Another Application of LP: Game Theory

Only short introduction here.

- Suppose we have 2 players playing a simple game.

Ex.: 2 advertizing companies sharing the market. Company C receives whatever share Comp. R doesn't get.

- Comp. R → 2 advertizing strategies
- Comp. C → 3 advertizing strategies

(Strategy is predetermined rule of actions in each possible situation)

Payoff table:

R \ C	Str. 1	Str. 2	Str. 3
Str. 1	30%	40%	60%
Str. 2	20%	10%	30%

Values in this table are market shares of R.

- R wants to maximize payoff;
  C wants to minimize payoff. (for R)

This kind of games are called
two-person, zero-sum games.

- How to solve this game?
- Suppose both players are conservative.
  - R wants to maximize worst outcome
  - C wants to minimize best outcome (for R).

R \ C	S1	S2	S3
S1	30*	40	60
S2	20	10	30

maximin = max $\begin{Bmatrix} 30 \\ 10 \end{Bmatrix}$ = 30

minimax = min {30, 40, 60} = 30

- maximin = minimax = 30; the corresponding entry in the table is called saddle point.
- This sol-n: (S1 for R, S1 for C) with payoff (30%, 70%) is a stable sol-n (equilibrium sol-n).

- Are maximin and minimax always equal?
  - Change 20% to 60% in the table.
  - New table:

R \ C	S1	S2	S3
S1	30	40	60
S2	60	10	30

maximin $= \max(30, 10) = 30$
(S1 for R)

minimax =
$\min(60, 40, 60) = 40$ (S2 for C)

- Here minimax $= 40 > 30 =$ maximin.
  (Generally, minimax $\geq$ maximin)
- There is no stable sol-n here (in pure strategies).
- Need to mix the strategies to achieve stability.

- Recall the last example:

R \ C	S1	S2	S3
S1	30	40	60
S2	60	10	30

minimax $= 40 > 30 =$ maximin

- No stable sol-n in pure strategies.
- How about randomizing (mixing strategies)?

  E.g., C selects 3 alternative strategies with probabilities x_1, x_2, x_3.

  $$x_1 + x_2 + x_3 = 1 \quad , \quad x_1, x_2, x_3 \geq 0$$

- Then the expected market share of R is

  $30x_1 + 40x_2 + 60x_3$ if R selects S1;

  $60x_1 + 10x_2 + 30x_3$ if R selects S2.

- C wants to minimize this expectation:

  $$\min \max \{30x_1 + 40x_2 + 60x_3 , 60x_1 + 10x_2 + 30x_3\}$$
  $$\text{s.t.} \sum_{i=1}^{3} x_i = 1 \quad , \quad x_i \geq 0$$

- This is not a linear program yet.

An equivalent LP is:

$$\begin{aligned} \min \quad & V \\ \text{s.t.} \quad & 30x_1 + 40x_2 + 60x_3 \leq V \\ & 60x_1 + 10x_2 + 30x_3 \leq V \\ & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \ \forall i \end{aligned} \qquad \text{(LP1)}$$

- The optimal sol-n to this LP:

  $x_1^* = \frac{1}{2}, \ x_2^* = \frac{1}{2}, \ x_3^* = 0, \ V^* = 35$

- Similar ideas for row player R result in the following LP:

$$\begin{aligned} \max \quad & W \\ \text{s.t.} \quad & 30y_1 + 60y_2 \geq W \\ & 40y_1 + 10y_2 \geq W \\ & 60y_1 + 30y_2 \geq W \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned} \qquad \text{(LP2)}$$

with optimal sol-n $y_1^* = \frac{5}{6}, \ y_2^* = \frac{1}{6}, \ W^* = 35$.

- $V^* = 35 = W^*$ $\longrightarrow$ stable solution
- This is not a coincidence. Generally, (LP1) and (LP2) are dual problems (check it as an exercise).
- **Theorem:** Optimal mixed strategies provide a stable sol-n with value $V^* = W^*$.