sides for the original Wyndor Glass Co. model

Constraint	Shadow Price	Current RHS	Allowable Increase	Allowable Decrease
Plant 1	0	4	∞	2
Plant 2	1.5	12	6	6
Plant 3	1	18	6	6

Analyzing Simultaneous Changes in Right-Hand Sides. When multiple b_i values are changed simultaneously, the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$ can again be used to see how the right-hand sides change in the final tableau. If all these right-hand sides still are nonnegative, the feasibility test will indicate that the revised solution provided by this tableau still is feasible. Since row 0 has not changed, being feasible implies that this solution also is optimal.

Although this approach works fine for checking the effect of a *specific* set of changes in the b_i, it does not give much insight into how far the b_i can be simultaneously changed from their original values before the revised solution will no longer be feasible. As part of postoptimality analysis, the management of an organization often is interested in investigating the effect of various changes in policy decisions (e.g., the amounts of resources being made available to the activities under consideration) that determine the right-hand sides. Rather than considering just one specific set of changes, management may want to explore *directions* of changes where some right-hand sides increase while others decrease. Shadow prices are invaluable for this kind of exploration. However, shadow prices remain valid for evaluating the effect of such changes on Z only within certain ranges of changes. For each b_i, the *allowable range to stay feasible* gives this range if *none* of the other b_i are changing at the same time. What do these *allowable ranges* become when some of the b_i are changing simultaneously?

A partial answer to this question is provided by the following 100 percent rule, which combines the *allowable changes* (increase or decrease) for the individual b_i that are given by the last two columns of a table like Table 6.22.

> **The 100 Percent Rule for Simultaneous Changes in Right-Hand Sides:** The shadow prices remain valid for predicting the effect of simultaneously changing the right-hand sides of some of the functional constraints as long as the changes are not too large. To check whether the changes are small enough, calculate for each change the percentage of the allowable change (increase or decrease) for that right-hand side to remain within its allowable range to stay feasible. If the *sum* of the percentage changes does *not* exceed 100 percent, the shadow prices definitely will still be valid. (If the sum *does* exceed 100 percent, then we cannot be sure.)

Example (Variation 3 of the Wyndor Model). To illustrate this rule, consider *Variation* 3 of the Wyndor Glass Co. model, which revises the original model by changing the right-hand side vector as follows:

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \rightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 15 \\ 15 \end{bmatrix}.$$

The calculations for the 100 percent rule in this case are

$$b_2\text{: } 12 \rightarrow 15. \quad \text{Percentage of allowable increase} = 100\left(\frac{15-12}{6}\right) = 50\%$$

$$b_3\text{: } 18 \rightarrow 15. \quad \text{Percentage of allowable decrease} = 100\left(\frac{18-15}{6}\right) = \underline{50\%}$$

$$\text{Sum} = 100\%$$

Since the sum of 100 percent barely does *not* exceed 100 percent, the shadow prices definitely are valid for predicting the effect of these changes on Z. In particular, since the shadow prices of b_2 and b_3 are 1.5 and 1, respectively, the resulting change in Z would be

$$\Delta Z = 1.5(3) + 1(-3) = 1.5,$$

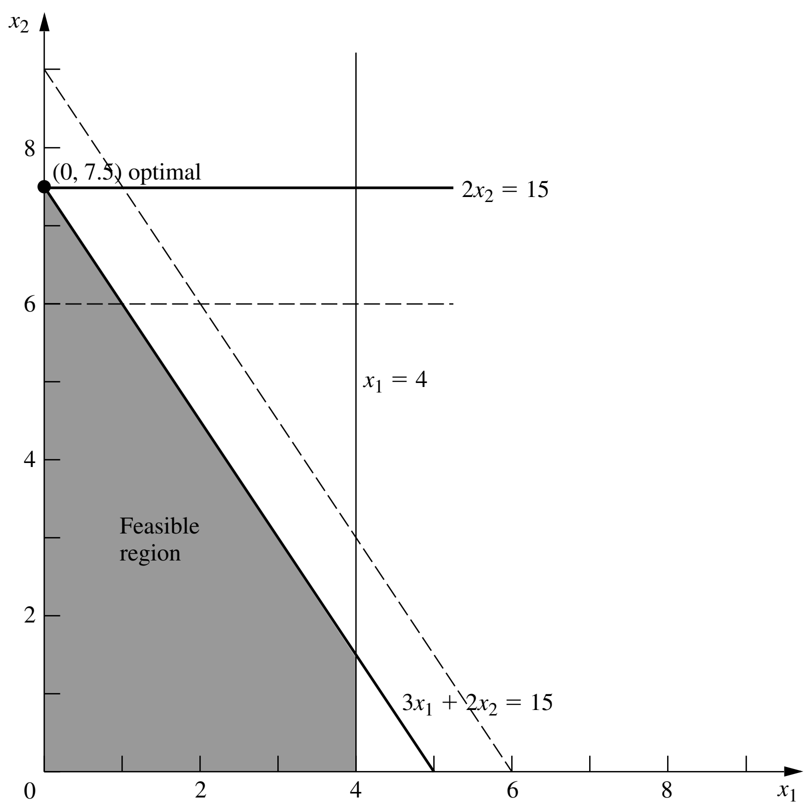
so Z^* would increase from 36 to 37.5.

Figure 6.4 shows the feasible region for this revised model. (The dashed lines show the original locations of the revised constraint boundary lines.) The optimal solution now is the CPF solution (0, 7.5), which gives

$$Z = 3x_1 + 5x_2 = 0 + 5(7.5) = 37.5,$$

just as predicted by the shadow prices. However, note what would happen if either b_2 were further increased above 15 or b_3 were further decreased below 15, so that the sum of the percentages of allowable changes would exceed 100 percent. This would cause the previously optimal corner-point solution to slide to the left of the x_2 axis ($x_1 < 0$), so this *infeasible* solution would no longer be optimal. Consequently, the old shadow prices would no longer be valid for predicting the new value of Z^*.

FIGURE 6.4
Feasible region for Variation 3 of the Wyndor Glass Co. model where $b_2 = 12 \rightarrow 15$ and $b_3 = 18 \rightarrow 15$.



Case 2*a*—Changes in the Coefficients of a Nonbasic Variable

Consider a particular variable x_j (fixed j) that is a nonbasic variable in the optimal solution shown by the final simplex tableau. In Case 2*a,* the only change in the current model is that one or more of the coefficients of this variable—$c_j, a_{1j}, a_{2j}, \ldots, a_{mj}$—have been changed. Thus, letting $\bar{c}_j$ and $\bar{a}_{ij}$ denote the new values of these parameters, with $\overline{\mathbf{A}}_j$ (column j of matrix $\overline{\mathbf{A}}$) as the vector containing the $\bar{a}_{ij}$, we have

$$c_j \longrightarrow \bar{c}_j, \qquad \mathbf{A}_j \longrightarrow \overline{\mathbf{A}}_j$$

for the revised model.

As described at the beginning of Sec. 6.5, duality theory provides a very convenient way of checking these changes. In particular, if the *complementary* basic solution $\mathbf{y}^*$ in the dual problem still satisfies the single dual constraint that has changed, then the original optimal solution in the primal problem *remains optimal* as is. Conversely, if $\mathbf{y}^*$ violates this dual constraint, then this primal solution is *no longer optimal.*

If the optimal solution has changed and you wish to find the new one, you can do so rather easily. Simply apply the fundamental insight to revise the x_j column (the only one that has changed) in the final simplex tableau. Specifically, the formulas in Table 6.17 reduce to the following:

$$\begin{aligned} &\text{Coefficient of } x_j \text{ in final row 0:} && z_j^* - \bar{c}_j = \mathbf{y}^*\overline{\mathbf{A}}_j - \bar{c}_j, \\ &\text{Coefficient of } x_j \text{ in final rows 1 to } m\text{:} && \mathbf{A}_j^* = \mathbf{S}^*\overline{\mathbf{A}}_j. \end{aligned}$$

With the current basic solution no longer optimal, the new value of $z_j^* - c_j$ now will be the one negative coefficient in row 0, so restart the simplex method with x_j as the initial entering basic variable.

Note that this procedure is a streamlined version of the general procedure summarized at the end of Sec. 6.6. Steps 3 and 4 (conversion to proper form from Gaussian elimination and the feasibility test) have been deleted as irrelevant, because the only column being changed in the revision of the final tableau (before reoptimization) is for the nonbasic variable x_j. Step 5 (optimality test) has been replaced by a quicker test of optimality to be performed right after step 1 (revision of model). It is only if this test reveals that the optimal solution has changed, and you wish to find the new one, that steps 2 and 6 (revision of final tableau and reoptimization) are needed.

Example (Variation 4 of the Wyndor Model). Since x_1 is nonbasic in the current optimal solution (see Table 6.21) for Variation 2 of the Wyndor Glass Co. model, the next step in its sensitivity analysis is to check whether any reasonable changes in the estimates of the coefficients of x_1 could still make it advisable to introduce product 1. The set of changes that goes as far as realistically possible to make product 1 more attractive would be to reset $c_1 = 4$ and $a_{31} = 2$. Rather than exploring each of these changes independently (as is often done in sensitivity analysis), we will consider them together. Thus, the changes under consideration are

$$c_1 = 3 \longrightarrow \bar{c}_1 = 4, \qquad \mathbf{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \longrightarrow \overline{\mathbf{A}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

These two changes in Variation 2 give us *Variation* 4 of the Wyndor model. Variation 4 actually is equivalent to Variation 1 considered in Sec. 6.6 and depicted in Fig. 6.2, since Variation 1 combined these two changes with the change in the original Wyndor model ($b_2 = 12 \rightarrow 24$) that gave Variation 2. However, the key difference from the treatment of Variation 1 in Sec. 6.6 is that the analysis of Variation 4 treats Variation 2 as being the original model, so our starting point is the final simplex tableau given in Table 6.21 where x_1 now is a nonbasic variable.

The change in a_{31} revises the feasible region from that shown in Fig. 6.3 to the corresponding region in Fig. 6.5. The change in c_1 revises the objective function from $Z = 3x_1 + 5x_2$ to $Z = 4x_1 + 5x_2$. Figure 6.5 shows that the optimal objective function line $Z = 45 = 4x_1 + 5x_2$ still passes through the current optimal solution (0, 9), so this solution remains optimal after these changes in a_{31} and c_1.

To use duality theory to draw this same conclusion, observe that the changes in c_1 and a_{31} lead to a single revised constraint for the dual problem, namely, the constraint that $a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \geq c_1$. Both this revised constraint and the current $\mathbf{y}^*$ (coefficients of the slack variables in row 0 of Table 6.21) are shown below.

$$y_1^* = 0, \qquad y_2^* = 0, \qquad y_3^* = \frac{5}{2},$$

$$y_1 + 3y_3 \geq 3 \longrightarrow y_1 + 2y_3 \geq 4,$$

$$0 + 2\left(\frac{5}{2}\right) \geq 4.$$

Since $\mathbf{y}^*$ *still* satisfies the revised constraint, the current primal solution (Table 6.21) is still optimal.

Because this solution is still optimal, there is no need to revise the x_j column in the final tableau (step 2). Nevertheless, we do so below for illustrative purposes.

$$z_1^* - \bar{c}_1 = \mathbf{y}^*\overline{\mathbf{A}}_1 - c_1 = [0, 0, \tfrac{5}{2}] \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 4 = 1.$$

$$\mathbf{A}_1^* = \mathbf{S}^*\overline{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

The fact that $z_1^* - \bar{c}_1 \geq 0$ again confirms the optimality of the current solution. Since $z_1^* - c_1$ is the surplus variable for the revised constraint in the dual problem, this way of testing for optimality is equivalent to the one used above.

This completes the analysis of the effect of changing the current model (Variation 2) to Variation 4. Because any larger changes in the original estimates of the coefficients of x_1 would be unrealistic, the OR team concludes that these coefficients are *insensitive* parameters in the current model. Therefore, they will be kept fixed at their best estimates shown in Table 6.21—$c_1 = 3$ and $a_{31} = 3$—for the remainder of the sensitivity analysis.

The Allowable Range to Stay Optimal. We have just described and illustrated how to analyze *simultaneous* changes in the coefficients of a nonbasic variable x_j. It is common practice in sensitivity analysis to also focus on the effect of changing just *one* param-

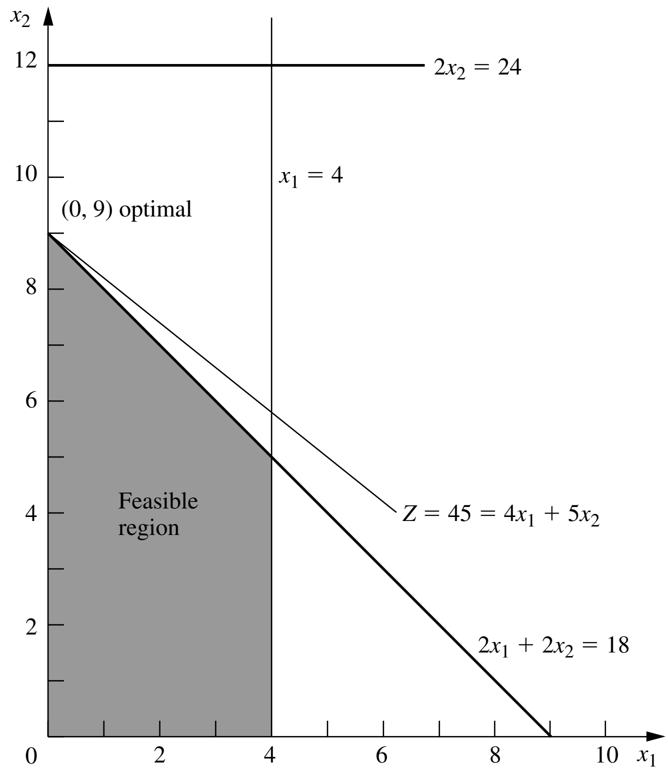


FIGURE 6.5
Feasible region for Variation 4 of the Wyndor model where Variation 2 (Fig. 6.3) has been revised so $a_{31} = 3 \rightarrow 2$ and $c_1 = 3 \rightarrow 4$.

eter, c_j. As introduced in Sec. 4.7, this involves streamlining the above approach to find the *allowable range to stay optimal* for c_j.

For any c_j, recall from Sec. 4.7 that its **allowable range to stay optimal** is the range of values over which the current optimal solution (as obtained by the simplex method for the current model before c_j is changed) remains optimal. (It is assumed that the change in this one c_j is the only change in the current model.) When x_j is a nonbasic variable for this solution, the solution remains optimal as long as $z_j^* - c_j \geq 0$, where $z_j^* = \mathbf{y}^*\mathbf{A}_j$ is a constant unaffected by any change in the value of c_j. Therefore, the allowable range to stay optimal for c_j can be calculated as $c_j \leq \mathbf{y}^*\mathbf{A}_j$.

For example, consider the current model (Variation 2) for the Wyndor Glass Co. problem summarized on the left side of Table 6.21, where the current optimal solution (with $c_1 = 3$) is given on the right side. When considering only the decision variables, x_1 and x_2, this optimal solution is $(x_1, x_2) = (0, 9)$, as displayed in Fig. 6.3. When just c_1 is changed, this solution remains optimal as long as

$$c_1 \leq \mathbf{y}^*\mathbf{A}_1 = [0, 0, \tfrac{5}{2}]\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = 7\tfrac{1}{2},$$

so $c_1 \leq 7\frac{1}{2}$ is the allowable range to stay optimal.

