Introduction to Duality Theory

- useful in Sensitivity Analysis
- give us another variation of Simplex (called Dual Simplex)
- nice mathematical theory

$$\begin{aligned} \max\ & X_1 + X_2 \\ \text{st}\ & 2X_1 + X_2 \le 6 \\ & 2X_1 + 4X_2 \le 12 \\ & X_1, X_2 \ge 0 \end{aligned}$$

Q: Is it possible to give a lower bound on the optimal value of this LP?

▾ Yes. Any feasible sol-n has a value that's lower than the optimal value.

E.g., for $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, value $= 0 < 4 =$ optimal value.

Q: Same question, but with upper bound.

▾ Use constraints to answer this:

i) $Z = X_1 + X_2 \le 2X_1 + X_2 \le 6$

so 6 is an upper bound

ii) $Z = X_1 + X_2 \le 2X_1 + 4X_2 \le 12$

12 is another, weaker upper bound

iii) $z = x_1 + x_2 \leq \frac{1}{3}(2x_1 + x_2) + \frac{1}{6}(2x_1 + 4x_2)$

$$\leq \frac{1}{3}\cdot 6 + \frac{1}{6}\cdot 12 = 2 + 2 = 4$$

this was the optimal value

Idea: Find best "constraint multipliers" which will give smallest possible upper bound.

- Define multipliers y_i, $i = 1, \dots, m$
  (one for each constraint)
- To give an upper bound, the multipliers should satisfy

$$2y_1 + 2y_2 \geq c_1 = 1$$
$$y_1 + 4y_2 \geq c_2 = 1$$

in our problem.

Our examples were (i) $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(ii) $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(iii) $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \end{pmatrix}$

- Also want these multipliers not to reverse the sign of the inequality:

$$y_i \geq 0, \ \forall i$$

- Then the resulting upper bound:

$$Z = c_1 \cdot x_1 + c_2 \cdot x_2 \le (2y_1 + 2y_2)\cdot x_1 + (y_1 + 4y_2)\cdot x_2$$

$$= (2x_1 + x_2)\cdot y_1 + (2x_1 + 4x_2)\cdot y_2 \le 6\cdot y_1 + 12\cdot y_2$$

- Summarizing,
  finding the best upper bound can be written as an LP:

$$\begin{aligned} \min \quad & 6y_1 + 12\cdot y_2 \\ \text{s.t.} \quad & 2y_1 + 2\cdot y_2 \ge 1 \\ & y_1 + 4\cdot y_2 \ge 1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

In general:

$$\begin{aligned} \max \quad & c^T x \\ \text{st} \quad & Ax \le b \\ & x \ge 0 \end{aligned} \longrightarrow \begin{aligned} \min \quad & b^T y \\ \text{st} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

"primal LP" "dual LP"

Observations:

- The dual LP has same data as the primal LP
- If primal LP has m constraints then dual LP has m variables
- If primal LP has n variables then dual LP has n constraints

- The roles of b and c are interchanged.

Duals of LPs in non-standard form

$$(P)\quad \begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array} \xrightarrow{\text{dual}} \begin{array}{ll} \min & b^T y \\ & A^T y \ge c \\ & y \ge 0 \end{array} \quad (D)$$

this is called **primal-dual pair** of LP problems.

- Question: What's the dual of the dual?

$$\left.\begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \ge c \\ & y \ge 0 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{ll} -\max & (-b)^T y \\ \text{s.t.} & (-A^T) y \le -c \\ & y \ge 0 \end{array}\right\} \xrightarrow{\text{dual}}$$

$$\left\{\begin{array}{ll} -\min & (-c)^T x \\ \text{s.t.} & (-A) x \ge -b \\ & x \ge 0 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}\right\}$$

Thus, the dual of the dual is **the primal**.

This is called primal-dual symmetry.

- Duals of LP's in non-standard form?
  - Tranform into standard form;
  - Take the "standard" dual;
  - manipulate further to simplify.

Example:

$$\left.\begin{array}{ll}\max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0\end{array}\right\} \longleftrightarrow \left\{\begin{array}{ll}\max & c^T x \\ \text{s.t.} & Ax \leq b \\ & Ax \geq b \\ & x \geq 0\end{array}\right\} \longleftrightarrow$$

$$\left\{\begin{array}{ll}\max & c^T x \\ \text{s.t.} & Ax \leq b \\ & -Ax \leq -b \\ & x \geq 0\end{array}\right\} \longleftrightarrow \left\{\begin{array}{ll}\max & c^T x \\ \text{s.t.} & \begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix} \\ & x \geq 0\end{array}\right\} \xrightarrow{\text{dual}}$$

$$\left\{\begin{array}{ll}\min & b^T y_1 - b^T y_2 \\ \text{s.t.} & (A^T \,\vdots\, -A^T)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq c \\ & y_1, y_2 \geq 0\end{array}\right\} \longleftrightarrow \left\{\begin{array}{ll}\min & b^T(y_1 - y_2) \\ \text{s.t} & A^T(y_1 - y_2) \geq c \\ & y_1, y_2 \geq 0\end{array}\right.$$

$$\longleftrightarrow \left\{\begin{array}{ll}\min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \text{ free}\end{array}\right\}, \text{ where } y = y_1 - y_2$$

Thus, $\left\{\begin{array}{c}\text{equality constraint} \\ \text{in primal}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{free variable} \\ \text{in dual}\end{array}\right\}$

Similarly, can get the following **dualization rules**.

(P) max	$\longleftrightarrow$	min (D)
"$\leq$"-constr.	$\longleftrightarrow$	"≥ 0" variable
"$\geq$"-constr.	$\longleftrightarrow$	"≤ 0" variable
"$=$"-constr.	$\longleftrightarrow$	free variable
"≥ 0"-var.	$\longleftrightarrow$	"$\geq$"-constr.
"≤ 0"-var.	$\longleftrightarrow$	"$\leq$"-constr.
free var.	$\longleftrightarrow$	"$=$"-constr.

Example:

$$\left.\begin{array}{rll}
\min & x_1 + x_2 - x_3 & \\
\text{s.t.} & 2x_1 + 2x_2 - 4x_3 = 0 & (y_1) \\
& 3x_1 - 2x_2 + 5x_3 \leq 5 & (y_2) \\
& 8x_1 - 7x_2 - 10x_3 \geq -17 & \\
& x_1 \text{ free} & \\
& x_2 \geq 0 & \\
& x_3 \leq 0 &
\end{array}\right\}
\xleftrightarrow{\text{dual}}
\left\{\begin{array}{rl}
\max & 0 \cdot y_1 + 5 \cdot y_2 - 17 \cdot y_3 \\
\text{s.t.} & 2 \cdot y_1 + 3 \cdot y_2 + 8 \cdot y_3 = 1 \\
& 2 \cdot y_1 - 2 \cdot y_2 - 7 \cdot y_3 \leq 1 \\
& -4 \cdot y_1 + 5 \cdot y_2 - 10 \cdot y_3 \geq -1 \\
& y_1 \text{ free} \\
& y_2 \leq 0 \\
& y_3 \geq 0
\end{array}\right.$$