Special Types of LP's

Network Models

- Min-Cost Flow Problem
  - Transportation Problem
  - Assignment Problem
  - Shortest Path Problem
  - Max Flow Problem

The model

Given: Graph $G=(V,E)$, $|V|=n$, $|E|=m$

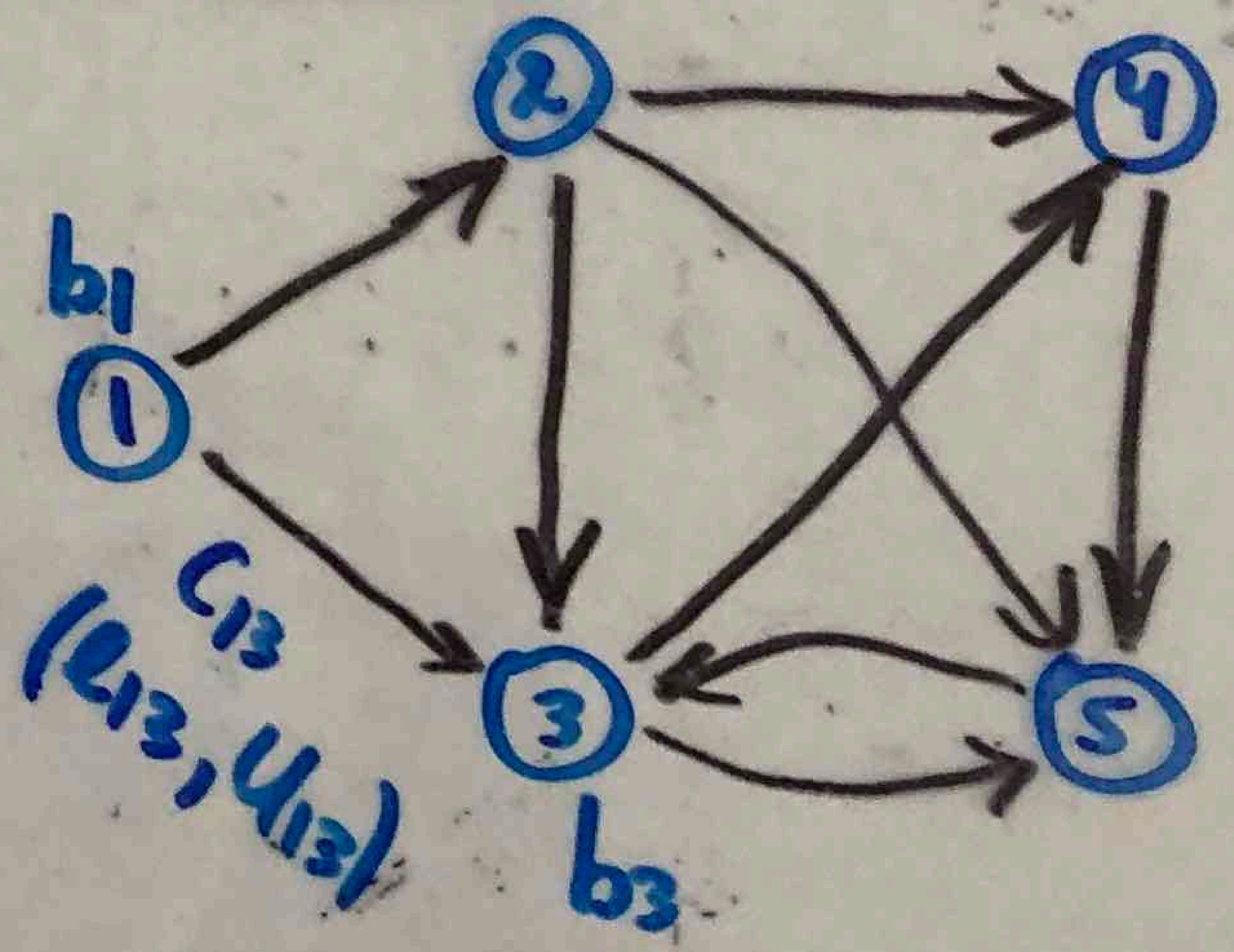
set of vertices (nodes) — set of edges (arcs)

cost function: $c: E \to \mathbb{R}$

upper bounds: $u: E \to \mathbb{R}$

lower bounds: $\ell: E \to \mathbb{R}$

demand/supply of nodes: $b: V \to \mathbb{R}$



$b_j < 0 \to$ demand at node j

$b_j > 0 \to$ supply at node j

$b_j = 0 \to$ j is transshipment node

Objective for "min-cost flow problem":

Find a feasible flow X on arcs such that cost is minimized.