An alternative to performing this vector multiplication is to note in Table 6.21 that $z_1^* - c_1 = \frac{9}{2}$ (the coefficient of x_1 in row 0) when $c_1 = 3$, so $z_1^* = 3 + \frac{9}{2} = 7\frac{1}{2}$. Since $z_1^* = \mathbf{y}^*\mathbf{A}_1$, this immediately yields the same allowable range.

Figure 6.3 provides graphical insight into why $c_1 \leq 7\frac{1}{2}$ is the allowable range. At $c_1 = 7\frac{1}{2}$, the objective function becomes $Z = 7.5x_1 + 5x_2 = 2.5(3x_1 + 2x_2)$, so the optimal objective line will lie on top of the constraint boundary line $3x_1 + 2x_2 = 18$ shown in the figure. Thus, at this endpoint of the allowable range, we have multiple optimal solutions consisting of the line segment between (0, 9) and (4, 3). If c_1 were to be increased any further ($c_1 > 7\frac{1}{2}$), only (4, 3) would be optimal. Consequently, we need $c_1 \leq 7\frac{1}{2}$ for (0, 9) to remain optimal.

For any nonbasic decision variable x_j, the value of $z_j^* - c_j$ sometimes is referred to as the **reduced cost** for x_j, because it is the minimum amount by which the unit *cost* of activity j would have to be *reduced* to make it worthwhile to undertake activity j (increase x_j from zero). Interpreting c_j as the unit profit of activity j (so reducing the unit cost increases c_j by the same amount), the value of $z_j^* - c_j$ thereby is the maximum allowable increase in c_j to keep the current BF solution optimal.

The sensitivity analysis information generated by linear programming software packages normally includes both the reduced cost and the allowable range to stay optimal for each coefficient in the objective function (along with the types of information displayed in Table 6.22). This was illustrated in Figs. 4.10, 4.12, and 4.13 for the Excel Solver and LINDO. Table 6.23 displays this information in a typical form for our current model (Variation 2 of the Wyndor Glass Co. model). The last three columns are used to calculate the allowable range to stay optimal for each coefficient, so these allowable ranges are

$$c_1 \leq 3 + 4.5 = 7.5,$$
$$c_2 \geq 5 - 3 = 2.$$

As was discussed in Sec. 4.7, if any of the allowable increases or decreases had turned out to be zero, this would have been a signpost that the optimal solution given in the table is only one of multiple optimal solutions. In this case, changing the corresponding coefficient a tiny amount beyond the zero allowed and re-solving would provide another optimal CPF solution for the original model.

Thus far, we have described how to calculate the type of information in Table 6.23 for only nonbasic variables. For a basic variable like x_2, the reduced cost automatically is 0. We will discuss how to obtain the allowable range to stay optimal for c_j when x_j is a basic variable under Case 3.

TABLE 6.23 Typical software output for sensitivity analysis of the objective function coefficients for Variation 2 of the Wyndor Glass Co. model

Variable	Value	Reduced Cost	Current Coefficient	Allowable Increase	Allowable Decrease
x_1	0	4.5	3	4.5	∞
x_2	9	0	5	∞	3

Analyzing Simultaneous Changes in Objective Function Coefficients. Regardless of whether x_j is a basic or nonbasic variable, the allowable range to stay optimal for c_j is valid only if this objective function coefficient is the only one being changed. However, when simultaneous changes are made in the coefficients of the objective function, a 100 percent rule is available for checking whether the original solution must still be optimal. Much like the 100 percent rule for simultaneous changes in right-hand sides, this 100 percent rule combines the *allowable changes* (increase or decrease) for the individual c_j that are given by the last two columns of a table like Table 6.23, as described below.

> **The 100 Percent Rule for Simultaneous Changes in Objective Function Coefficients:** If simultaneous changes are made in the coefficients of the objective function, calculate for each change the percentage of the allowable change (increase or decrease) for that coefficient to remain within its allowable range to stay optimal. If the *sum* of the percentage changes does *not* exceed 100 percent, the original optimal solution definitely will still be optimal. (If the sum *does* exceed 100 percent, then we cannot be sure.)

Using Table 6.23 (and referring to Fig. 6.3 for visualization), this 100 percent rule says that (0, 9) will remain optimal for Variation 2 of the Wyndor Glass Co. model even if we simultaneously increase c_1 from 3 and decrease c_2 from 5 as long as these changes are not too large. For example, if c_1 is increased by 1.5 ($33\frac{1}{3}$ percent of the allowable change), then c_2 can be decreased by as much as 2 ($66\frac{2}{3}$ percent of the allowable change). Similarly, if c_1 is increased by 3 ($66\frac{2}{3}$ percent of the allowable change), then c_2 can only be decreased by as much as 1 ($33\frac{1}{3}$ percent of the allowable change). These maximum changes revise the objective function to either $Z = 4.5x_1 + 3x_2$ or $Z = 6x_1 + 4x_2$, which causes the optimal objective function line in Fig. 6.3 to rotate clockwise until it coincides with the constraint boundary equation $3x_1 + 2x_2 = 18$.

In general, when objective function coefficients change in the *same* direction, it is possible for the percentages of allowable changes to sum to more than 100 percent without changing the optimal solution. We will give an example at the end of the discussion of Case 3.

Case 2*b*—Introduction of a New Variable

After solving for the optimal solution, we may discover that the linear programming formulation did not consider all the attractive alternative activities. Considering a new activity requires introducing a new variable with the appropriate coefficients into the objective function and constraints of the current model—which is Case 2*b*.

The convenient way to deal with this case is to treat it just as if it were Case 2*a*! This is done by pretending that the new variable x_j actually was in the original model with all its coefficients equal to zero (so that they still are zero in the final simplex tableau) and that x_j is a nonbasic variable in the current BF solution. Therefore, if we change these zero coefficients to their actual values for the new variable, the procedure (including any reoptimization) does indeed become identical to that for Case 2*a*.

In particular, all you have to do to check whether the current solution still is optimal is to check whether the complementary basic solution $\mathbf{y}^*$ satisfies the one new dual constraint that corresponds to the new variable in the primal problem. We already have described this approach and then illustrated it for the Wyndor Glass Co. problem in Sec. 6.5.

Case 3—Changes in the Coefficients of a Basic Variable

Now suppose that the variable x_j (fixed j) under consideration is a *basic* variable in the optimal solution shown by the final simplex tableau. Case 3 assumes that the only changes in the current model are made to the coefficients of this variable.

Case 3 differs from Case 2*a* because of the requirement that a simplex tableau be in proper form from Gaussian elimination. This requirement allows the column for a nonbasic variable to be anything, so it does not affect Case 2*a*. However, for Case 3, the basic variable x_j must have a coefficient of 1 in its row of the simplex tableau and a coefficient of 0 in every other row (including row 0). Therefore, after the changes in the x_j column of the final simplex tableau have been calculated,[1] it probably will be necessary to apply Gaussian elimination to restore this form, as illustrated in Table 6.20. In turn, this step probably will change the value of the current basic solution and may make it either infeasible or nonoptimal (so reoptimization may be needed). Consequently, all the steps of the overall procedure summarized at the end of Sec. 6.6 are required for Case 3.

Before Gaussian elimination is applied, the formulas for revising the x_j column are the same as for Case 2*a,* as summarized below.

Coefficient of x_j in final row 0: $\quad z_j^* - \bar{c}_j = \mathbf{y}^*\bar{\mathbf{A}}_j - \bar{c}_j.$
Coefficient of x_j in final rows 1 to m: $\quad \mathbf{A}_j^* = \mathbf{S}^*\bar{\mathbf{A}}_j.$

Example (Variation 5 of the Wyndor Model). Because x_2 is a basic variable in Table 6.21 for Variation 2 of the Wyndor Glass Co. model, sensitivity analysis of its coefficients fits Case 3. Given the current optimal solution ($x_1 = 0$, $x_2 = 9$), product 2 is the *only* new product that should be introduced, and its production rate should be relatively large. Therefore, the key question now is whether the initial estimates that led to the coefficients of x_2 in the current model (Variation 2) could have *overestimated* the attractiveness of product 2 so much as to invalidate this conclusion. This question can be tested by checking the *most pessimistic* set of reasonable estimates for these coefficients, which turns out to be $c_2 = 3$, $a_{22} = 3$, and $a_{32} = 4$. Consequently, the changes to be investigated (Variation 5 of the Wyndor model) are

$$c_2 = 5 \longrightarrow \bar{c}_2 = 3, \qquad \mathbf{A}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \bar{\mathbf{A}}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}.$$

The graphical effect of these changes is that the feasible region changes from the one shown in Fig. 6.3 to the one in Fig. 6.6. The optimal solution in Fig. 6.3 is $(x_1, x_2) = (0, 9)$, which is the corner-point solution lying at the intersection of the $x_1 = 0$ and $3x_1 + 2x_2 = 18$ constraint boundaries. With the revision of the constraints, the corre-

[1]For the relatively sophisticated reader, we should point out a possible pitfall for Case 3 that would be discovered at this point. Specifically, the changes in the initial tableau can destroy the linear independence of the columns of coefficients of basic variables. This event occurs only if the unit coefficient of the basic variable x_j in the final tableau has been changed to zero at this point, in which case more extensive simplex method calculations must be used for Case 3.

sponding corner-point solution in Fig. 6.6 is $(0, \frac{9}{2})$. However, this solution no longer is optimal, because the revised objective function of $Z = 3x_1 + 3x_2$ now yields a new optimal solution of $(x_1, x_2) = (4, \frac{3}{2})$.

Analysis of Variation 5. Now let us see how we draw these same conclusions algebraically. Because the only changes in the model are in the coefficients of x_2, the *only* resulting changes in the final simplex tableau (Table 6.21) are in the x_2 column. Therefore, the above formulas are used to recompute just this column.

$$z_2 - \bar{c}_2 = \mathbf{y}^*\bar{\mathbf{A}}_2 - \bar{c}_2 = [0, 0, \tfrac{5}{2}]\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 3 = 7.$$

$$\mathbf{A}_2^* = \mathbf{S}^*\bar{\mathbf{A}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

FIGURE 6.6
Feasible region for Variation 5 of the Wyndor model where Variation 2 (Fig. 6.3) has been revised so $c_2 = 5 \rightarrow 3$, $a_{22} = 2 \rightarrow 3$, and $a_{32} = 2 \rightarrow 4$.

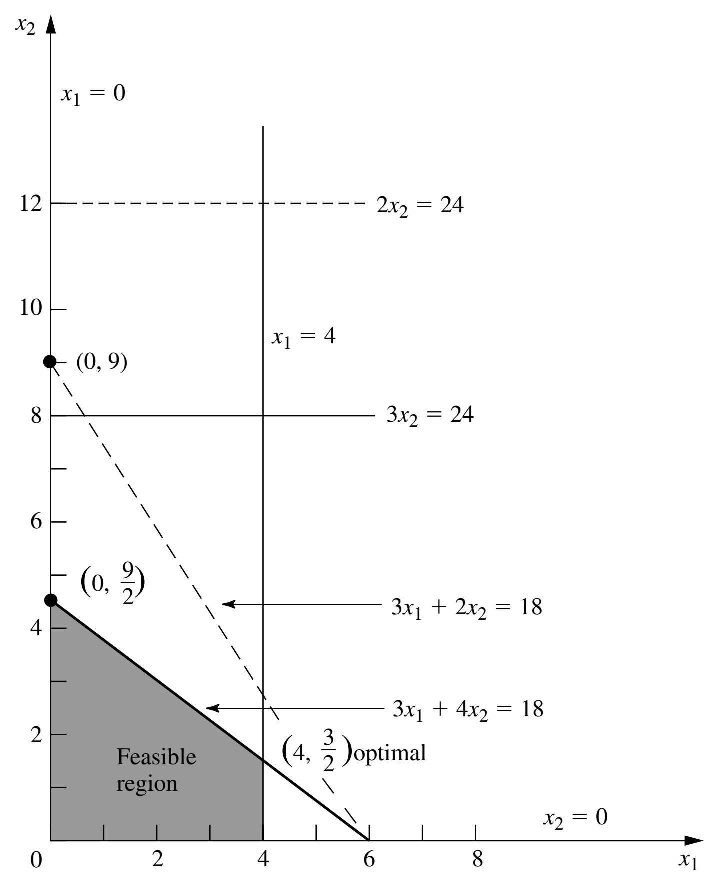


TABLE 6.24 Sensitivity analysis procedure applied to Variation 5 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Revised final tableau	Z	(0)	1	$\frac{9}{2}$	7	0	0	$\frac{5}{2}$	45
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	2	0	0	$\frac{1}{2}$	9
	x_4	(3)	0	-3	-1	0	1	-1	6
Converted to proper form	Z	(0)	1	$-\frac{3}{4}$	0	0	0	$\frac{3}{4}$	$\frac{27}{2}$
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	$\frac{3}{4}$	1	0	0	$\frac{1}{4}$	$\frac{9}{2}$
	x_4	(3)	0	$-\frac{9}{4}$	0	0	1	$-\frac{3}{4}$	$\frac{21}{2}$
New final tableau after reoptimization (only one iteration of the simplex method needed in this case)	Z	(0)	1	0	0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{33}{2}$
	x_1	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$
	x_4	(3)	0	0	0	$\frac{9}{4}$	1	$-\frac{3}{4}$	$\frac{39}{2}$

(Equivalently, incremental analysis with $\Delta c_2 = -2$, $\Delta a_{22} = 1$, and $\Delta a_{32} = 2$ can be used in the same way to obtain this column.)

The resulting revised final tableau is shown at the top of Table 6.24. Note that the new coefficients of the basic variable x_2 do not have the required values, so the conversion to proper form from Gaussian elimination must be applied next. This step involves dividing row 2 by 2, subtracting 7 times the new row 2 from row 0, and adding the new row 2 to row 3.

The resulting second tableau in Table 6.24 gives the new value of the current basic solution, namely, $x_3 = 4$, $x_2 = \frac{9}{2}$, $x_4 = \frac{21}{2}$ ($x_1 = 0$, $x_5 = 0$). Since all these variables are nonnegative, the solution is still feasible. However, because of the negative coefficient of x_1 in row 0, we know that it is no longer optimal. Therefore, the simplex method would be applied to this tableau, with this solution as the initial BF solution, to find the new optimal solution. The initial entering basic variable is x_1, with x_3 as the leaving basic variable. Just one iteration is needed in this case to reach the new optimal solution $x_1 = 4$, $x_2 = \frac{3}{2}$, $x_4 = \frac{39}{2}$ ($x_3 = 0$, $x_5 = 0$), as shown in the last tableau of Table 6.24.

All this analysis suggests that c_2, a_{22}, and a_{32} are relatively sensitive parameters. However, additional data for estimating them more closely can be obtained only by conducting a pilot run. Therefore, the OR team recommends that production of product 2 be initiated immediately on a small scale ($x_2 = \frac{3}{2}$) and that this experience be used to guide the decision on whether the remaining production capacity should be allocated to product 2 or product 1.

The Allowable Range to Stay Optimal. For Case 2*a,* we described how to find the allowable range to stay optimal for any c_j such that x_j is a nonbasic variable for the current optimal solution (before c_j is changed). When x_j is a basic variable instead, the procedure is somewhat more involved because of the need to convert to proper form from Gaussian elimination before testing for optimality.

