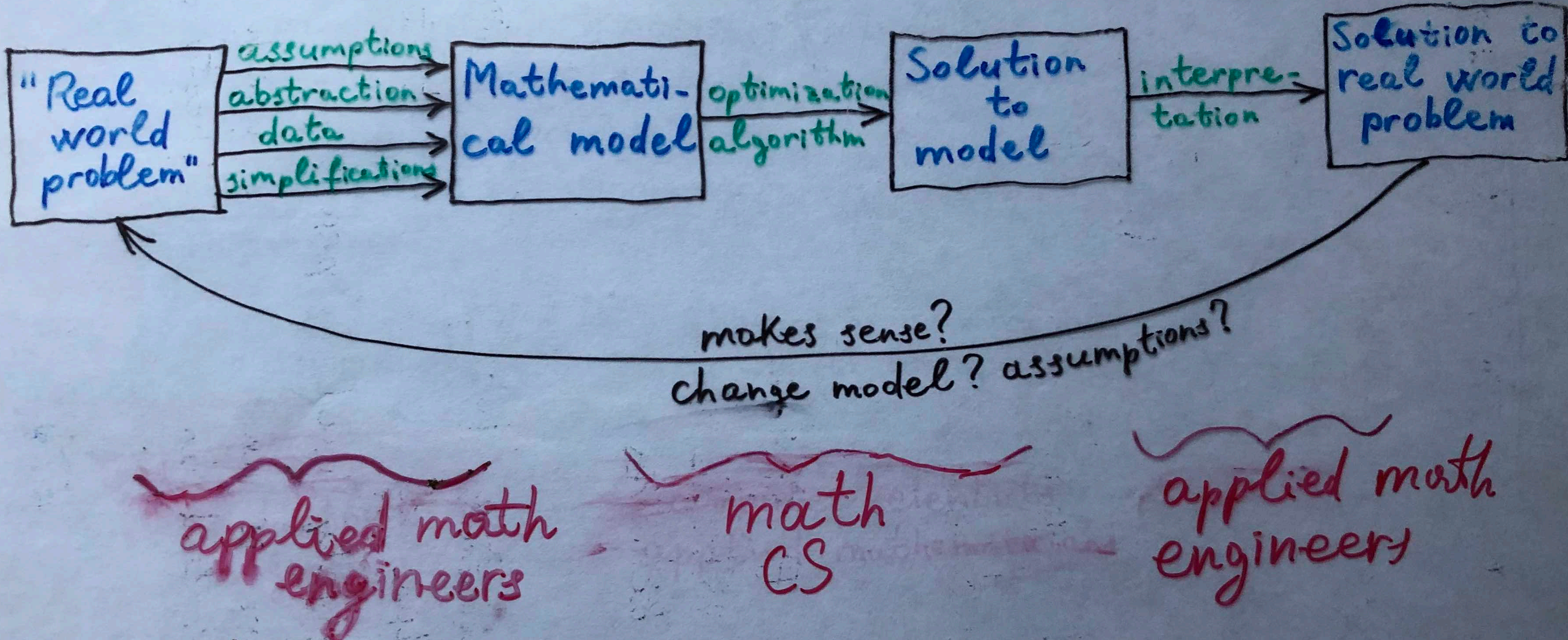
A (schematic) view of the modeling/optimization process



In this class:

- special case of mathematical model
- some simple modeling
- use modeling language (optional)
- learn solution process (algorithm)

Mathematical models in optimization

in general, look like the following:

$$\begin{aligned} &\text{min or max} \quad f(x) && \leftarrow \text{"objective function"} \\ &\text{subject to} \quad a(x) \geq 0 && \leftarrow \text{"functional constraints"} \\ & \qquad\qquad\quad x \in S && \leftarrow \text{"set constraints"} \end{aligned}$$

$\uparrow$ some set

$f: \mathbb{R}^n \to \mathbb{R}$, $a: \mathbb{R}^n \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^n$

x is called the <u>decision variable</u>.

- In words,
  goal is to find a vector x that
  - satisfies the constraints ($a(x) \geq 0, x \in S$)
  - achieves max (min) objective function value.

- In the <u>general</u> form, very hard to solve.
- In this class a special case that
  - is widely used in practice
  - there is an algorithm which will find an "optimal" (best) solution.

