Last time
- weak duality
- strong duality
- Duality Theorem
- complementary slackness

Today
- Reinterpretation of Simplex based on Duality
- Dual Simplex

Recall the Simplex tableau of current basis B in terms of original data:

Basic var.	X_B	X_N	RHS
Z	0	$C_B^T B^{-1} A - C_N^T$	$C_B^T B^{-1} b$
X_B	I	$B^{-1} N$	$B^{-1} b$

Q: Can we get a compementary dual sol-n to the current primal BFS from this tableau?

▼ For complementarity need $C^T x = b^T y$.

▼ Obj. f-n value at current BFS X is $C_B^T B^{-1} b$.

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▼ Obj. f-n value at $y^T = C_B^T B^{-1}$ is
$$y^T b = C_B^T B^{-1} b \quad \text{too}.$$

▼ Thus, $y^T = C_B^T B^{-1}$ is a dual complementary sol-n to current primal BFS. Can we read that dual sol-n from current tableau?

- If X_j is slack and nonbasic then $C_j = 0$, $A_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← row of X_j

So the coefficint of X_j in row 0:
$$C_B^T B^{-1} A_j - C_j = C_B^T B^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (C_B^T B^{-1})_j$$

- If X_j is slack and basic then based on fundamental insight
$$B^{-1} = \begin{array}{c c} & X_j \\ X_j & \left[\begin{matrix} & 0 & \\ & \vdots & \\ & 0 & \\ \cdots & 1 & \cdots \\ & 0 & \end{matrix}\right] \end{array}. \quad \text{Also} \quad C_B^T = (\ldots \overset{X_j}{0} \ldots$$

So $(C_B^T B^{-1})_j = (\ldots \overset{X_j}{0} \ldots) \begin{array}{c c} & \\ X_j & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{array} = 0 =$

= coefficient of X_j in row 0

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▼ Summarizing, the coefficients of slack variables in row 0 give a compementary dual sol-n to the current primal BFS.

▼ Similarly, we can get that the coefficients of original variables in row 0 give the surplus variables of the compementary dual sol-n.

Example:

(P)
$$\max \; X_1 + X_2$$
$$\text{st} \quad 2X_1 + X_2 \le 6$$
$$2X_1 + 4X_2 \le 12$$
$$X_1, X_2 \ge 0$$

(D)
$$\min \; 6y_1 + 12y_2$$
$$2y_1 + 2y_2 \ge 1$$
$$y_1 + 4y_2 \ge 1$$
$$y_1, y_2 \ge 0$$

First Tableau:

	X_1 (original)	X_2 (original)	X_3 (slack)	X_4 (slack)	RHS
Z	-1	-1	0	0	0
X_3	2	1	1	0	6
X_4	2	4^*	0	1	12

primal BFS $= \begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \end{pmatrix}$, complementary dual sol-n $= \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$ ← infeasible

$Z = 0$ for both

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Second Tableau:

	X_1	X_2	X_3	X_4	RHS
Z	$-1/2$	0	0	$1/4$	3
X_3	$3/2$	0	1	$-1/4$	3
X_2	$1/2$	1	0	$1/4$	3

primal BFS $= \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}$, complementary dual sol-n $= \begin{pmatrix} 0 \\ 1/4 \\ -1/2 \\ 0 \end{pmatrix}$ ← infeas.

$Z = 3$ for both

Third Tableau:

	X_1	X_2	X_3	X_4	RHS
Z	0	0	$1/3$	$1/6$	4
X_1	1	0	$2/3$	$-1/6$	2
X_2	0	1	$-1/3$	$1/3$	2

primal BFS $= \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, complem. dual sol-n $= \begin{pmatrix} 1/3 \\ 1/6 \\ 0 \\ 0 \end{pmatrix}$, $Z = 4$

primal and dual feasible ⇒ both of them are optimal

Reinterpretation of Simplex Method

- At each iteration we have primal and dual complementary solutions.
- Keep the primal feasibility and work towards dual feasibility

Dual Simplex Algorithm

- Suppose we have a pair of complementary solutions such that
  - dual is feasible
  - primal is infeasible.