To illustrate the procedure, consider Variation 5 of the Wyndor Glass Co. model (with $c_2 = 3$, $a_{22} = 3$, $a_{23} = 4$) that is graphed in Fig. 6.6 and solved in Table 6.24. Since x_2 is a basic variable for the optimal solution (with $c_2 = 3$) given at the bottom of this table, the steps needed to find the allowable range to stay optimal for c_2 are the following:

1. Since x_2 is a basic variable, note that its coefficient in the new final row 0 (see the bottom tableau in Table 6.24) is automatically $z_2^* - c_2 = 0$ before c_2 is changed from its current value of 3.
2. Now increment $c_2 = 3$ by Δc_2 (so $c_2 = 3 + \Delta c_2$). This changes the coefficient noted in step 1 to $z_2^* - c_2 = -\Delta c_2$, which changes row 0 to

$$\text{Row 0} = \left[0,\ -\Delta c_2,\ \frac{3}{4},\ 0,\ \frac{3}{4}\ \ \frac{33}{2}\right].$$

3. With this coefficient now not zero, we must perform elementary row operations to restore proper form from Gaussian elimination. In particular, add to row 0 the product, Δc_2 times row 2, to obtain the new row 0, as shown below.

$$\begin{array}{rl}
 & \left[0,\ -\Delta c_2,\ \frac{3}{4},\qquad 0,\ \frac{3}{4} \qquad\qquad \frac{33}{2}\right] \\
+ & \left[0,\quad \Delta c_2,\ -\frac{3}{4}\Delta c_2,\ 0,\ \frac{1}{4}\Delta c_2 \qquad \frac{3}{2}\Delta c_2\right] \\
\hline
\text{New row 0} = & \left[0,\ \ 0,\ \frac{3}{4} - \frac{3}{4}\Delta c_2,\ 0,\ \frac{3}{4} + \frac{1}{4}\Delta c_2\ \ \ \frac{33}{2} + \frac{3}{2}\Delta c_2\right]
\end{array}$$

4. Using this new row 0, solve for the range of values of Δc_2 that keeps the coefficients of the nonbasic variables (x_3 and x_5) nonnegative.

$$\frac{3}{4} - \frac{3}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \frac{3}{4} \geq \frac{3}{4}\Delta c_2 \quad \Rightarrow \quad \Delta c_2 \leq 1.$$

$$\frac{3}{4} + \frac{1}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \frac{1}{4}\Delta c_2 \geq -\frac{3}{4} \quad \Rightarrow \quad \Delta c_2 \geq -3.$$

Thus, the range of values is $-3 \leq \Delta c_2 \leq 1$.

5. Since $c_2 = 3 + \Delta c_2$, add 3 to this range of values, which yields

$$0 \leq c_2 \leq 4$$

as the allowable range to stay optimal for c_2.

With just two decision variables, this allowable range can be verified graphically by using Fig. 6.6 with an objective function of $Z = 3x_1 + c_2x_2$. With the current value of $c_2 = 3$, the optimal solution is $(4, \frac{3}{2})$. When c_2 is increased, this solution remains optimal only for $c_2 \leq 4$. For $c_2 \geq 4$, $(0, \frac{9}{2})$ becomes optimal (with a tie at $c_2 = 4$), because of the constraint boundary $3x_1 + 4x_2 = 18$. When c_2 is decreased instead, $(4, \frac{3}{2})$ remains optimal only for $c_2 \geq 0$. For $c_2 \leq 0$, (4, 0) becomes optimal because of the constraint boundary $x_1 = 4$.

In a similar manner, the allowable range to stay optimal for c_1 (with c_2 fixed at 3) can be derived either algebraically or graphically to be $c_1 \geq \frac{9}{4}$. (Problem 6.7-13 asks you to verify this both ways.)

Thus, the *allowable decrease* for c_1 from its current value of 3 is only $\frac{3}{4}$. However, it is possible to decrease c_1 by a larger amount without changing the optimal solution if c_2 also decreases sufficiently. For example, suppose that *both* c_1 and c_2 are decreased by 1 from their current value of 3, so that the objective function changes from $Z = 3x_1 + 3x_2$ to $Z = 2x_1 + 2x_2$. According to the 100 percent rule for simultaneous changes in objective function coefficients, the percentages of allowable changes are $133\frac{1}{3}$ percent and $33\frac{1}{3}$ percent, respectively, which sum to far over 100 percent. However, the slope of the objective function line has not changed at all, so $(4, \frac{3}{2})$ still is optimal.

Case 4—Introduction of a New Constraint

In this case, a new constraint must be introduced to the model after it has already been solved. This case may occur because the constraint was overlooked initially or because new considerations have arisen since the model was formulated. Another possibility is that the constraint was deleted purposely to decrease computational effort because it appeared to be less restrictive than other constraints already in the model, but now this impression needs to be checked with the optimal solution actually obtained.

To see if the current optimal solution would be affected by a new constraint, all you have to do is to check directly whether the optimal solution satisfies the constraint. If it does, then it would still be the *best feasible solution* (i.e., the optimal solution), even if the constraint were added to the model. The reason is that a new constraint can only eliminate some previously feasible solutions without adding any new ones.

If the new constraint does eliminate the current optimal solution, and if you want to find the new solution, then introduce this constraint into the final simplex tableau (as an additional row) *just* as if this were the initial tableau, where the usual additional variable (slack variable or artificial variable) is designated to be the basic variable for this new row. Because the new row probably will have *nonzero* coefficients for some of the other basic variables, the conversion to proper form from Gaussian elimination is applied next, and then the reoptimization step is applied in the usual way.

Just as for some of the preceding cases, this procedure for Case 4 is a streamlined version of the general procedure summarized at the end of Sec. 6.6. The only question to be addressed for this case is whether the previously optimal solution still is *feasible,* so step 5 (optimality test) has been deleted. Step 4 (feasibility test) has been replaced by a much quicker test of feasibility (does the previously optimal solution satisfy the new constraint?) to be performed right after step 1 (revision of model). It is only if this test provides a negative answer, and you wish to reoptimize, that steps 2, 3, and 6 are used (revision of final tableau, conversion to proper form from Gaussian elimination, and reoptimization).

Example (Variation 6 of the Wyndor Model). To illustrate this case, we consider Variation 6 of the Wyndor Glass Co. model, which simply introduces the new constraint

$$2x_1 + 3x_2 \leq 24$$

into the Variation 2 model given in Table 6.21. The graphical effect is shown in Fig. 6.7. The previous optimal solution (0, 9) violates the new constraint, so the optimal solution changes to (0, 8).

To analyze this example algebraically, note that (0, 9) yields $2x_1 + 3x_2 = 27 > 24$, so this previous optimal solution is no longer feasible. To find the new optimal solution, add the new constraint to the current final simplex tableau as just described, with the slack variable x_6 as its initial basic variable. This step yields the first tableau shown in Table 6.25. The conversion to proper form from Gaussian elimination then requires subtracting from the new row the product, 3 times row 2, which identifies the current basic solution $x_3 = 4$, $x_2 = 9$, $x_4 = 6$, $x_6 = -3$ ($x_1 = 0$, $x_5 = 0$), as shown in the second tableau. Applying the dual simplex method (described in Sec. 7.1) to this tableau then leads in just one iteration (more are sometimes needed) to the new optimal solution in the last tableau of Table 6.25.

FIGURE 6.7
Feasible region for Variation 6 of the Wyndor model where Variation 2 (Fig. 6.3) has been revised by adding the new constraint, $2x_1 + 3x_2 \leq 24$.

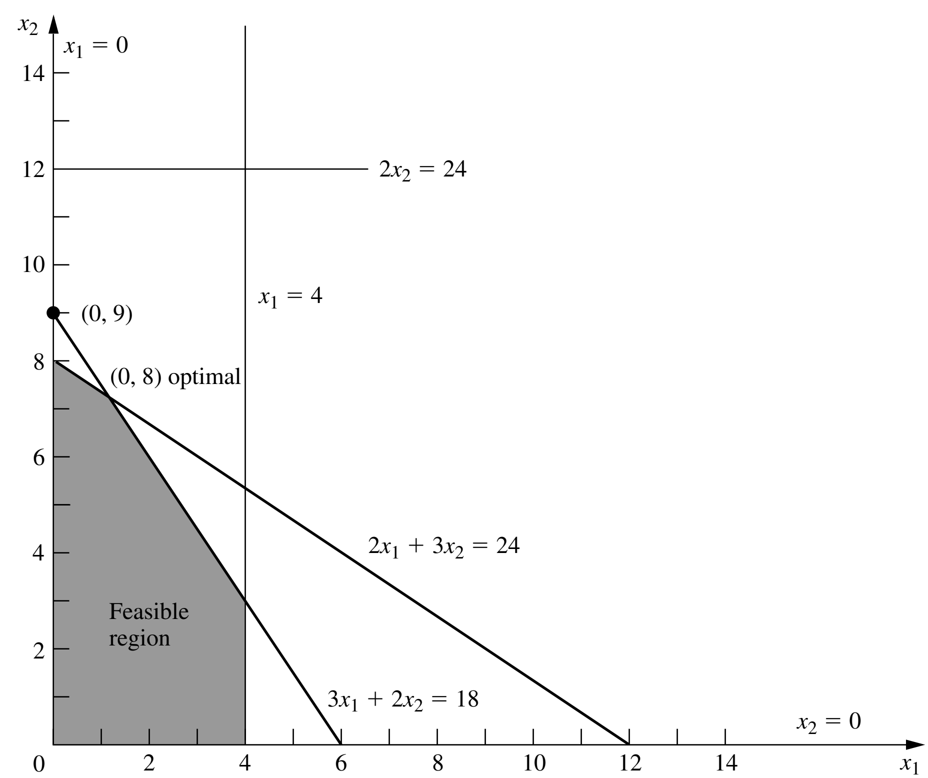


TABLE 6.25 Sensitivity analysis procedure applied to Variation 6 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Revised final tableau	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	0	45
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	9
	x_4	(3)	0	-3	0	0	1	-1	0	6
	x_6	New	0	2	3	0	0	0	1	24
Converted to proper form	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	0	45
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	9
	x_4	(3)	0	-3	0	0	1	-1	0	6
	x_6	New	0	$-\frac{5}{2}$	0	0	0	$-\frac{3}{2}$	1	-3
New final tableau after reoptimization (only one iteration of dual simplex method needed in this case)	Z	(0)	1	$\frac{1}{3}$	0	0	0	0	$\frac{5}{3}$	40
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{2}{3}$	1	0	0	0	$\frac{1}{3}$	8
	x_4	(3)	0	$-\frac{4}{3}$	0	0	1	0	$-\frac{2}{3}$	8
	x_5	New	0	$\frac{5}{3}$	0	0	0	1	$-\frac{2}{3}$	2

Systematic Sensitivity Analysis—Parametric Programming

So far we have described how to test specific changes in the model parameters. Another common approach to sensitivity analysis is to vary one or more parameters continuously over some interval(s) to see when the optimal solution changes.

For example, with Variation 2 of the Wyndor Glass Co. model, rather than beginning by testing the specific change from $b_2 = 12$ to $\bar{b}_2 = 24$, we might instead set

$$\bar{b}_2 = 12 + \theta$$

and then vary θ continuously from 0 to 12 (the maximum value of interest). The geometric interpretation in Fig. 6.3 is that the $2x_2 = 12$ constraint line is being shifted upward to $2x_2 = 12 + \theta$, with θ being increased from 0 to 12. The result is that the original optimal CPF solution (2, 6) shifts up the $3x_1 + 2x_2 = 18$ constraint line toward $(-2, 12)$. This corner-point solution remains optimal as long as it is still feasible ($x_1 \geq 0$), after which (0, 9) becomes the optimal solution.

The algebraic calculations of the effect of having $\Delta b_2 = \theta$ are directly analogous to those for the Case 1 example where $\Delta b_2 = 12$. In particular, we use the expressions for Z^* and $\mathbf{b}^*$ given for Case 1,

$$Z^* = \mathbf{y}^*\overline{\mathbf{b}}$$
$$\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}}$$

where $\overline{\mathbf{b}}$ now is

$$\overline{\mathbf{b}} = \begin{bmatrix} 4 \\ 12+\theta \\ 18 \end{bmatrix}$$

and where $\mathbf{y}^*$ and $\mathbf{S}^*$ are given in the boxes in the middle tableau in Table 6.19. These equations indicate that the optimal solution is

$$\begin{aligned} Z^* &= 36 + \frac{3}{2}\theta \\ x_3 &= 2 + \frac{1}{3}\theta \\ x_2 &= 6 + \frac{1}{2}\theta \\ x_1 &= 2 - \frac{1}{3}\theta \end{aligned} \qquad (x_4 = 0,\ x_5 = 0)$$

for θ small enough that this solution still is feasible, i.e., for $\theta \le 6$. For $\theta > 6$, the dual simplex method (described in Sec. 7.1) yields the tableau shown in Table 6.21 except for the value of x_4. Thus, $Z = 45$, $x_3 = 4$, $x_2 = 9$ (along with $x_1 = 0$, $x_5 = 0$), and the expression for $\mathbf{b}^*$ yields

$$x_4 = b_3^* = 0(4) + 1(12 + \theta) - 1(18) = -6 + \theta.$$

This information can then be used (along with other data not incorporated into the model on the effect of increasing b_2) to decide whether to retain the original optimal solution and, if not, how much to increase b_2.

In a similar way, we can investigate the effect on the optimal solution of varying several parameters simultaneously. When we vary just the b_i parameters, we express the new value $\overline{b}_i$ in terms of the original value b_i as follows:

$$\overline{b}_i = b_i + \alpha_i\theta, \qquad \text{for } i = 1, 2, \ldots, m,$$

where the α_i values are input constants specifying the desired rate of increase (positive or negative) of the corresponding right-hand side as θ is increased.

For example, suppose that it is possible to shift some of the production of a current Wyndor Glass Co. product from Plant 2 to Plant 3, thereby increasing b_2 by decreasing b_3. Also suppose that b_3 decreases twice as fast as b_2 increases. Then

$$\overline{b}_2 = 12 + \theta$$
$$\overline{b}_3 = 18 - 2\theta,$$

where the (nonnegative) value of θ measures the amount of production shifted. (Thus, $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = -2$ in this case.) In Fig. 6.3, the geometric interpretation is that as θ is increased from 0, the $2x_2 = 12$ constraint line is being pushed up to $2x_2 = 12 + \theta$ (ignore the $2x_2 = 24$ line) and simultaneously the $3x_1 + 2x_2 = 18$ constraint line is being pushed down to $3x_1 + 2x_2 = 18 - 2\theta$. The original optimal CPF solution (2, 6) lies at the intersection of the $2x_2 = 12$ and $3x_1 + 2x_2 = 18$ lines, so shifting these lines causes this corner-point solution to shift. However, with the objective function of $Z = 3x_1 + 5x_2$, this corner-point solution will remain optimal as long as it is still feasible ($x_1 \geq 0$).

An algebraic investigation of simultaneously changing b_2 and b_3 in this way again involves using the formulas for Case 1 (treating θ as representing an unknown number) to calculate the resulting changes in the final tableau (middle of Table 6.19), namely,

$$Z^* = \mathbf{y}^*\overline{\mathbf{b}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 12 + \theta \\ 18 - 2\theta \end{bmatrix} = 36 - \tfrac{1}{2}\theta,$$

$$\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 12 + \theta \\ 18 - 2\theta \end{bmatrix} = \begin{bmatrix} 2 + \theta \\ 6 + \frac{1}{2}\theta \\ 2 - \theta \end{bmatrix}.$$

Therefore, the optimal solution becomes

$$\begin{aligned} Z^* &= 36 - \frac{1}{2}\theta \\ x_3 &= 2 + \theta \qquad\qquad (x_4 = 0, \quad x_5 = 0) \\ x_2 &= 6 + \frac{1}{2}\theta \\ x_1 &= 2 - \theta \end{aligned}$$

for θ small enough that this solution still is feasible, i.e., for $\theta \leq 2$. (Check this conclusion in Fig. 6.3.) However, the fact that Z decreases as θ increases from 0 indicates that the best choice for θ is $\theta = 0$, so none of the possible shifting of production should be done.