Classification of Mathematical Programming Problems

Stochastic vs **Deterministic**

Stochastic (have probabilistic information on data)

Deterministic (assume data is certain, no uncertainty)

- integer, discrete ($S = \mathbb{N}^n$ or $S = \{0,1\}^n$)
- continuous (f, a are differentiable functions, $S = \mathbb{R}^n$)
  - linear (f, a are linear)
  - nonlinear (f, a nonlinear)

A simple first example

Furniture manufacturer

- production of 1 table requires 5 ft pine, 2 ft oak, 3 hrs labor
  1 chair 1 —"—, 3 —"—, 2 —"—
  1 desk 9 —"—, 4 —"—, 5 —"—
  1 bookcase 12 —"—, 1 —"—, 10 —"—

- for 1 week have
  - 1500 ft pine
  - 1000 ft oak
  - 20 employees who will each work 40 hrs

- market data:

	profit	demand
table	$12/unit	40
chair	$5/unit	130
desk	$15/unit	30
bookcase	$10/unit	–

Goal: find production schedule for 1 week that will maximize profit.

Production schedule = # tables to be produced → X_T
chairs —"— → X_C
desks —"— → X_D
bookcases —"— → X_B

ALWAYS DEFINE VARIABLES PROPERLY!

Incorporate all information into mathematical model:

- Objective: maximize profit:

$$\max \; \left. 12 \cdot X_T + 5 \cdot X_c + 15 \cdot X_D + 10 \cdot X_B \right\} \text{objective function}$$

- Constraints/requirements:

$$\left.\begin{array}{rl} \text{pine:} & 5 \cdot X_T + 1 \cdot X_c + 9 \cdot X_D + 12 \cdot X_B \leq 1500 \\ \text{oak:} & 2 \cdot X_T + 3 \cdot X_c + 4 \cdot X_D + 1 \cdot X_B \leq 1000 \\ \text{labor:} & 3 \cdot X_T + 2 \cdot X_c + 5 \cdot X_D + 10 \cdot X_B \leq 800 \end{array}\right\} \text{can't exceed available resources}$$

$$\left.\begin{array}{l} X_c \geq 130 \\ X_T \geq 40 \\ X_D \geq 30 \\ X_B \geq 0 \end{array}\right\} \text{want to satisfy demand}$$

This is a special kind of optimization problem: the objective functions and the constraints are linear in X.

This MP is called a

Linear Programming Problem (LP)

General form of LPs

$$
\begin{aligned}
\text{max/min} \quad & c_1 \cdot x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{subject to} \quad & a_{11} \cdot x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
& a_{21} \cdot x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m \\
& x_1 \geq 0, \; x_2 \geq 0, \; \ldots, \; x_n \geq 0 \\
& x \in \mathbb{R}^n
\end{aligned}
$$

This is the standard form of LP.

Can also have equality or $\geq$ constraints.

Assumptions of LP

- proportionality (no discounts, $c_i x_i^2$ not allowed)
- additivity ($x_i \cdot x_2$ not allowed)

$\Big\}$ linearity

- divisibility (fractional values allowed)
- certainty (of demand, etc.)

(read more in H&L 3.3)

Other ways of writing LPs

1) $\max \sum_{i=1}^{n} c_i x_i$

s.t. $\sum_{i=1}^{n} a_{ki} x_i \leq b_k \quad \forall k \in 1..m$

$x_i \geq 0 \quad \forall i \in 1..n$

This summation form is used in AMPL.

2) matrix form

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$\max \quad c^T x$

s.t. $Ax \leq b$

$x \geq 0$

Input in LP is (A, b, c) → given data, parameters

Output —"— is x → variables

- Programming = Planning in this context, origins go back to military logistics in WWII (1940's)
- In a survey of Fortune 500 firms, 85% of those responding said that they had used LP

LP applications

- production planning (inventory models, ...)
- finance (portfolio management, ...)
- scheduling (airline, production, ...)
- networks (transportation, telecommunications, ...)

...

Goals in this class

- practise modelling process: given "real world problem", formulate as a MP problem (here mostly LP); become aware of underlying assumptions & limitations
- learn a general algorithm (solution procedure) for LP problems (Known as the "Simplex method")
- learn to view LP & the algorithm from different perspectives: geometry, algebra
- perform sensitivity analysis:

  what if data changes?

  how much will solution change, if at all?
- look into applications & special cases of LP
- learn to justify answers in a mathematically precise way
- introduction to nonlinear programming