Last time

- Introduction to Duality
- Primal-Dual Symmetry
- Dualization Rules

Today

Primal-Dual Relationships

$$(P)\quad \left.\begin{array}{rl} \max & c^T x \\ \text{st} & Ax \le b \\ & x \ge 0 \end{array}\right\} \qquad \left\{\begin{array}{rl} \min & b^T y \\ & A^T y \ge c \\ & y \ge 0 \end{array}\right. \quad (D)$$

- Suppose $\hat{x}$ is feasible for (P); and $\hat{y}$ is feasible for (D).
- Then

$$A\hat{x} \le b \quad (1) \qquad A^T\hat{y} \ge c \quad (3)$$
$$\hat{x} \ge 0 \quad (2) \qquad \hat{y} \ge 0 \quad (4)$$

- Compare obj. f-n values at $\hat{x}$ and $\hat{y}$:

$$c^T\hat{x} \underset{(2),(3)}{\le} (A^T\hat{y})^T\hat{x} = \hat{y}^T A\hat{x} \underset{(1),(4)}{\le} \hat{y}^T b = b^T\hat{y}$$

- Thus,

Weak Duality Property:

If $\hat{x}$ is feasible for (P) (maximization problem)
$\hat{y}$ is feasible for (D) (minimization problem)
Then $c^T\hat{x} \le b^T\hat{y}$

Ex.: (Recall our problem from last lecture)

$\hat{x} = (0,3)$ feasible for (P);

$\hat{y} = (1,0)$ feasible for (D).

$$c^T\hat{x} = 0+3 = 3 < 6 = 6\cdot 1 + 12\cdot 0 = b^T\hat{y}$$

- **Strong Duality Property:**

If x^* is optimal for (P),
y^* is optimal for (D),
then $c^Tx^* = b^Ty^*$.

Ex.: In the same problem,

$x^* = (2,2)$ optimal for (P);

$y^* = (1/3, 1/6)$ optimal for (D).

$$c^Tx^* = 2+2 = 4 = 6\cdot\tfrac{1}{3} + 12\cdot\tfrac{1}{6} = b^Ty^*$$

- What if (P) (or (D)) doesn't have optimal solution?
  - Recall that the following 3 cases are possible for any LP:
    - has optimal solution(s);
    - is unbounded;
    - is infeasible.
  - Suppose (P) is in one of these 3 cases. What can we say about (D)?

	(D) ∃ opt.	unbounded	infeasible
(P) ∃ opt.	① ✓	② ✗	⑤ ✗
unbounded	② ✗	② ✗	③ ✓
infeasible	⑤ ✗	③ ✓	④ ✓

① can happen (see our ex.)

② cannot happen by weak duality

③ can happen by weak duality

④ can happen (see ex. below)

⑤ cannot happen; we'll see later

Ex.: (both (P) and (D) infeasible)

$$\begin{array}{ll} \max & 8x_1+5x_2 \\ & x_1+x_2 \geq 7 \\ & x_1+x_2 \leq 6 \\ & x_1 \geq 0,\ x_2 \leq 0 \end{array} \qquad \begin{array}{ll} \min & 7y_1+6y_2 \\ & y_1+y_2 \geq 8 \\ & y_1+y_2 \leq 5 \\ & y_1 \leq 0,\ y_2 \geq 0 \end{array}$$

- Duality Theorem: The following are only possible relationships between (P) and (D):
  - a) both (P) and (D) have optimal sol-ns, and their optimal values are the same.
  - b) (P) is unbounded, (D) is infeasible.
  - c) (P) is infeasible, (D) is unbounded.
  - d) both (P) and (D) are infeable.

- **Def-n:** A primal-dual pair of solutions $(\hat{x}, \hat{y})$ is called <u>complementary</u> if their obj. f-n values coincide, i.e., if $c^T\hat{x} = b^T\hat{y}$.