The approach to varying several c_j parameters simultaneously is similar. In this case, we express the new value $\overline{c}_j$ in terms of the original value of c_j as

$$\overline{c}_j = c_j + \alpha_j\theta, \qquad \text{for } j = 1, 2, \ldots, n,$$

where the α_j are input constants specifying the desired rate of increase (positive or negative) of c_j as θ is increased.

To illustrate this case, reconsider the sensitivity analysis of c_1 and c_2 for the Wyndor Glass Co. problem that was performed earlier in this section. Starting with Variation 2 of the Wyndor model presented in Table 6.21 and Fig. 6.3, we separately considered the effect of changing c_1 from 3 to 4 (its most optimistic estimate) and c_2 from 5 to 3 (its most pessimistic estimate). Now we can simultaneously consider both changes, as well as various intermediate cases with smaller changes, by setting

$$\overline{c}_1 = 3 + \theta \qquad \text{and} \qquad \overline{c}_2 = 5 - 2\theta,$$

where the value of θ measures the *fraction* of the maximum possible change that is made. The result is to replace the original objective function $Z = 3x_1 + 5x_2$ by a *function* of θ

$$Z(\theta) = (3 + \theta)x_1 + (5 - 2\theta)x_2,$$

so the optimization now can be performed for any desired (fixed) value of θ between 0 and 1. By checking the effect as θ increases from 0 to 1, we can determine just when and how the optimal solution changes as the error in the original estimates of these parameters increases.

Considering these changes simultaneously is especially appropriate if there are factors that cause the parameters to change together. Are the two products competitive in some sense, so that a larger-than-expected unit profit for one implies a smaller-than-expected unit profit for the other? Are they both affected by some exogenous factor, such as the advertising emphasis of a competitor? Is it possible to simultaneously change both unit profits through appropriate shifting of personnel and equipment?

In the feasible region shown in Fig. 6.3, the geometric interpretation of changing the objective function from $Z = 3x_1 + 5x_2$ to $Z(\theta) = (3 + \theta)x_1 + (5 - 2\theta)x_2$ is that we are changing the *slope* of the original objective function line ($Z = 45 = 3x_1 + 5x_2$) that passes through the optimal solution (0, 9). If θ is increased enough, this slope will change sufficiently that the optimal solution will switch from (0, 9) to another CPF solution (4, 3). (Check graphically whether this occurs for $\theta \leq 1$.)

The algebraic procedure for dealing simultaneously with these two changes ($\Delta c_1 = \theta$ and $\Delta c_2 = -2\theta$) is shown in Table 6.26. Although the changes now are expressed in terms of θ rather than specific numerical amounts, θ is treated just as an unknown number. The table displays just the relevant rows of the tableaux involved (row 0 and the row for the basic variable x_2). The first tableau shown is just the final tableau for the current version of the model (before c_1 and c_2 are changed) as given in Table 6.21. Refer to the formulas in Table 6.17. The only changes in the *revised* final tableau shown next are that Δc_1 and Δc_2 are subtracted from the row 0 coefficients of x_1 and x_2, respectively. To convert this tableau to proper form from Gaussian elimination, we subtract 2θ times row 2 from row 0, which yields the last tableau shown. The expressions in terms of θ for the coeffi-

TABLE 6.26 Dealing with $\Delta c_1 = \theta$ and $\Delta c_2 = -2\theta$ for Variation 2 of the Wyndor model as given in Table 6.21

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Final tableau	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
Revised final tableau when $\Delta c_1 = \theta$ and $\Delta c_2 = -2\theta$	$Z(\theta)$	(0)	1	$\frac{9}{2} - \theta$	2θ	0	0	$\frac{5}{2}$	45
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
Converted to proper form	$Z(\theta)$	(0)	1	$\frac{9}{2} - 4\theta$	0	0	0	$\frac{5}{2} - \theta$	$45 - 18\theta$
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9

cients of nonbasic variables x_1 and x_5 in row 0 of this tableau show that the current BF solution remains optimal for $\theta \leq \frac{9}{8}$. Because $\theta = 1$ is the maximum realistic value of θ, this indicates that c_1 and c_2 together are insensitive parameters with respect to the Variation 2 model in Table 6.21. There is no need to try to estimate these parameters more closely unless other parameters change (as occurred for Variation 5 of the Wyndor model).

As we discussed in Sec. 4.7, this way of continuously varying several parameters simultaneously is referred to as *parametric linear programming.* Section 7.2 presents the complete parametric linear programming procedure (including identifying new optimal solutions for larger values of θ) when just the c_j parameters are being varied and then when just the b_i parameters are being varied. Some linear programming software packages also include routines for varying just the coefficients of a single variable or just the parameters of a single constraint. In addition to the other applications discussed in Sec. 4.7, these procedures provide a convenient way of conducting sensitivity analysis systematically.

6.8 CONCLUSIONS

Every linear programming problem has associated with it a dual linear programming problem. There are a number of very useful relationships between the original (primal) problem and its dual problem that enhance our ability to analyze the primal problem. For example, the economic interpretation of the dual problem gives shadow prices that measure the marginal value of the resources in the primal problem and provides an interpretation of the simplex method. Because the simplex method can be applied directly to either problem in order to solve both of them simultaneously, considerable computational effort sometimes can be saved by dealing directly with the dual problem. Duality theory, including the dual simplex method for working with superoptimal basic solutions, also plays a major role in sensitivity analysis.

The values used for the parameters of a linear programming model generally are just estimates. Therefore, sensitivity analysis needs to be performed to investigate what happens if these estimates are wrong. The fundamental insight of Sec. 5.3 provides the key to performing this investigation efficiently. The general objectives are to identify the sensitive parameters that affect the optimal solution, to try to estimate these sensitive parameters more closely, and then to select a solution that remains good over the range of likely values of the sensitive parameters. This analysis is a very important part of most linear programming studies.

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LEARNING AIDS FOR THIS CHAPTER IN YOUR OR COURSEWARE

A Demonstration Example in OR Tutor:

Sensitivity Analysis

Interactive Routines:

Enter or Revise a General Linear Programming Model
Solve Interactively by the Simplex Method
Sensitivity Analysis

An Excel Add-In:

Premium Solver

Files (Chapter 3) for Solving the Wyndor Example:

Excel File
LINGO/LINDO File
MPL/CPLEX File

See Appendix 1 for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The demonstration example listed above may be helpful.
I: We suggest that you use the corresponding interactive routine listed above (the printout records your work).
C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem automatically.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

6.1-1. Construct the primal-dual table and the dual problem for each of the following linear programming models fitting our standard form.
(a) Model in Prob. 4.1-6
(b) Model in Prob. 4.7-8

6.1-2.* Construct the dual problem for each of the following linear programming models fitting our standard form.
(a) Model in Prob. 3.1-5
(b) Model in Prob. 4.7-6

6.1-3. Consider the linear programming model in Prob. 4.5-4.
(a) Construct the primal-dual table and the dual problem for this model.
(b) What does the fact that Z is unbounded for this model imply about its dual problem?

6.1-4. For each of the following linear programming models, give your recommendation on which is the more efficient way (probably) to obtain an optimal solution: by applying the simplex method directly to this primal problem or by applying the simplex method directly to the dual problem instead. Explain.
(a) Maximize $\quad Z = 10x_1 - 4x_2 + 7x_3,$

subject to

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &\le 25 \\ x_1 - 2x_2 + 3x_3 &\le 25 \\ 5x_1 + x_2 + 2x_3 &\le 40 \\ x_1 + x_2 + x_3 &\le 90 \\ 2x_1 - x_2 + x_3 &\le 20 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

(b) Maximize $Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5,$

subject to

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &\le 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &\le 15 \end{aligned}$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, 3, 4, 5.$$

6.1-5. Consider the following problem.

Maximize $Z = -x_1 - 2x_2 - x_3,$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 12 \\ x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

(a) Construct the dual problem.
(b) Use duality theory to show that the optimal solution for the primal problem has $Z \le 0$.

6.1-6. Consider the following problem.

Maximize $Z = 2x_1 + 6x_2 + 9x_3,$

subject to

$$\begin{aligned} x_1 + x_3 &\le 3 \qquad \text{(resource 1)} \\ x_2 + 2x_3 &\le 5 \qquad \text{(resource 2)} \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

(a) Construct the dual problem for this primal problem.
(b) Solve the dual problem graphically. Use this solution to identify the shadow prices for the resources in the primal problem.
C **(c)** Confirm your results from part (*b*) by solving the primal problem automatically by the simplex method and then identifying the shadow prices.

6.1-7. Follow the instructions of Prob. 6.1-6 for the following problem.

Maximize $Z = x_1 - 3x_2 + 2x_3,$

subject to

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\le 6 \qquad \text{(resource 1)} \\ -x_2 + 2x_3 &\le 4 \qquad \text{(resource 2)} \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

6.1-8. Consider the following problem.

Maximize $Z = x_1 + 2x_2,$

subject to

$$\begin{aligned} -x_1 + x_2 &\le -2 \\ 4x_1 + x_2 &\le 4 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

(a) Demonstrate graphically that this problem has no feasible solutions.
(b) Construct the dual problem.
(c) Demonstrate graphically that the dual problem has an unbounded objective function.

6.1-9. Construct and graph a primal problem with two decision variables and two functional constraints that has feasible solutions and an unbounded objective function. Then construct the dual problem and demonstrate graphically that it has no feasible solutions.

6.1-10. Construct a pair of primal and dual problems, each with two decision variables and two functional constraints, such that both problems have no feasible solutions. Demonstrate this property graphically.

6.1-11. Construct a pair of primal and dual problems, each with two decision variables and two functional constraints, such that the primal problem has no feasible solutions and the dual problem has an unbounded objective function.

6.1-12. Use the weak duality property to prove that if both the primal and the dual problem have feasible solutions, then both must have an optimal solution.

6.1-13. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Use only this definition of the dual problem for a primal problem in this form to prove each of the following results.
(a) The weak duality property presented in Sec. 6.1.
(b) If the primal problem has an unbounded feasible region that permits increasing Z indefinitely, then the dual problem has no feasible solutions.

6.1-14. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Let $\mathbf{y}^*$ denote the optimal solution for this dual problem. Suppose that $\mathbf{b}$ is then replaced by $\overline{\mathbf{b}}$. Let $\overline{\mathbf{x}}$ denote the optimal solution for the new primal problem. Prove that

$$\mathbf{c}\overline{\mathbf{x}} \le \mathbf{y}^*\overline{\mathbf{b}}.$$

6.1-15. For any linear programming problem in our standard form and its dual problem, label each of the following statements as true or false and then justify your answer.
(a) The sum of the number of functional constraints and the number of variables (before augmenting) is the same for both the primal and the dual problems.
(b) At each iteration, the simplex method simultaneously identifies a CPF solution for the primal problem and a CPF solution for the dual problem such that their objective function values are the same.
(c) If the primal problem has an unbounded objective function, then the optimal value of the objective function for the dual problem must be zero.

6.2-1. Consider the simplex tableaux for the Wyndor Glass Co. problem given in Table 4.8. For each tableau, give the economic interpretation of the following items:
(a) Each of the coefficients of the slack variables (x_3, x_4, x_5) in row 0
(b) Each of the coefficients of the decision variables (x_1, x_2) in row 0
(c) The resulting choice for the entering basic variable (or the decision to stop after the final tableau)

6.3-1.* Consider the following problem.

$$\text{Maximize} \quad Z = 6x_1 + 8x_2,$$

subject to

$$\begin{aligned} 5x_1 + 2x_2 &\le 20 \\ x_1 + 2x_2 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

(a) Construct the dual problem for this primal problem.
(b) Solve both the primal problem and the dual problem graphically. Identify the CPF solutions and corner-point infeasible solutions for both problems. Calculate the objective function values for all these solutions.
(c) Use the information obtained in part (*b*) to construct a table listing the complementary basic solutions for these problems. (Use the same column headings as for Table 6.9.)
I **(d)** Work through the simplex method step by step to solve the primal problem. After each iteration (including iteration 0), identify the BF solution for this problem and the complementary basic solution for the dual problem. Also identify the corresponding corner-point solutions.

6.3-2. Consider the model with two functional constraints and two variables given in Prob. 4.1-5. Follow the instructions of Prob. 6.3-1 for this model.

6.3-3. Consider the primal and dual problems for the Wyndor Glass Co. example given in Table 6.1. Using Tables 5.5, 5.6, 6.8, and 6.9, construct a new table showing the eight sets of nonbasic variables for the primal problem in column 1, the corresponding sets of associated variables for the dual problem in column 2, and the set of nonbasic variables for each complementary basic solution in the dual problem in column 3. Explain why this table demonstrates the complementary slackness property for this example.

6.3-4. Suppose that a primal problem has a *degenerate* BF solution (one or more basic variables equal to zero) as its optimal solution. What does this degeneracy imply about the dual problem? Why? Is the converse also true?

6.3-5. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 - 4x_2,$$

subject to

$$x_1 - x_2 \le 1$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

(a) Construct the dual problem, and then find its optimal solution by inspection.
(b) Use the complementary slackness property and the optimal solution for the dual problem to find the optimal solution for the primal problem.
(c) Suppose that c_1, the coefficient of x_1 in the primal objective function, actually can have any value in the model. For what values of c_1 does the dual problem have no feasible solutions? For these values, what does duality theory then imply about the primal problem?

6.3-6. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 7x_2 + 4x_3,$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10 \\ 3x_1 + 3x_2 + 2x_3 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

(a) Construct the dual problem for this primal problem.
(b) Use the dual problem to demonstrate that the optimal value of Z for the primal problem cannot exceed 25.
(c) It has been conjectured that x_2 and x_3 should be the basic variables for the optimal solution of the primal problem. Directly derive this basic solution (and Z) by using Gaussian elimination. Simultaneously derive and identify the complementary ba-

sic solution for the dual problem by using Eq. (0) for the primal problem. Then draw your conclusions about whether these two basic solutions are optimal for their respective problems.

I **(d)** Solve the dual problem graphically. Use this solution to identify the basic variables and the nonbasic variables for the optimal solution of the primal problem. Directly derive this solution, using Gaussian elimination.

6.3-7.* Reconsider the model of Prob. 6.1-4*b*.
(a) Construct its dual problem.
(b) Solve this dual problem graphically.
(c) Use the result from part (*b*) to identify the nonbasic variables and basic variables for the optimal BF solution for the primal problem.
(d) Use the results from part (*c*) to obtain the optimal solution for the primal problem directly by using Gaussian elimination to solve for its basic variables, starting from the initial system of equations [excluding Eq. (0)] constructed for the simplex method and setting the nonbasic variables to zero.
(e) Use the results from part (*c*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the primal problem, and then use these equations to find this solution.

6.3-8. Consider the model given in Prob. 5.3-13.
(a) Construct the dual problem.
(b) Use the given information about the basic variables in the optimal primal solution to identify the nonbasic variables and basic variables for the optimal dual solution.
(c) Use the results from part (*b*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the dual problem, and then use these equations to find this solution.
(d) Solve the dual problem graphically to verify your results from part (*c*).

6.3-9. Consider the model given in Prob. 3.1-4.
(a) Construct the dual problem for this model.
(b) Use the fact that $(x_1, x_2) = (13, 5)$ is optimal for the primal problem to identify the nonbasic variables and basic variables for the optimal BF solution for the dual problem.
(c) Identify this optimal solution for the dual problem by directly deriving Eq. (0) corresponding to the optimal primal solution identified in part (*b*). Derive this equation by using Gaussian elimination.
(d) Use the results from part (*b*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the dual problem. Verify your optimal dual solution from part (*c*) by checking to see that it satisfies this system of equations.

