

Another Application of LP:

Game Theory

Only short introduction here.

See [link] for a full course on this subject.

- Suppose we have 2 players playing a simple game.

Ex.: 2 advertizing companies sharing the market.
Company C receives whatever share Comp. R doesn't get.

▼ Comp. R → 2 advertizing strategies

▼ Comp. C → 3 advertizing strategies

(Strategy is predetermined rule of actions in each possible situation)

Payoff table:

R \ C	Str. 1	Str. 2	Str. 3
Str. 1	30%	40%	60%
Str. 2	20%	10%	30%

Values in this table are market shares of R.

- ▼ R wants to maximize payoff;
- C wants to minimize payoff. (for R)

This kind of games are called two-person, zero-sum games.

- How to solve this game?
- Suppose both players are conservative.
 - ▼ R wants to maximize worst outcome
 - ▼ C wants to minimize best outcome (for R).

R \ C	S1	S2	S3
S1	30*	40	60
S2	20	10	30

$$\text{maximin} = \max \begin{Bmatrix} 30 \\ 10 \end{Bmatrix} = 30$$

$$\begin{array}{c} \text{minimax} \\ \parallel \\ \text{min} \\ \{30, 40, 60\} = 30 \end{array}$$

▼ $\text{maximin} = \text{minimax} = 30$; the corresponding entry in the table is called saddle point.

▼ This sol-n: (S1 for R, S1 for C) with payoff (30%, 70%) is a stable sol-n (equilibrium sol-n).

• Are maximin and minimax always equal?

▼ Change 20% to 60% in the table.

▼ New table:

R\C	S1	S2	S3
S1	30	40	60
S2	60	10	30

maximin = $\max(30, 10) = 30$
 (S1 for R)

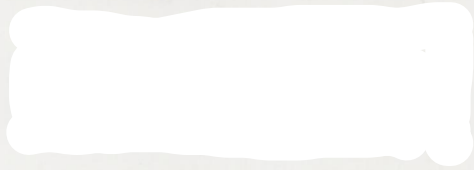
minimax = $\min(60, 40, 60) = 40$ (S2 for C)

▼ Here $\text{minimax} = 40 > 30 = \text{maximin}$.

(Generally, $\text{minimax} \geq \text{maximin}$)

▼ There is no stable soln here (in pure strategies).

▼ Need to mix the strategies to achieve stability.



Recall the last example:

R \ C	S1	S2	S3
S1	30	40	60
S2	60	10	30

minimax = 40 > 30 = maximin

- ▼ No stable sol-n in pure strategies.
- ▼ How about randomizing (mixing strategies)?

E.g., C selects 3 alternative strategies with probabilities x_1, x_2, x_3 .

$$x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0$$

- ▼ Then the expected market share of R is
 - $30x_1 + 40x_2 + 60x_3$ if R selects S1;
 - $60x_1 + 10x_2 + 30x_3$ if R selects S2.

▼ C wants to minimize this expectation:

$$\min \max \{ 30x_1 + 40x_2 + 60x_3, 60x_1 + 10x_2 + 30x_3 \}$$

$$\text{s.t. } \sum_{i=1}^3 x_i = 1, \quad x_i \geq 0$$

▼ This is not a linear program yet.

An equivalent LP is:

(recall the techniques of HVG, R.P.)

$$\begin{aligned} \min \quad & V \\ \text{s.t.} \quad & 30X_1 + 40X_2 + 60X_3 \leq V \\ & 60X_1 + 10X_2 + 30X_3 \leq V \\ & X_1 + X_2 + X_3 = 1 \\ & X_i \geq 0 \quad \forall i \end{aligned} \quad (\text{LP1})$$

▼ The optimal sol-n to this LP:

$$X_1^* = \frac{1}{2}, \quad X_2^* = \frac{1}{2}, \quad X_3^* = 0, \quad V^* = 35$$

▼ Similar ideas for row player R result in the following LP:

$$\begin{aligned} \max \quad & W \\ \text{s.t.} \quad & 30y_1 + 60y_2 \geq W \\ & 40y_1 + 10y_2 \geq W \\ & 60y_1 + 30y_2 \geq W \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned} \quad (\text{LP2})$$

with optimal sol-n $y_1^* = \frac{5}{6}, \quad y_2^* = \frac{1}{6}, \quad W^* = 35.$

▼ $V^* = 35 = W^* \rightarrow$ stable solution

▼ This is not a coincidence. Generally, (LP1) and (LP2) are dual problems (check it as an exercise).

▼ Theorem: Optimal mixed strategies provide a stable sol-n with value $V^* = W^*$.