  E.g., this situation might arise when RHS is changed (recall sensitivity analysis)

- How to proceed?

  Idea: Maintain dual feasibility and complementarity, while striving for primal feasibility.

- How to implement this?

  Apply (implicitly) primal simplex algorithm to the dual problem, while working with the primal tableau.

Outline of Dual Simplex

Step 1: Optimality test (= feasibility test): If primal feasible stop.

Step 2: Choice of leaving variable: if not primal feasible then choose basic variable with most negative value to leave basis.

Step 3: Infeasibility test: if coefficients of all nonbasic variables in the row of leavi[ng] variable (pivot row) are ≥ 0, stop. Primal problem is infeasible.

Step 4: Choice of entering variable: choose as entering variable that with the ratio (row 0 coeff.)/(coef. in pivot row) closest to zero, considering only nonbasic variables with (coef. in pivot row) negative.

Step 5: As in primal simplex method, use row operations to get in proper form for new basic variables. Go to Step 1.

Notes:
- pivot number negative
- value of entering variable will be positive
- obj. f-n value will go down (if no dual degeneracy)

Ex.:
$$\begin{aligned} \min\ & 4y_1 + 7y_2 \\ \text{st}\ & 2y_1 + y_2 \geq 5 \\ & 3y_1 + 2y_2 \geq 2 \\ & y_1 + 3y_2 \geq 5 \qquad y_1, y_2 \geq 0 \end{aligned} \qquad (P)$$

Standard form:
$$\begin{aligned} \max\ & -4y_1 - 7y_2 \\ \text{st}\ & -2y_1 - y_2 \leq -5 \\ & -3y_1 - 2y_2 \leq -2 \\ & -y_1 - 3y_2 \leq -5 \qquad y_1, y_2 \geq 0 \end{aligned}$$

B.V.	y_1	y_2	y_3	y_4	y_5	RHS	
Z	4	7	0	0	0	0	
y_3	-2^*	-1	1	0	0	-5	← most neg.
y_4	-3	-2	0	1	0	-2	ratios: $\frac{4}{-2}, \frac{7}{-1}$
y_5	-1	-3	0	0	1	-5	↖closest to 0

	y_1	y_2	y_3	y_4	y_5	RHS	
Z	0	5	2	0	0	-10	
y_1	1	$1/2$	$-1/2$	0	0	$5/2$	ratios:
y_4	0	$-1/2$	$-3/2$	1	0	$11/2$	$\frac{5}{-5/2}, \frac{2}{-1/2}$
y_5	0	$-5/2^*$	$-1/2$	0	1	$-5/2$	↖ closest to 0

	y_1	y_2	y_3	y_4	y_5	RHS	
Z	0	0	1	0	2	-15	
y_1	1	0	$-3/5$	0	$1/5$	2	primal
y_4	0	0	$-7/5$	1	$-1/5$	6	feasible
y_2	0	1	$1/5$	0	$-2/5$	1	→ done

$y^* = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}$, with obj. f-n value -15

Ex. (cont.):

To get more insight how dual simplex works let's solve the dual problem by primal simplex and draw parallels between corresponding iterations:

$$(D) \qquad \begin{aligned} \max\ & 5x_1 + 2x_2 + 5x_3 \\ & 2x_1 + 3x_2 + x_3 \leq 4 \\ & x_1 + 2x_2 + 3x_3 \leq 7 \qquad x_i \geq 0\ \forall i \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	RHS
Z	-5	-2	-5	0	0	0
x_4	2^*	3	1	1	0	4
x_5	1	2	3	0	1	7

	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	$1/2$	$-5/2$	$5/2$	0	10
x_1	1	$3/2$	$1/2$	$1/2$	0	2
x_5	0	$1/2$	$5/2^*$	$-1/2$	1	5

	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	6	0	2	1	15
x_1	1	$7/5$	0	$3/5$	$-1/5$	1
x_3	0	$1/5$	1	$-1/5$	$2/5$	2