6.3-10. Suppose that you also want information about the dual problem when you apply the revised simplex method (see Sec. 5.2) to the primal problem in our standard form.
(a) How would you identify the optimal solution for the dual problem?
(b) After obtaining the BF solution at each iteration, how would you identify the complementary basic solution in the dual problem?

6.4-1. Consider the following problem.

$$\text{Maximize} \quad Z = x_1 + x_2,$$

subject to

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ 2x_1 + x_2 &\geq 2 \end{aligned}$$

and

$$x_2 \geq 0 \qquad (x_1 \text{ unconstrained in sign}).$$

(a) Use the SOB method to construct the dual problem.
(b) Use Table 6.12 to convert the primal problem to our standard form given at the beginning of Sec. 6.1, and construct the corresponding dual problem. Then show that this dual problem is equivalent to the one obtained in part (*a*).

6.4-2. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Use only this definition of the dual problem for a primal problem in this form to prove each of the following results.
(a) If the functional constraints for the primal problem $\mathbf{Ax} \leq \mathbf{b}$ are changed to $\mathbf{Ax} = \mathbf{b}$, the only resulting change in the dual problem is to *delete* the nonnegativity constraints, $\mathbf{y} \geq \mathbf{0}$. (*Hint:* The constraints $\mathbf{Ax} = \mathbf{b}$ are equivalent to the set of constraints $\mathbf{Ax} \leq \mathbf{b}$ *and* $\mathbf{Ax} \geq \mathbf{b}$.)
(b) If the functional constraints for the primal problem $\mathbf{Ax} \leq \mathbf{b}$ are changed to $\mathbf{Ax} \geq \mathbf{b}$, the only resulting change in the dual problem is that the nonnegativity constraints $\mathbf{y} \geq \mathbf{0}$ are replaced by nonpositivity constraints $\mathbf{y} \leq \mathbf{0}$, where the current dual variables are interpreted as the negative of the original dual variables. (*Hint:* The constraints $\mathbf{Ax} \geq \mathbf{b}$ are equivalent to $-\mathbf{Ax} \leq -\mathbf{b}$.)
(c) If the nonnegativity constraints for the primal problem $\mathbf{x} \geq \mathbf{0}$ are deleted, the only resulting change in the dual problem is to replace the functional constraints $\mathbf{yA} \geq \mathbf{c}$ by $\mathbf{yA} = \mathbf{c}$. (*Hint:* A variable unconstrained in sign can be replaced by the difference of two nonnegative variables.)

6.4-3.* Construct the dual problem for the linear programming problem given in Prob. 4.6-4.

6.4-4. Consider the following problem.

$$\text{Minimize} \quad Z = x_1 + 2x_2,$$

subject to

$$\begin{aligned} -2x_1 + x_2 &\geq 1 \\ x_1 - 2x_2 &\geq 1 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0.$

(a) Construct the dual problem.
(b) Use graphical analysis of the dual problem to determine whether the primal problem has feasible solutions and, if so, whether its objective function is bounded.

6.4-5. Consider the two versions of the dual problem for the radiation therapy example that are given in Tables 6.15 and 6.16. Review in Sec. 6.4 the general discussion of why these two versions are completely equivalent. Then fill in the details to verify this equivalency by proceeding step by step to convert the version in Table 6.15 to equivalent forms until the version in Table 6.16 is obtained.

6.4-6. For each of the following linear programming models, use the SOB method to construct its dual problem.
(a) Model in Prob. 4.6-3
(b) Model in Prob. 4.6-8
(c) Model in Prob. 4.6-18

6.4-7. Consider the model with equality constraints given in Prob. 4.6-2.
(a) Construct its dual problem.
(b) Demonstrate that the answer in part (*a*) is correct (i.e., equality constraints yield dual variables without nonnegativity constraints) by first converting the primal problem to our standard form (see Table 6.12), then constructing its dual problem, and next converting this dual problem to the form obtained in part (*a*).

6.4-8.* Consider the model without nonnegativity constraints given in Prob. 4.6-16.
(a) Construct its dual problem.
(b) Demonstrate that the answer in part (*a*) is correct (i.e., variables without nonnegativity constraints yield equality constraints in the dual problem) by first converting the primal problem to our standard form (see Table 6.12), then constructing its dual problem, and finally converting this dual problem to the form obtained in part (*a*).

6.4-9. Consider the dual problem for the Wyndor Glass Co. example given in Table 6.1. Demonstrate that *its* dual problem is the primal problem given in Table 6.1 by going through the conversion steps given in Table 6.13.

6.4-10. Consider the following problem.

Minimize $\quad Z = -x_1 - 3x_2,$

subject to

$$\begin{aligned} x_1 - 2x_2 &\leq 2 \\ -x_1 + x_2 &\leq 4 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0.$

(a) Demonstrate graphically that this problem has an unbounded objective function.
(b) Construct the dual problem.
(c) Demonstrate graphically that the dual problem has no feasible solutions.

6.5-1. Consider the model of Prob. 6.7-1. Use duality theory directly to determine whether the current basic solution remains optimal after each of the following independent changes.
(a) The change in part (*e*) of Prob. 6.7-1
(b) The change in part (*g*) of Prob. 6.7-1

6.5-2. Consider the model of Prob. 6.7-3. Use duality theory directly to determine whether the current basic solution remains optimal after each of the following independent changes.
(a) The change in part (*c*) of Prob. 6.7-3
(b) The change in part (*f*) of Prob. 6.7-3

6.5-3. Consider the model of Prob. 6.7-4. Use duality theory directly to determine whether the current basic solution remains optimal after each of the following independent changes.
(a) The change in part (*b*) of Prob. 6.7-4
(b) The change in part (*d*) of Prob. 6.7-4

6.5-4. Reconsider part (*d*) of Prob. 6.7-6. Use duality theory directly to determine whether the original optimal solution is still optimal.

6.6-1.* Consider the following problem.

Maximize $\quad Z = 3x_1 + x_2 + 4x_3,$

subject to

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 &\leq 25 \\ 3x_1 + 4x_2 + 5x_3 &\leq 20 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$

The corresponding final set of equations yielding the optimal solution is

$$\begin{aligned} &(0) \quad Z + 2x_2 + \tfrac{1}{5}x_4 + \tfrac{3}{5}x_5 = 17 \\ &(1) \quad x_1 - \tfrac{1}{3}x_2 + \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5 = \tfrac{5}{3} \\ &(2) \quad x_2 + x_3 - \tfrac{1}{5}x_4 + \tfrac{2}{5}x_5 = 3. \end{aligned}$$

(a) Identify the optimal solution from this set of equations.
(b) Construct the dual problem.

(c) Identify the optimal solution for the dual problem from the final set of equations. Verify this solution by solving the dual problem graphically.

(d) Suppose that the original problem is changed to

Maximize $Z = 3x_1 + 3x_2 + 4x_3,$

subject to

$$6x_1 + 2x_2 + 5x_3 \le 25$$
$$3x_1 + 3x_2 + 5x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Use duality theory to determine whether the previous optimal solution is still optimal.

(e) Use the fundamental insight presented in Sec. 5.3 to identify the new coefficients of x_2 in the final set of equations after it has been adjusted for the changes in the original problem given in part (*d*).

(f) Now suppose that the only change in the original problem is that a new variable x_{new} has been introduced into the model as follows:

Maximize $Z = 3x_1 + x_2 + 4x_3 + 2x_{new},$

subject to

$$6x_1 + 3x_2 + 5x_3 + 3x_{new} \le 25$$
$$3x_1 + 4x_2 + 5x_3 + 2x_{new} \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_{new} \ge 0.$$

Use duality theory to determine whether the previous optimal solution, along with $x_{new} = 0$, is still optimal.

(g) Use the fundamental insight presented in Sec. 5.3 to identify the coefficients of x_{new} as a nonbasic variable in the final set of equations resulting from the introduction of x_{new} into the original model as shown in part (*f*).

D,I **6.6-2.** Reconsider the model of Prob. 6.6-1. You are now to conduct sensitivity analysis by *independently* investigating each of the following six changes in the original model. For each change, use the sensitivity analysis procedure to revise the given final set of equations (in tableau form) and convert it to proper form from Gaussian elimination. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

(a) Change the right-hand side of constraint 1 to $b_1 = 15$.

(b) Change the right-hand side of constraint 2 to $b_2 = 5$.

(c) Change the coefficient of x_2 in the objective function to $c_2 = 4$.

(d) Change the coefficient of x_3 in the objective function to $c_3 = 3$.

(e) Change the coefficient of x_2 in constraint 2 to $a_{22} = 1$.

(f) Change the coefficient of x_1 in constraint 1 to $a_{11} = 10$.

D,I **6.6-3.** Consider the following problem.

Minimize $W = 5y_1 + 4y_2,$

subject to

$$4y_1 + 3y_2 \ge 4$$
$$2y_1 + y_2 \ge 3$$
$$y_1 + 2y_2 \ge 1$$
$$y_1 + y_2 \ge 2$$

and

$$y_1 \ge 0, \quad y_2 \ge 0.$$

Because this primal problem has more functional constraints than variables, suppose that the simplex method has been applied directly to its dual problem. If we let x_5 and x_6 denote the slack variables for this dual problem, the resulting final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	3	0	2	0	1	1	9
x_2	(1)	0	1	1	−1	0	1	−1	1
x_4	(2)	0	2	0	3	1	−1	2	3

For each of the following independent changes in the original primal model, you now are to conduct sensitivity analysis by directly investigating the effect on the dual problem and then inferring the complementary effect on the primal problem. For each change, apply the procedure for sensitivity analysis summarized at the end of Sec. 6.6 to the dual problem (do *not* reoptimize), and then give your conclusions as to whether the current basic solution for the primal problem still is feasible and whether it still is optimal. Then check your conclusions by a direct graphical analysis of the primal problem.

(a) Change the objective function to $W = 3y_1 + 5y_2$.

(b) Change the right-hand sides of the functional constraints to 3, 5, 2, and 3, respectively.

(c) Change the first constraint to $2y_1 + 4y_2 \ge 7$.

(d) Change the second constraint to $5y_1 + 2y_2 \ge 10$.

D,I **6.7-1.*** Consider the following problem.

Maximize $Z = -5x_1 + 5x_2 + 13x_3,$

subject to

$$-x_1 + x_2 + 3x_3 \le 20$$
$$12x_1 + 4x_2 + 10x_3 \le 90$$

and

$$x_j \ge 0 \quad (j = 1, 2, 3).$$

If we let x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{lrcl}
(0) & Z \qquad\qquad + 2x_3 + 5x_4 & = & 100 \\
(1) & -x_1 + x_2 + 3x_3 + \ x_4 & = & 20 \\
(2) & 16x_1 \qquad - 2x_3 - 4x_4 + x_5 & = & 10.
\end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following nine changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

(a) Change the right-hand side of constraint 1 to

$$b_1 = 30.$$

(b) Change the right-hand side of constraint 2 to

$$b_2 = 70.$$

(c) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}.$$

(d) Change the coefficient of x_3 in the objective function to

$$c_3 = 8.$$

(e) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

(f) Change the coefficients of x_2 to

$$\begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}.$$

(g) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}.$$

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \le 50$. (Denote its slack variable by x_6.)

(i) Change constraint 2 to

$$10x_1 + 5x_2 + 10x_3 \le 100.$$

6.7-2.* Reconsider the model of Prob. 6.7-1. Suppose that we now want to apply parametric linear programming analysis to this problem. Specifically, the right-hand sides of the functional constraints are changed to

$$20 + 2\theta \qquad \text{(for constraint 1)}$$

and

$$90 - \theta \qquad \text{(for constraint 2)},$$

where θ can be assigned any positive or negative values.

Express the basic solution (and Z) corresponding to the original optimal solution as a function of θ. Determine the lower and upper bounds on θ before this solution would become infeasible.

D,I **6.7-3.** Consider the following problem.

$$\text{Maximize} \qquad Z = 2x_1 - x_2 + x_3,$$

subject to

$$\begin{aligned}
3x_1 + x_2 + \ x_3 &\le 60 \\
x_1 - x_2 + 2x_3 &\le 10 \\
x_1 + x_2 - \ x_3 &\le 20
\end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	0	0	$\frac{3}{2}$	0	$\frac{3}{2}$	$\frac{1}{2}$	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	15
x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	5

Now you are to conduct sensitivity analysis by *independently* investigating each of the following six changes in the original model. For each change, use the sensitivity analysis procedure to revise this final tableau and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides

$$\text{from} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix}.$$

(b) Change the coefficients of x_1

$$\text{from} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

(c) Change the coefficients of x_3

$$\text{from} \quad \begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}.$$

(d) Change the objective function to $Z = 3x_1 - 2x_2 + 3x_3$.

(e) Introduce a new constraint $3x_1 - 2x_2 + x_3 \leq 30$. (Denote its slack variable by x_7.)

(f) Introduce a new variable x_8 with coefficients

$$\begin{bmatrix} c_8 \\ a_{18} \\ a_{28} \\ a_{38} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix}.$$

D,I **6.7-4.** Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 7x_2 - 3x_3,$$

subject to

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &\leq 30 \\ x_1 + 4x_2 - x_3 &\leq 10 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

By letting x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{lrcl} (0) & Z + x_2 + x_3 + 2x_5 &=& 20 \\ (1) & -x_2 + 5x_3 + x_4 - x_5 &=& 20 \\ (2) & x_1 + 4x_2 - x_3 + x_5 &=& 10. \end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following seven changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}.$$

(b) Change the coefficients of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}.$$

(c) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

(d) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}.$$

(e) Change the objective function to $Z = x_1 + 5x_2 - 2x_3$.

(f) Introduce a new constraint $3x_1 + 2x_2 + 3x_3 \leq 25$.

(g) Change constraint 2 to $x_1 + 2x_2 + 2x_3 \leq 35$.

6.7-5. Reconsider the model of Prob. 6.7-4. Suppose that we now want to apply parametric linear programming analysis to this problem. Specifically, the right-hand sides of the functional constraints are changed to

$$30 + 3\theta \qquad \text{(for constraint 1)}$$

and

$$10 - \theta \qquad \text{(for constraint 2)},$$

where θ can be assigned any positive or negative values.

Express the basic solution (and Z) corresponding to the original optimal solution as a function of θ. Determine the lower and upper bounds on θ before this solution would become infeasible.

D,I **6.7-6.** Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 - x_2 + x_3,$$

subject to

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 &\leq 15 \\ -x_1 + x_2 + x_3 &\leq 3 \\ x_1 - x_2 + x_3 &\leq 4 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

If we let x_4, x_5, and x_6 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{lllr}
(0) & Z & + 2x_3 + x_4 + x_5 & = 18 \\
(1) & & x_2 + 5x_3 + x_4 + 3x_5 & = 24 \\
(2) & & 2x_3 + x_5 + x_6 & = 7 \\
(3) & x_1 & + 4x_3 + x_4 + 2x_5 & = 21.
\end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following eight changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}.$$

(b) Change the coefficient of x_3 in the objective function to $c_3 = 2$.

(c) Change the coefficient of x_1 in the objective function to $c_1 = 3$.

(d) Change the coefficients of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

(e) Change the coefficients of x_1 and x_2 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \\ 2 \end{bmatrix},$$

respectively.

(f) Change the objective function to $Z = 5x_1 + x_2 + 3x_3$.

(g) Change constraint 1 to $2x_1 - x_2 + 4x_3 \leq 12$.

(h) Introduce a new constraint $2x_1 + x_2 + 3x_3 \leq 60$.