<u>Ex.:</u> $\hat{x} = (0,3)$ and $\hat{y} = (0, 1/4)$ are complementary in our example:

$$c^T\hat{x} = 0 + 3 = 3 = 6\cdot 0 + 12 \cdot 1/4 = b^T\hat{y}$$

<u>Note</u> that $\hat{y}$ is <u>not</u> feasible for (D):

$$2\cdot 0 + 2\cdot 1/4 = 1/2 < 1$$

- <u>Optimality Criterion:</u>

  x is optimal for (P) if

  a) x is feasible for (P) (primal feasibility)

  b) $\exists\, y$ feasible for (D) (dual feasibility)

  c) $c^Tx = b^Ty$ (complementarity)

- Let's examine more closely when we have complementarity for feasible x and y.
  - Recall <u>weak duality</u> inequalities:

$$c^Tx \le (A^Ty)^Tx = y^TAx \le b^Ty$$

  - Get <u>equality</u> in this inequality if

$$c^Tx = (A^Ty)^Tx \quad \text{and} \quad y^TAx = y^Tb$$

$$\Leftrightarrow \quad \underset{\ge 0}{x^T}(\underset{\ge 0}{A^Ty - c}) = 0 \quad \text{and} \quad \underset{\ge 0}{y^T}(\underset{\ge 0}{b - Ax}) = 0$$

$$\Leftrightarrow \quad x_j \cdot \left(\sum_{i=1}^{m} a_{ij} y_i - c_j\right) = 0 \quad \forall j = 1, \dots, n$$

and $$y_i \cdot \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) = 0 \quad \forall i = 1, \dots, m$$

- Thus, for complementary solutions:

  - $x_j > 0 \Rightarrow \sum_{i=1}^{m} a_{ij} y_i - c_j = 0$

    (primal var. >0 $\Rightarrow$ corresponding dual constraint is tight)

  - $\sum_{i=1}^{m} a_{ij} y_i - c_j > 0 \Rightarrow x_j = 0$

    (dual constraint not tight $\Rightarrow$ corresponding pr. var. $= 0$)

  - $y_i > 0 \Rightarrow b_i - \sum_{j=1}^{n} a_{ij} x_j = 0$

    (dual var. >0 $\Rightarrow$ corresponding primal constraint is tight)

  - $b_i - \sum_{j=1}^{n} a_{ij} \cdot x_j > 0 \Rightarrow y_i = 0$

    (primal constraint not tight $\Rightarrow$ corresponding dual var. $= 0$)

COMPLEMENTARY SLACKNESS

- Optimality Criterion (Restated)

  x is optimal for (P) if

  a) x is feasible for (P)

  b) $\exists\, y$ feasible for (D)

  c) x and y satisfy compementary slackness conditions

Example:

(P)

$$\max \quad x_1 + x_2$$
$$\text{s.t.} \quad 2x_1 + x_2 \le 6 \quad (y_1)$$
$$2x_1 + 4x_2 \le 12 \quad (y_2)$$
$$x_1, x_2 \ge 0$$

(D)

$$\min \quad 6y_1 + 12 \cdot y_2$$
$$\text{s.t.} \quad 2y_1 + 2 \cdot y_2 \ge 1 \quad (x_1)$$
$$y_1 + 4y_2 \ge 1 \quad (x_2)$$
$$y_1, y_2 \ge 0$$

- optimal sol-n for (P) : $x^* = \binom{2}{2}$, $z^* = 4$
- How to get optimal sol-n for (D) using this information?
- Use complementary slackness:

$$x_1 = 2 > 0 \Rightarrow 2y_1 + 2 \cdot y_2 = 1$$
$$x_2 = 2 > 0 \Rightarrow y_1 + 4 \cdot y_2 = 1$$

- Solve the system:

$$\begin{cases} 2 \cdot y_1 + 2 \cdot y_2 = 1 \\ y_1 + 4 \cdot y_2 = 1 \end{cases} \rightarrow \begin{cases} y_1 = \frac{1}{3} \\ y_2 = \frac{1}{6} \end{cases}$$

- Check:

$y^* = \left(\frac{1}{3}, \frac{1}{6}\right) \ge 0$ and

$$6 \cdot \frac{1}{3} + 12 \cdot \frac{1}{4} = 4 = z^*$$

- $y^* = \left(\frac{1}{3}, \frac{1}{6}\right)$ is optimal for (D)

Note:
- $\frac{1}{3}$ and $\frac{1}{6}$ were the best constraint multipliers (giving the best upper bound)
- $\frac{1}{3}$ and $\frac{1}{6}$ appeared also in the

final Simplex tableau for (P):

Basic var.	x_1	x_2	x_3	x_4	RHS
Z	0	0	$1/3$	$1/6$	4
x_1	1	0	$2/3$	$-1/6$	2
x_2	0	1	$-1/3$	$1/3$	2

- Is it coincidence?

We'll see next time that it's not.