6.7-7. One of the products of the G. A. Tanner Company is a special kind of toy that provides an estimated unit profit of \$3. Because of a large demand for this toy, management would like to increase its production rate from the current level of 1,000 per day. However, a limited supply of two subassemblies (A and B) from vendors makes this difficult. Each toy requires two subassemblies of type A, but the vendor providing these subassemblies would only be able to increase its supply rate from the current 2,000 per day to a maximum of 3,000 per day. Each toy requires only one subassembly of type B, but the vendor providing these subassemblies would be unable to increase its supply rate above the current level of 1,000 per day.

Because no other vendors currently are available to provide these subassemblies, management is considering initiating a new production process internally that would simultaneously produce an equal number of subassemblies of the two types to supplement the supply from the two vendors. It is estimated that the company's cost for producing one subassembly of each type would be \$2.50 more than the cost of purchasing these subassemblies from the two vendors. Management wants to determine both the production rate of the toy and the production rate of each pair of subassemblies (one A and one B) that would maximize the total profit.

The following table summarizes the data for the problem.

	Resource Usage per Unit of Each Activity		
Resource	**Produce Toys**	**Produce Subassemblies**	**Amount of Resource Available**
Subassembly A	2	−1	3,000
Subassembly B	1	−1	1,000
Unit profit	\$3	−\$2.50	

(a) Formulate a linear programming model for this problem and use the graphical method to obtain its optimal solution.

C **(b)** Use a software package based on the simplex method to solve for an optimal solution.

C **(c)** Since the stated unit profits for the two activities are only estimates, management wants to know how much each of these estimates can be off before the optimal solution would change. Begin exploring this question for the first activity (producing toys) by using the same software package to resolve for an optimal solution and total profit as the unit profit for this activity increases in 50-cent increments from \$2.00 to \$4.00. What conclusion can be drawn about how much the estimate of this unit profit can differ in each direction from its original value of \$3.00 before the optimal solution would change?

C **(d)** Repeat part (*c*) for the second activity (producing subassemblies) by re-solving as the unit profit for this activity increases in 50-cent increments from −\$3.50 to −\$1.50 (with the unit profit for the first activity fixed at \$3).

C **(e)** Use the same software package to generate the usual output (as in Table 6.23) for sensitivity analysis of the unit profits.

Use this output to obtain the allowable range to stay optimal for each unit profit.

(f) Use graphical analysis to verify the allowable ranges obtained in part (*e*).

(g) For each of the 16 combinations of unit profits considered in parts (*c*) and (*d*) where both unit profits differ from their original estimates, use the 100 percent rule for simultaneous changes in objective function coefficients to determine if the original optimal solution must still be optimal.

(h) For each of the combinations of unit profits considered in part (*g*) where it was found that the original optimal solution is not guaranteed to still be optimal, use graphical analysis to determine whether this solution is still optimal.

6.7-8. Reconsider Prob. 6.7-7. After further negotiations with each vendor, management of the G. A. Tanner Co. has learned that either of them would be willing to consider increasing their supply of their respective subassemblies over the previously stated maxima (3,000 subassemblies of type A per day and 1,000 of type B per day) if the company would pay a small premium over the regular price for the extra subassemblies. The size of the premium for each type of subassembly remains to be negotiated. The demand for the toy being produced is sufficiently high that 2,500 per day could be sold if the supply of subassemblies could be increased enough to support this production rate. Assume that the original estimates of unit profits given in Prob. 6.7-7 are accurate.

(a) Formulate a linear programming model for this problem with the original maximum supply levels and the additional constraint that no more than 2,500 toys should be produced per day. Then use the graphical method to obtain its optimal solution.

C **(b)** Use a software package based on the simplex method to solve for an optimal solution.

C **(c)** Without considering the premium, use the same software package to determine the shadow price for the subassembly A constraint by solving the model again after increasing the maximum supply by 1. Use this shadow price to determine the maximum premium that the company should be willing to pay for each subassembly of this type.

C **(d)** Repeat part (*c*) for the subassembly B constraint.

C **(e)** Estimate how much the maximum supply of subassemblies of type A could be increased before the shadow price (and the corresponding premium) found in part (*c*) would no longer be valid by using the same software package to re-solve for an optimal solution and the total profit (excluding the premium) as the maximum supply increases in increments of 100 from 3,000 to 4,000.

C **(f)** Repeat part (*e*) for subassemblies of type B by re-solving as the maximum supply increases in increments of 100 from 1,000 to 2,000.

C **(g)** Use the same software package to generate the usual output (as in Table 6.23) for sensitivity analysis of the supplies being made available of the subassemblies. Use this output to obtain the allowable range to stay feasible for each subassembly supply.

(h) Use graphical analysis to verify the allowable ranges obtained in part (*g*).

(i) For each of the four combinations where the maximum supply of subassembly A is either 3,500 or 4,000 and the maximum supply of subassembly B is either 1,500 or 2,000, use the 100 percent rule for simultaneous changes in right-hand sides to determine whether the original shadow prices definitely will still be valid.

(j) For each of the combinations considered in part (*i*) where it was found that the original shadow prices are not guaranteed to still be valid, use graphical analysis to determine whether these shadow prices actually are still valid for predicting the effect of changing the right-hand sides.

C **6.7-9** Consider the Distribution Unlimited Co. problem presented in Sec. 3.4 and summarized in Fig. 3.13.

Although Fig. 3.13 gives estimated unit costs for shipping through the various shipping lanes, there actually is some uncertainty about what these unit costs will turn out to be. Therefore, before adopting the optimal solution given at the end of Sec. 3.4, management wants additional information about the effect of inaccuracies in estimating these unit costs.

Use a computer package based on the simplex method to generate sensitivity analysis information preparatory to addressing the following questions.

(a) Which of the unit shipping costs given in Fig. 3.13 has the smallest margin for error without invalidating the optimal solution given in Sec. 3.4? Where should the greatest effort be placed in estimating the unit shipping costs?

(b) What is the allowable range to stay optimal for each of the unit shipping costs?

(c) How should these allowable ranges be interpreted to management?

(d) If the estimates change for more than one of the unit shipping costs, how can you use the generated sensitivity analysis information to determine whether the optimal solution might change?

C **6.7-10.** Consider the Union Airways problem presented in Sec. 3.4, including the data given in Table 3.19.

Management is about to begin negotiations on a new contract with the union that represents the company's customer service agents. This might result in some small changes in the daily costs per agent given in Table 3.19 for the various shifts. Several possible changes listed below are being considered separately. In each case, management would like to know whether the change might result in the original optimal solution (given in Sec. 3.4) no longer being optimal. Answer this question in parts (*a*) to (*e*) by using a software package based on the simplex method to generate sensitivity analysis information. If the optimal solution might change, use the software package to re-solve for the optimal solution.

(a) The daily cost per agent for Shift 2 changes from \$160 to \$165.
(b) The daily cost per agent for Shift 4 changes from \$180 to \$170.
(c) The changes in parts (*a*) and (*b*) both occur.
(d) The daily cost per agent increases by \$4 for shifts 2, 4, and 5, but decreases by \$4 for shifts 1 and 3.
(e) The daily cost per agent increases by 2 percent for each shift.

6.7-11. Consider the following problem.

Maximize $Z = c_1x_1 + c_2x_2,$

subject to

$$2x_1 - x_2 \le b_1$$
$$x_1 - x_2 \le b_2$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Let x_3 and x_4 denote the slack variables for the respective functional constraints. When $c_1 = 3$, $c_2 = -2$, $b_1 = 30$, and $b_2 = 10$, the simplex method yields the following final simplex tableau.

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	0	0	1	1	40
x_2	(1)	0	0	1	1	−2	10
x_1	(2)	0	1	0	1	−1	20

(a) Use graphical analysis to determine the allowable range to stay optimal for c_1 and c_2.
(b) Use algebraic analysis to derive and verify your answers in part (*a*).
(c) Use graphical analysis to determine the allowable range to stay feasible for b_1 and b_2.
(d) Use algebraic analysis to derive and verify your answers in part (*c*)
C **(e)** Use a software package based on the simplex method to find these allowable ranges.

6.7-12. Consider Variation 5 of the Wyndor Glass Co. model (see Fig. 6.6 and Table 6.24), where the changes in the parameter values given in Table 6.21 are $\bar{c}_2 = 3$, $\bar{a}_{22} = 3$, and $\bar{a}_{32} = 4$. Use the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$ to find the allowable range to stay feasible for each b_i. Then interpret each allowable range graphically.

6.7-13. Consider Variation 5 of the Wyndor Glass Co. model (see Fig. 6.6 and Table 6.24), where the changes in the parameter values given in Table 6.21 are $\bar{c}_2 = 3$, $\bar{a}_{22} = 3$, and $\bar{a}_{32} = 4$. Verify both algebraically and graphically that the allowable range to stay optimal for c_1 is $c_1 \ge \frac{9}{4}$.

6.7-14. Consider the following problem.

Maximize $Z = 3x_1 + x_2 + 2x_3,$

subject to

$$x_1 - x_2 + 2x_3 \le 20$$
$$2x_1 + x_2 - x_3 \le 10$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4 and x_5 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	8	0	0	3	4	100
x_3	(1)	0	3	0	1	1	1	30
x_2	(2)	0	5	1	0	1	2	40

(a) Perform sensitivity analysis to determine which of the 11 parameters of the model are sensitive parameters in the sense that *any* change in just that parameter's value will change the optimal solution.
(b) Use algebraic analysis to find the allowable range to stay optimal for each c_j.
(c) Use algebraic analysis to find the allowable range to stay feasible for each b_i.
C **(d)** Use a software package based on the simplex method to find these allowable ranges.

6.7-15. For the problem given in Table 6.21, find the allowable range to stay optimal for c_2. Show your work algebraically, using the tableau given in Table 6.21. Then justify your answer from a geometric viewpoint, referring to Fig. 6.3.

6.7-16.* For the original Wyndor Glass Co. problem, use the last tableau in Table 4.8 to do the following.
(a) Find the allowable range to stay feasible for each b_i.
(b) Find the allowable range to stay optimal for c_1 and c_2.
C **(c)** Use a software package based on the simplex method to find these allowable ranges.

6.7-17. For Variation 6 of the Wyndor Glass Co. model presented in Sec. 6.7, use the last tableau in Table 6.25 to do the following.
(a) Find the allowable range to stay feasible for each b_i.
(b) Find the allowable range to stay optimal for c_1 and c_2.
C **(c)** Use a software package based on the simplex method to find these allowable ranges.

6.7-18. Ken and Larry, Inc., supplies its ice cream parlors with three flavors of ice cream: chocolate, vanilla, and banana. Because of extremely hot weather and a high demand for its products, the company has run short of its supply of ingredients: milk, sugar, and cream. Hence, they will not be able to fill all the orders received from their retail outlets, the ice cream parlors. Owing to these circumstances, the company has decided to choose the amount of each flavor to produce that will maximize total profit, given the constraints on supply of the basic ingredients.

The chocolate, vanilla, and banana flavors generate, respectively, \$1.00, \$0.90, and \$0.95 of profit per gallon sold. The company has only 200 gallons of milk, 150 pounds of sugar, and 60 gallons of cream left in its inventory. The linear programming formulation for this problem is shown below in algebraic form.

Let C = gallons of chocolate ice cream produced,
V = gallons of vanilla ice cream produced,
B = gallons of banana ice cream produced.

$$\text{Maximize} \quad \text{profit} = 1.00\,C + 0.90\,V + 0.95\,B,$$

subject to

$$\begin{aligned} \text{Milk:} &\quad 0.45\,C + 0.50\,V + 0.40\,B \le 200 \text{ gallons} \\ \text{Sugar:} &\quad 0.50\,C + 0.40\,V + 0.40\,B \le 150 \text{ pounds} \\ \text{Cream:} &\quad 0.10\,C + 0.15\,V + 0.20\,B \le 60 \text{ gallons} \end{aligned}$$

and

$$C \ge 0, \qquad V \ge 0, \qquad B \ge 0.$$

This problem was solved using the Excel Solver. The spreadsheet (already solved) and the sensitivity report are shown below. [*Note*: The numbers in the sensitivity report for the milk constraint are missing on purpose, since you will be asked to fill in these numbers in part (f).]

	A	B	C	D	E	F	G
1		Resource Usage Per Unit of Each Activity					
2			Activity				Resource
3	Resource	Chocolate	Vanilla	Banana	Totals		Available
4	Milk	0.45	0.5	0.4	180	≤	200
5	Sugar	0.5	0.4	0.4	150	≤	150
6	Cream	0.1	0.15	0.2	60	≤	60
7	Unit Profit	1	0.9	0.95	\$341.25		
8	Solution	0	300	75			

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$8	Solution Chocolate	0	-0.0375	1	0.0375	1E+30
\$C\$8	Solution Vanilla	300	0	0.9	0.05	0.0125
\$D\$8	Solution Banana	75	0	0.95	0.021429	0.05

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$4	Milk Totals					
\$E\$5	Sugar Totals	150	1.875	150	10	30
\$E\$6	Cream Totals	60	1	60	15	3.75

For each of the following parts, answer the question as specifically and completely as is possible without solving the problem again on the Excel Solver. *Note:* Each part is independent (i.e., any change made to the model in one part does not apply to any other parts).

(a) What is the optimal solution and total profit?
(b) Suppose the profit per gallon of banana changes to $1.00. Will the optimal solution change, and what can be said about the effect on total profit?
(c) Suppose the profit per gallon of banana changes to 92 cents. Will the optimal solution change, and what can be said about the effect on total profit?
(d) Suppose the company discovers that 3 gallons of cream have gone sour and so must be thrown out. Will the optimal solution change, and what can be said about the effect on total profit?
(e) Suppose the company has the opportunity to buy an additional 15 pounds of sugar at a total cost of $15. Should they? Explain.
(f) Fill in all the sensitivity report information for the milk constraint, given just the optimal solution for the problem. Explain how you were able to deduce each number.

6.7-19. David, LaDeana, and Lydia are the sole partners and workers in a company which produces fine clocks. David and LaDeana each are available to work a maximum of 40 hours per week at the company, while Lydia is available to work a maximum of 20 hours per week.

The company makes two different types of clocks: a grandfather clock and a wall clock. To make a clock, David (a mechanical engineer) assembles the inside mechanical parts of the clock while LaDeana (a woodworker) produces the hand-carved wood casings. Lydia is responsible for taking orders and shipping the clocks. The amount of time required for each of these tasks is shown below.

	Time Required	
Task	**Grandfather Clock**	**Wall Clock**
Assemble clock mechanism	6 hours	4 hours
Carve wood casing	8 hours	4 hours
Shipping	3 hours	3 hours

Each grandfather clock built and shipped yields a profit of $300, while each wall clock yields a profit of $200.

The three partners now want to determine how many clocks of each type should be produced per week to maximize the total profit.

(a) Formulate a linear programming model for this problem.
(b) Use the graphical method to solve the model.
C **(c)** Use a software package based on the simplex method to solve the model.
C **(d)** Use this same software package to generate sensitivity analysis information.
(e) Use this sensitivity analysis information to determine whether the optimal solution must remain optimal if the estimate of the unit profit for grandfather clocks is changed from $300 to $375 (with no other changes in the model).
(f) Repeat part (*e*) if, in addition to this change in the unit profit for grandfather clocks, the estimated unit profit for wall clocks also changes from $200 to $175.
(g) Use graphical analysis to verify your answers in parts (*e*) and (*f*).
(h) To increase the total profit, the three partners have agreed that one of them will slightly increase the maximum number of hours available to work per week. The choice of which one will be based on which one would increase the total profit the most. Use the sensitivity analysis information to make this choice. (Assume no change in the original estimates of the unit profits.)
(i) Explain why one of the shadow prices is equal to zero.
(j) Can the shadow prices given in the sensitivity analysis information be validly used to determine the effect if Lydia were to change her maximum number of hours available to work per week from 20 to 25? If so, what would be the increase in the total profit?
(k) Repeat part (*j*) if, in addition to the change for Lydia, David also were to change his maximum number of hours available to work per week from 40 to 35.
(l) Use graphical analysis to verify your answer in part (*k*).

C **6.7-20.** Consider the Union Airways problem presented in Sec. 3.4, including the data given in Table 3.19.

Management now is considering increasing the level of service provided to customers by increasing one or more of the numbers in the rightmost column of Table 3.19 for the minimum number of agents needed in the various time periods. To guide them in making this decision, they would like to know what impact this change would have on total cost.

Use a software package based on the simplex method to generate sensitivity analysis information in preparation for addressing the following questions.

(a) Which of the numbers in the rightmost column of Table 3.19 can be increased without increasing total cost? In each case, indicate how much it can be increased (if it is the only one being changed) without increasing total cost.
(b) For each of the other numbers, how much would the total cost increase per increase of 1 in the number? For each answer, in-

dicate how much the number can be increased (if it is the only one being changed) before the answer is no longer valid.

(c) Do your answers in part (*b*) definitely remain valid if all the numbers considered in part (*b*) are simultaneously increased by 1?

(d) Do your answers in part (*b*) definitely remain valid if all 10 numbers are simultaneously increased by 1?

(e) How far can all 10 numbers be simultaneously increased by the same amount before your answers in part (*b*) may no longer be valid?

6.7-21. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 5x_2,$$

subject to

$$x_1 + 2x_2 \le 10$$
$$x_1 + 3x_2 \le 12$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

Let x_3 and x_4 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	0	0	1	1	22
x_1	(1)	0	1	0	3	−2	6
x_2	(2)	0	0	1	−1	1	2

While doing postoptimality analysis, you learn that all four b_i and c_j values used in the original model just given are accurate only to within ±50 percent. In other words, their ranges of *likely values* are $5 \le b_1 \le 15$, $6 \le b_2 \le 18$, $1 \le c_1 \le 3$, and $2.5 \le c_2 \le 7.5$. Your job now is to perform sensitivity analysis to determine for each parameter individually (assuming the other three parameters equal their values in the original model) whether this uncertainty might affect either the feasibility or the optimality of the above basic solution (perhaps with new values for the basic variables). Specifically, determine the allowable range to stay feasible for each b_i and the allowable range to stay optimal for each c_j. Then, for each parameter and its range of likely values, indicate which part of this range lies within the allowable range and which parts correspond to values for which the current basic solution will no longer be both feasible and optimal.

(a) Perform this sensitivity analysis graphically on the original model.

(b) Now perform this sensitivity analysis as described and illustrated in Sec. 6.7 for b_1 and c_1.

(c) Repeat part (*b*) for b_2.

(d) Repeat part (*b*) for c_2.

6.7-22. Reconsider Prob. 6.7-21. Now use a software package based on the simplex method to generate sensitivity analysis information preparatory to doing parts (*a*) and (*c*) below.

C **(a)** Suppose that the estimates for c_1 and c_2 are correct but the estimates for both b_1 and b_2 are incorrect. Consider the following four cases where the true values of b_1 and b_2 differ from their estimates by the same percentage: (1) both b_1 and b_2 are smaller than their estimates, (2) both b_1 and b_2 are larger than their estimates, (3) b_1 is smaller and b_2 is larger than their estimates, and (4) b_1 is larger and b_2 is smaller than their estimates. For each of these cases, use the 100 percent rule for simultaneous changes in right-hand sides to determine how large the percentage error can be while guaranteeing that the original shadow prices still will be valid.

(b) For each of the four cases considered in part (*a*), start with the final simplex tableau given in Prob. 6.7-21 and use algebraic analysis based on the fundamental insight presented in Sec. 5.3 to determine how large the percentage error can be without invalidating the original shadow prices.

C **(c)** Suppose that the estimates for b_1 and b_2 are correct but the estimates for both c_1 and c_2 are incorrect. Consider the following four cases where the true values of c_1 and c_2 differ from their estimates by the same percentage: (1) both c_1 and c_2 are smaller than their estimates, (2) both c_1 and c_2 are larger than their estimates, (3) c_1 is smaller and c_2 is larger than their estimates, and (4) c_1 is larger and c_2 is smaller than their estimates. For each of these cases, use the 100 percent rule for simultaneous changes in objective function coefficients to determine how large the percentage error can be while guaranteeing that the original optimal solution must still be optimal.

(d) For each of the four cases considered in part (*c*), start with the final simplex tableau given in Prob. 6.7-21 and use algebraic analysis based on the fundamental insight presented in Sec. 5.3 to determine how large the percentage error can be without invalidating the original optimal solution.

6.7-23. Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + 8x_3,$$

subject to

$$2x_1 + 3x_2 + 5x_3 \le 9$$
$$x_1 + 2x_2 + 3x_3 \le 5$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Let x_4 and x_5 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	1	0	1	1	14
x_1	(1)	0	1	−1	0	3	−5	2
x_3	(2)	0	0	1	1	−1	2	1

While doing postoptimality analysis, you learn that some of the parameter values used in the original model just given are just rough estimates, where the range of likely values in each case is within ±50 percent of the value used here. For each of these following parameters, perform sensitivity analysis to determine whether this uncertainty might affect either the feasibility or the optimality of the above basic solution. Specifically, for each parameter, determine the allowable range of values for which the current basic solution (perhaps with new values for the basic variables) will remain both feasible and optimal. Then, for each parameter and its range of likely values, indicate which part of this range lies within the allowable range and which parts correspond to values for which the current basic solution will no longer be both feasible and optimal.

(a) Parameter b_2
(b) Parameter c_2
(c) Parameter a_{22}
(d) Parameter c_3
(e) Parameter a_{12}
(f) Parameter b_1

6.7-24. Consider Variation 5 of the Wyndor Glass Co. model presented in Sec. 6.7, where $\bar{c}_2 = 3$, $\bar{a}_{22} = 3$, $\bar{a}_{32} = 4$, and where the other parameters are given in Table 6.21. Starting from the resulting final tableau given at the bottom of Table 6.24, construct a table like Table 6.26 to perform parametric linear programming analysis, where

$$c_1 = 3 + \theta \qquad \text{and} \qquad c_2 = 3 + 2\theta.$$

How far can θ be increased above 0 before the current basic solution is no longer optimal?

6.7-25. Reconsider the model of Prob. 6.7-6. Suppose that you now have the option of making trade-offs in the profitability of the first two activities, whereby the objective function coefficient of x_1 can be increased by any amount by simultaneously decreasing the objective function coefficient of x_2 by the same amount. Thus, the alternative choices of the objective function are

$$Z(\theta) = (2 + \theta)x_1 - (1 + \theta)x_2 + x_3,$$

where any nonnegative value of θ can be chosen.

Construct a table like Table 6.26 to perform parametric linear programming analysis on this problem. Determine the upper bound on θ before the original optimal solution would become nonoptimal. Then determine the best choice of θ over this range.

6.7-26. Consider the following parametric linear programming problem.

$$\text{Maximize} \quad Z(\theta) = (10 - 4\theta)x_1 + (4 - \theta)x_2 + (7 + \theta)x_3,$$

subject to

$$3x_1 + x_2 + 2x_3 \le 7 \quad \text{(resource 1)},$$
$$2x_1 + x_2 + 3x_3 \le 5 \quad \text{(resource 2)},$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0,$$

where θ can be assigned any positive or negative values. Let x_4 and x_5 be the slack variables for the respective constraints. After we apply the simplex method with $\theta = 0$, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	0	3	2	2	24
x_1	(1)	0	1	0	−1	1	−1	2
x_2	(2)	0	0	1	5	−2	3	1

(a) Determine the range of values of θ over which the above BF solution will remain optimal. Then find the best choice of θ within this range.
(b) Given that θ is within the range of values found in part (*a*), find the allowable range to stay feasible for b_1 (the available amount of resource 1). Then do the same for b_2 (the available amount of resource 2).
(c) Given that θ is within the range of values found in part (*a*), identify the shadow prices (as a function of θ) for the two resources. Use this information to determine how the optimal value of the objective function would change (as a function of θ) if the available amount of resource 1 were decreased by 1 and the available amount of resource 2 simultaneously were increased by 1.
(d) Construct the dual of this parametric linear programming problem. Set $\theta = 0$ and solve this dual problem graphically to find the corresponding shadow prices for the two resources of the primal problem. Then find these shadow prices as a function of θ [within the range of values found in part (*a*)] by algebraically solving for this same optimal CPF solution for the dual problem as a function of θ.

6.7-27. Consider the following parametric linear programming problem.

Maximize $\quad Z(\theta) = 2x_1 + 4x_2 + 5x_3,$

subject to

$$x_1 + 3x_2 + 2x_3 \le 5 + \theta$$
$$x_1 + 2x_2 + 3x_3 \le 6 + 2\theta$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0,$$

where θ can be assigned any positive or negative values. Let x_4 and x_5 be the slack variables for the respective functional constraints. After we apply the simplex method with $\theta = 0$, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	0	0	1	0	1	1	11
x_1	(1)	1	1	5	0	3	-2	3
x_3	(2)	2	0	-1	1	-1	1	1

(a) Express the BF solution (and Z) given in this tableau as a function of θ. Determine the lower and upper bounds on θ before this optimal solution would become infeasible. Then determine the best choice of θ between these bounds.
(b) Given that θ is between the bounds found in part (*a*), determine the allowable range to stay optimal for c_1 (the coefficient of x_1 in the objective function).

6.7-28. Consider the following parametric linear programming problem, where the parameter θ must be nonnegative:

Maximize $\quad Z(\theta) = (5 + 2\theta)x_1 + (2 - \theta)x_2 + (3 + \theta)x_3,$

subject to

$$4x_1 + x_2 \ge 5 + 5\theta$$
$$3x_1 + x_2 + 2x_3 = 10 - 10\theta$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4 be the surplus variable for the first functional constraint, and let $\bar{x}_5$ and $\bar{x}_6$ be the artificial variables for the respective functional constraints. After we apply the simplex method with the Big M method and with $\theta = 0$, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	(0)	1	1	0	1	0	M	$M+2$	20
x_2	(1)	0	3	1	2	0	0	1	10
x_4	(2)	0	-1	0	2	1	-1	1	5

(a) Use the fundamental insight (Sec. 5.3) to revise this tableau to reflect the inclusion of the parameter θ in the original model. Show the complete tableau needed to apply the feasibility test and the optimality test for any value of θ. Express the corresponding basic solution (and Z) as a function of θ.
(b) Determine the range of nonnegative values of θ over which this basic solution is feasible.
(c) Determine the range of nonnegative values of θ over which this basic solution is both feasible and optimal. Determine the best choice of θ over this range.

6.7-29. Consider the following problem.

Maximize $\quad Z = 10x_1 + 4x_2,$

subject to

$$3x_1 + x_2 \le 30$$
$$2x_1 + x_2 \le 25$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Let x_3 and x_4 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	0	0	2	2	110
x_2	(1)	0	0	1	-2	3	15
x_1	(2)	0	1	0	1	-1	5

Now suppose that both of the following changes are made simultaneously in the original model:

1. The first constraint is changed to $4x_1 + x_2 \le 40$.
2. Parametric programming is introduced to change the objective function to the alternative choices of

   $Z(\theta) = (10 - 2\theta)x_1 + (4 + \theta)x_2,$

   where any nonnegative value of θ can be chosen.

(a) Construct the resulting revised final tableau (as a function of θ), and then convert this tableau to proper form from Gaussian elimination. Use this tableau to identify the new optimal solution that applies for either $\theta = 0$ or sufficiently small values of θ.

(b) What is the upper bound on θ before this optimal solution would become nonoptimal?

(c) Over the range of θ from zero to this upper bound, which choice of θ gives the largest value of the objective function?

6.7-30. Consider the following problem.

Maximize $\quad Z = 9x_1 + 8x_2 + 5x_3,$

subject to

$$2x_1 + 3x_2 + x_3 \leq 4$$
$$5x_1 + 4x_2 + 3x_3 \leq 11$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4 and x_5 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	2	0	2	1	19
x_1	(1)	0	1	5	0	3	−1	1
x_3	(2)	0	0	−7	1	−5	2	2

D,I **(a)** Suppose that a new technology has become available for conducting the first activity considered in this problem. If the new technology were adopted to replace the existing one, the coefficients of x_1 in the model would change

$$\text{from} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 5 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 18 \\ 3 \\ 6 \end{bmatrix}.$$

Use the sensitivity analysis procedure to investigate the potential effect and desirability of adopting the new technology. Specifically, assuming it were adopted, construct the resulting revised final tableau, convert this tableau to proper form from Gaussian elimination, and then reoptimize (if necessary) to find the new optimal solution.

(b) Now suppose that you have the option of mixing the old and new technologies for conducting the first activity. Let θ denote the fraction of the technology used that is from the new technology, so $0 \leq \theta \leq 1$. Given θ, the coefficients of x_1 in the model become

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 9 + 9\theta \\ 2 + \theta \\ 5 + \theta \end{bmatrix}.$$

Construct the resulting revised final tableau (as a function of θ), and convert this tableau to proper form from Gaussian elimination. Use this tableau to identify the current basic solution as a function of θ. Over the allowable values of $0 \leq \theta \leq 1$, give the range of values of θ for which this solution is both feasible and optimal. What is the best choice of θ within this range?

6.7-31. Consider the following problem.

Maximize $\quad Z = 3x_1 + 5x_2 + 2x_3,$

subject to

$$-2x_1 + 2x_2 + x_3 \leq 5$$
$$3x_1 + x_2 - x_3 \leq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4 and x_5 be the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	20	0	9	7	115
x_1	(1)	0	1	3	0	1	1	15
x_3	(2)	0	0	8	1	3	2	35

Parametric linear programming analysis now is to be applied simultaneously to the objective function and right-hand sides, where the model in terms of the new parameter is the following:

Maximize $\quad Z(\theta) = (3 + 2\theta)x_1 + (5 + \theta)x_2 + (2 - \theta)x_3,$

subject to

$$-2x_1 + 2x_2 + x_3 \leq 5 + 6\theta$$
$$3x_1 + x_2 - x_3 \leq 10 - 8\theta$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Construct the resulting revised final tableau (as a function of θ), and convert this tableau to proper form from Gaussian elimination.

Use this tableau to identify the current basic solution as a function of θ. For $\theta \geq 0$, give the range of values of θ for which this solution is both feasible and optimal. What is the best choice of θ within this range?

6.7-32. Consider the Wyndor Glass Co. problem described in Sec. 3.1. Suppose that, in addition to considering the introduction of two new products, management now is considering changing the production rate of a certain old product that is still profitable. Refer to Table 3.1. The number of production hours per week used per unit production rate of this old product is 1, 4, and 3 for Plants 1, 2, and 3, respectively. Therefore, if we let θ denote the *change* (positive or negative) in the production rate of this old product, the right-hand sides of the three functional constraints in Sec. 3.1 become $4 - \theta$, $12 - 4\theta$, and $18 - 3\theta$, respectively. Thus, choosing a negative value of θ would free additional capacity for producing more of the two new products, whereas a positive value would have the opposite effect.

(a) Use a parametric linear programming formulation to determine the effect of different choices of θ on the optimal solution for the product mix of the two new products given in the final tableau of Table 4.8. In particular, use the fundamental insight of Sec. 5.3 to obtain expressions for Z and the basic variables x_3, x_2, and x_1 in terms of θ, assuming that θ is sufficiently close to zero that this "final" basic solution still is feasible and thus optimal for the given value of θ.

(b) Now consider the broader question of the choice of θ along with the product mix for the two new products. What is the breakeven unit profit for the old product (in comparison with the two new products) below which its production rate should be decreased ($\theta < 0$) in favor of the new products and above which its production rate should be increased ($\theta > 0$)?

(c) If the unit profit is above this breakeven point, how much can the old product's production rate be increased before the final BF solution would become infeasible?

(d) If the unit profit is below this breakeven point, how much can the old product's production rate be decreased (assuming its previous rate was larger than this decrease) before the final BF solution would become infeasible?

6.7-33. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 - x_2 + 3x_3,$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &\geq 1 \\ 2x_2 + x_3 &\leq 2 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Suppose that the Big M method (see Sec. 4.6) is used to obtain the initial (artificial) BF solution. Let $\bar{x}_4$ be the artificial slack variable for the first constraint, x_5 the surplus variable for the second constraint, $\bar{x}_6$ the artificial variable for the second constraint, and x_7 the slack variable for the third constraint. The corresponding final set of equations yielding the optimal solution is

$$\begin{array}{lllllllll} (0) & Z & + 5x_2 & & + (M+2)\bar{x}_4 & & + M\bar{x}_6 & + x_7 & = 8 \\ (1) & & x_1 - x_2 & & + \bar{x}_4 & & & - x_7 & = 1 \\ (2) & & 2x_2 + x_3 & & & & & + x_7 & = 2 \\ (3) & & 3x_2 & & + \bar{x}_4 & + x_5 & - \bar{x}_6 & & = 2. \end{array}$$

Suppose that the original objective function is changed to $Z = 2x_1 + 3x_2 + 4x_3$ and that the original third constraint is changed to $2x_2 + x_3 \leq 1$. Use the sensitivity analysis procedure to revise the final set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

CASE 6.1 CONTROLLING AIR POLLUTION

Refer to Sec. 3.4 (subsection entitled "Controlling Air Pollution") for the Nori & Leets Co. problem. After the OR team obtained an optimal solution, we mentioned that the team then conducted sensitivity analysis. We now continue this story by having you retrace the steps taken by the OR team, after we provide some additional background.

The values of the various parameters in the original formulation of the model are given in Tables 3.12, 3.13, and 3.14. Since the company does not have much prior experience with the pollution abatement methods under consideration, the cost estimates given in Table 3.14 are fairly rough, and each one could easily be off by as much as 10 percent in either direction. There also is some uncertainty about the parameter values given in Table 3.13, but less so than for Table 3.14. By contrast, the values in Table 3.12 are policy standards, and so are prescribed constants.

However, there still is considerable debate about where to set these policy standards on the required reductions in the emission rates of the various pollutants. The numbers in Table 3.12 actually are preliminary values tentatively agreed upon before learning what the total cost would be to meet these standards. Both the city and company officials agree that the final decision on these policy standards should be based on the *trade-off* between costs and benefits. With this in mind, the city has concluded that each 10 percent increase in the policy standards over the current values (all the numbers in Table 3.12) would be worth \$3.5 million to the city. Therefore, the city has agreed to reduce the company's tax payments to the city by \$3.5 million for *each* 10 percent reduction in the policy standards (up to 50 percent) that is accepted by the company.

Finally, there has been some debate about the *relative* values of the policy standards for the three pollutants. As indicated in Table 3.12, the required reduction for particulates now is less than half of that for either sulfur oxides or hydrocarbons. Some have argued for decreasing this disparity. Others contend that an even greater disparity is justified because sulfur oxides and hydrocarbons cause considerably more damage than particulates. Agreement has been reached that this issue will be reexamined after information is obtained about which trade-offs in policy standards (increasing one while decreasing another) are available without increasing the total cost.

(a) Use any available linear programming software to solve the model for this problem as formulated in Sec. 3.4. In addition to the optimal solution, obtain the additional output provided for performing postoptimality analysis (e.g., the Sensitivity Report when using Excel). This output provides the basis for the following steps.

(b) Ignoring the constraints with no uncertainty about their parameter values (namely, $x_j \le 1$ for $j = 1, 2, \ldots, 6$), identify the parameters of the model that should be classified as *sensitive parameters.* (*Hint:* See the subsection "Sensitivity Analysis" in Sec. 4.7.) Make a resulting recommendation about which parameters should be estimated more closely, if possible.

(c) Analyze the effect of an inaccuracy in estimating each cost parameter given in Table 3.14. If the true value is 10 percent *less* than the estimated value, would this alter the optimal solution? Would it change if the true value were 10 percent *more* than the estimated value? Make a resulting recommendation about where to focus further work in estimating the cost parameters more closely.

(d) Consider the case where your model has been converted to maximization form before applying the simplex method. Use Table 6.14 to construct the corresponding dual problem, and use the output from applying the simplex method to the primal problem to identify an optimal solution for this dual problem. If the primal problem had been left in minimization form, how would this affect the form of the dual problem and the sign of the optimal dual variables?

(e) For each pollutant, use your results from part (d) to specify the rate at which the total cost of an optimal solution would change with any small change in the required reduction in the annual emission rate of the pollutant. Also specify how much this required reduction can be changed (up or down) without affecting the rate of change in the total cost.

(f) For each unit change in the policy standard for particulates given in Table 3.12, determine the change in the opposite direction for sulfur oxides that would keep the total cost of an optimal solution unchanged. Repeat this for hydrocarbons instead of sulfur oxides. Then do it for a simultaneous and equal change for both sulfur oxides and hydrocarbons in the opposite direction from particulates.

(g) Letting θ denote the percentage increase in all the policy standards given in Table 3.12, formulate the problem of analyzing the effect of simultaneous proportional increases in these standards as a parametric linear programming problem. Then use your results from part (*e*) to determine the rate at which the total cost of an optimal solution would increase with a small increase in θ from zero.

(h) Use the simplex method to find an optimal solution for the parametric linear programming problem formulated in part (*g*) for each $\theta = 10, 20, 30, 40, 50$. Considering the tax incentive offered by the city, use these results to determine which value of θ (including the option of $\theta = 0$) should be chosen to minimize the company's total cost of both pollution abatement and taxes.

(i) For the value of θ chosen in part (*h*), repeat parts (*e*) and (*f*) so that the decision makers can make a final decision on the *relative* values of the policy standards for the three pollutants.

CASE 6.2 FARM MANAGEMENT

The Ploughman family owns and operates a 640-acre farm that has been in the family for several generations. The Ploughmans always have had to work hard to make a decent living from the farm and have had to endure some occasional difficult years. Stories about earlier generations overcoming hardships due to droughts, floods, etc., are an important part of the family history. However, the Ploughmans enjoy their self-reliant lifestyle and gain considerable satisfaction from continuing the family tradition of successfully living off the land during an era when many family farms are being abandoned or taken over by large agricultural corporations.

John Ploughman is the current manager of the farm while his wife Eunice runs the house and manages the farm's finances. John's father, Grandpa Ploughman, lives with them and still puts in many hours working on the farm. John and Eunice's older children, Frank, Phyllis, and Carl, also are given heavy chores before and after school.

The entire famiy can produce a total of 4,000 person-hours worth of labor during the winter and spring months and 4,500 person-hours during the summer and fall. If any of these person-hours are not needed, Frank, Phyllis, and Carl will use them to work on a neighboring farm for $5 per hour during the winter and spring months and $5.50 per hour during the summer and fall.

The farm supports two types of livestock: dairy cows and laying hens, as well as three crops: soybeans, corn, and wheat. (All three are cash crops, but the corn also is a feed crop for the cows and the wheat also is used for chicken feed.) The crops are harvested during the late summer and fall. During the winter months, John, Eunice, and Grandpa make a decision about the mix of livestock and crops for the coming year.

Currently, the family has just completed a particularly successful harvest which has provided an investment fund of $20,000 that can be used to purchase more livestock. (Other money is available for ongoing expenses, including the next planting of crops.) The family currently has 30 cows valued at $35,000 and 2,000 hens valued at $5,000. They wish to keep all this livestock and perhaps purchase more. Each new cow would cost $1,500, and each new hen would cost $3.

Over a year's time, the value of a herd of cows will decrease by about 10 percent and the value of a flock of hens will decrease by about 25 percent due to aging.

Each cow will require 2 acres of land for grazing and 10 person-hours of work per month, while producing a net annual cash income of $850 for the family. The corresponding figures for each hen are: no significant acreage, 0.05 person-hour per month, and an annual net cash income of $4.25. The chicken house can accommodate a maximum of 5,000 hens, and the size of the barn limits the herd to a maximum of 42 cows.

For each acre planted in each of the three crops, the following table gives the number of person-hours of work that will be required during the first and second halves of the year, as well as a rough estimate of the crop's net value (in either income or savings in purchasing feed for the livestock).

Data per acre planted

	Soybeans	Corn	Wheat
Winter and spring, person-hours	1.0	0.9	0.6
Summer and fall, person-hours	1.4	1.2	0.7
Net value	$70	$60	$40

To provide much of the feed for the livestock, John wants to plant at least 1 acre of corn for each cow in the coming year's herd and at least 0.05 acre of wheat for each hen in the coming year's flock.

John, Eunice, and Grandpa now are discussing how much acreage should be planted in each of the crops and how many cows and hens to have for the coming year. Their objective is to maximize the family's monetary worth at the end of the coming year (the *sum* of the net income from the livestock for the coming year *plus* the net value of the crops for the coming year *plus* what remains from the investment fund *plus* the value of the livestock at the end of the coming year *plus* any income from working on a neighboring farm, *minus* living expenses of $40,000 for the year).

(a) Identify verbally the components of a linear programming model for this problem.
(b) Formulate this model. (Either an algebraic or a spreadsheet formulation is acceptable.)
(c) Obtain an optimal solution and generate the additional output provided for performing postoptimality analysis (e.g., the Sensitivity Report when using Excel). What does the model predict regarding the family's monetary worth at the end of the coming year?
(d) Find the allowable range to stay optimal for the net value per acre planted for each of the three crops.

The above estimates of the net value per acre planted in each of the three crops assumes good weather conditions. Adverse weather conditions would harm the crops and greatly reduce the resulting value. The scenarios particularly feared by the family are a drought, a flood, an early frost, *both* a drought and an early frost, and *both* a flood and an early frost. The estimated net values for the year under these scenarios are shown on the next page.

	Net Value per Acre Planted		
Scenario	**Soybeans**	**Corn**	**Wheat**
Drought	−$10	−$15	0
Flood	$15	$20	$10
Early frost	$50	$40	$30
Drought and early frost	−$15	−$20	−$10
Flood and early frost	$10	$10	$ 5

(e) Find an optimal solution under each scenario after making the necessary adjustments to the linear programming model formulated in part (*b*). In each case, what is the prediction regarding the family's monetary worth at the end of the year?

(f) For the optimal solution obtained under each of the six scenarios [including the good weather scenario considered in parts (*a*) to (*d*)], calculate what the family's monetary worth would be at the end of the year if each of the other five scenarios occur instead. In your judgment, which solution provides the best balance between yielding a large monetary worth under good weather conditions and avoiding an overly small monetary worth under adverse weather conditions.

Grandpa has researched what the weather conditions were in past years as far back as weather records have been kept, and obtained the following data.

Scenario	Frequency
Good weather	40%
Drought	20%
Flood	10%
Early frost	15%
Drought and early frost	10%
Flood and early frost	5%

With these data, the family has decided to use the following approach to making its planting and livestock decisions. Rather than the optimistic approach of assuming that good weather conditions will prevail [as done in parts (*a*) to (*d*)], the *average* net value under all weather conditions will be used for each crop (weighting the net values under the various scenarios by the frequencies in the above table).

(g) Modify the linear programming model formulated in part (*b*) to fit this new approach.

(h) Repeat part (*c*) for this modified model.

(i) Use a shadow price obtained in part (*h*) to analyze whether it would be worthwhile for the family to obtain a bank loan with a 10 percent interest rate to purchase more livestock now beyond what can be obtained with the $20,000 from the investment fund.

(j) For each of the three crops, use the postoptimality analysis information obtained in part (*h*) to identify how much latitude for error is available in estimating the net value per acre planted for that crop without changing the optimal solution. Which two net values need to be estimated most carefully? If both estimates are incorrect simultaneously, how close do the estimates need to be to guarantee that the optimal solution will not change?

This problem illustrates a kind of situation that is frequently faced by various kinds of organizations. To describe the situation in general terms, an organization faces an uncertain future where any one of a number of scenarios may unfold. Which one will occur depends on conditions that are outside the control of the organization. The organization needs to choose the levels of various activities, but the unit contribution of each activity to the overall measure of performance is greatly affected by which scenario unfolds. Under these circumstances, what is the best mix of activities?

(k) Think about specific situations outside of farm management that fit this description. Describe one.

CASE 6.3 ASSIGNING STUDENTS TO SCHOOLS (REVISITED)

Reconsider Case 4.3.

The Springfield School Board still has the policy of providing bussing for all middle school students who must travel more than approximately 1 mile. Another current policy is to allow splitting residential areas among multiple schools if this will reduce the total bussing cost. (This latter policy will be reversed in Case 12.4.) However, before adopting a bussing plan based on parts (*a*) and (*b*) of Case 4.3, the school board now wants to conduct some postoptimality analysis.

(a) If you have not already done so for parts (*a*) and (*b*) of Case 4.3, formulate and solve a linear programming model for this problem. (Either an algebraic or a spreadsheet formulation is acceptable.)
(b) Generate a sensitivity analysis report with the same software package as used in part (*a*).

One concern of the school board is the ongoing road construction in area 6. These construction projects have been delaying traffic considerably and are likely to affect the cost of bussing students from area 6, perhaps increasing them as much as 10 percent.

(c) Use the report from part (*b*) to check how much the bussing cost from area 6 to school 1 can increase (assuming no change in the costs for the other schools) before the current optimal solution would no longer be optimal. If the allowable increase is less than 10 percent, re-solve to find the new optimal solution with a 10 percent increase.
(d) Repeat part (*c*) for school 2 (assuming no change in the costs for the other schools).
(e) Now assume that the bussing cost from area 6 would increase by the same percentage for all the schools. Use the report from part (*b*) to determine how large this percentage can be before the current optimal solution might no longer be optimal. If the allowable increase is less than 10 percent, re-solve to find the new optimal solution with a 10 percent increase.

The school board has the option of adding portable classrooms to increase the capacity of one or more of the middle schools for a few years. However, this is a costly move that the board would consider only if it would significantly decrease bussing costs. Each portable classroom holds 20 students and has a leasing cost of $2,500 per year. To analyze this option, the school board decides to assume that the road construction in area 6 will wind down without significantly increasing the bussing costs from that area.

(f) For each school, use the corresponding shadow price from the report obtained in part (*b*) to determine whether it would be worthwhile to add any portable classrooms.
(g) For each school where it is worthwhile to add any portable classrooms, use the report from part (*b*) to determine how many could be added before the shadow price would no longer be valid (assuming this is the only school receiving portable classrooms).
(h) If it would be worthwhile to add portable classrooms to more than one school, use the report from part (*b*) to determine the combinations of the number to add for which the shadow prices definitely would still be valid. Then use the shadow prices to determine which of these combinations is best in terms of minimizing the total cost of bussing students and leasing portable classrooms. Re-solve to find the corresponding optimal solution for assigning students to schools.
(i) If part (*h*) was applicable, modify the best combination of portable classrooms found there by adding one more to the school with the most favorable shadow price. Find the corresponding optimal solution for assigning students to schools and generate the corresponding sensitivity analysis report. Use this information to assess whether the plan developed in part (*h*) is the best one available for minimizing the total cost of bussing students and leasing portable classrooms. If not, find the best plan.