Feasible means:

- supply & demand are satisfied at each node (flow conservation constraints)
- $\ell \leq X \leq u$ (bound constraints)

- What's the corresponding LP model?

decision variables X_{ij}: flow to send from node i to j. $\forall\, i \to j \in E$

LP formulation:

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \underbrace{\sum_{k:(j,k)\in E} X_{j,k}}_{\text{outflow from } j} - \underbrace{\sum_{i:(i,j)\in E} X_{ij}}_{\text{inflow to } j} = b_j \quad \forall \text{node } j$$

(flow conservation constraints)

$$\ell_{ij} \leq X_{ij} \leq u_{ij}, \quad \forall \text{arc } i \to j$$

(bound constraints)

- # of variables = # of arcs
- # of constraints = # of nodes (cons.-of-flow)
- Constraint matrix looks like:

$$
\begin{array}{c|ccccccccc|}
 & x_{12} & x_{13} & x_{23} & x_{24} & x_{25} & x_{34} & x_{35} & x_{45} & x_{53} \\
\hline
1 & 1 & 1 & & & & & & & \\
2 & -1 & & 1 & 1 & 1 & & & & \\
3 & & -1 & -1 & & & 1 & 1 & & -1 \\
4 & & & & -1 & & -1 & & 1 & \\
5 & & & & & -1 & & -1 & -1 & 1 \\
\hline
\end{array}
$$

- This is a very special LP, has "graph structure":
  - constraint matrix has "-1" and "+1" only
  - each column has 2 entries: one +1 and one -1
  - rows add up to 0 → have a <u>redundant constraint</u>
- <u>Feasible solutions property</u>: The "min-cost flow" problem is feasible only if

$$\sum_{i=1}^{n} b_i = 0.$$

- **Integer solutions property:** If b_i and l_{ij}, u_{ij} are all integers then in every BF sol-n all basic variables also have integer values

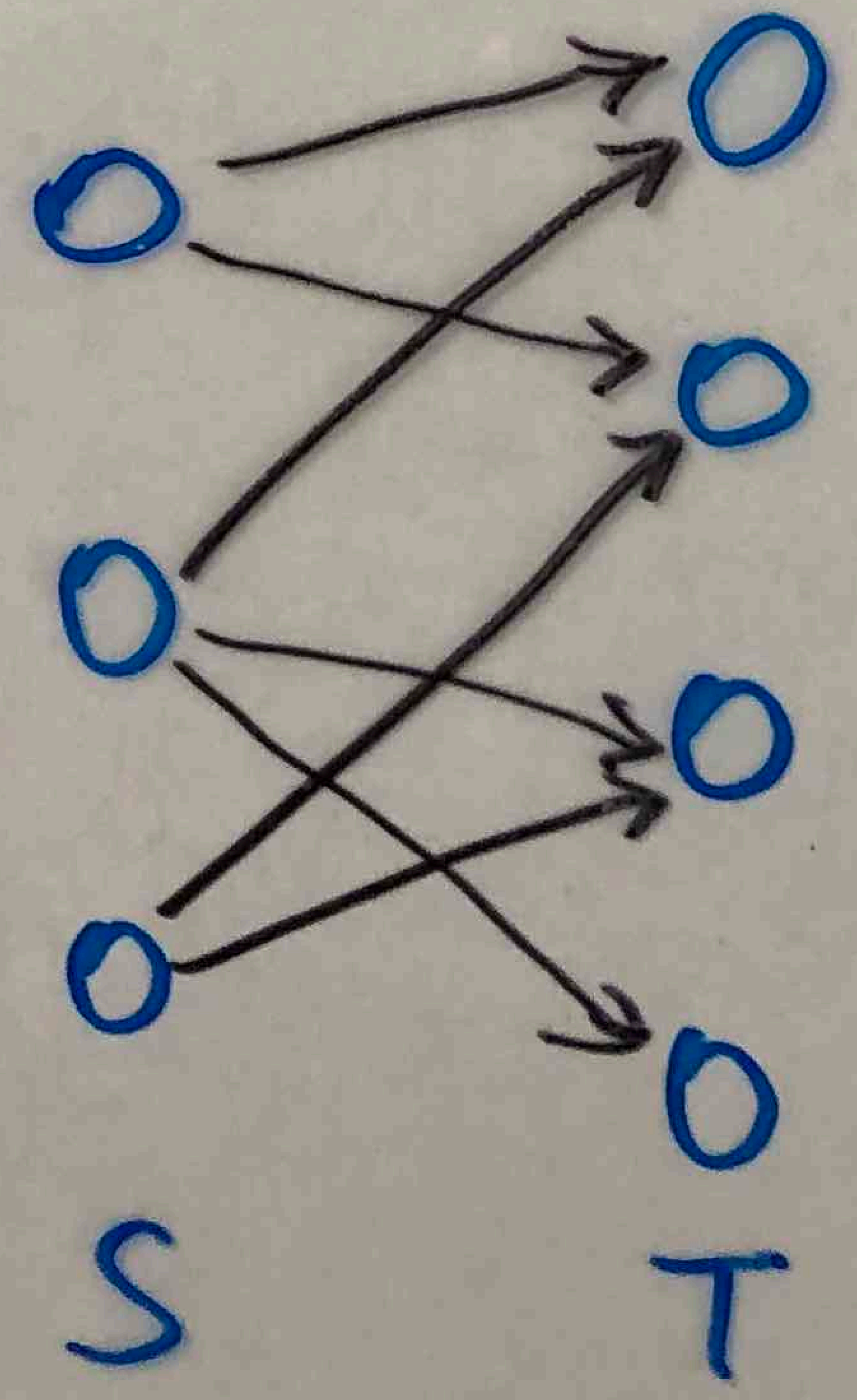
Special Cases:

Transportation Problem

The set of nodes is divided into 2 groups: sources + sinks.

$$V = \underbrace{S}_{\text{sources}} \cup \underbrace{T}_{\text{sinks}}$$

- each source is a supply node: $b_j > 0$
- each sink is a demand node: $b_j < 0$
- each arc of the graph goes from a ~~sink~~ source to a ~~source~~ sink.

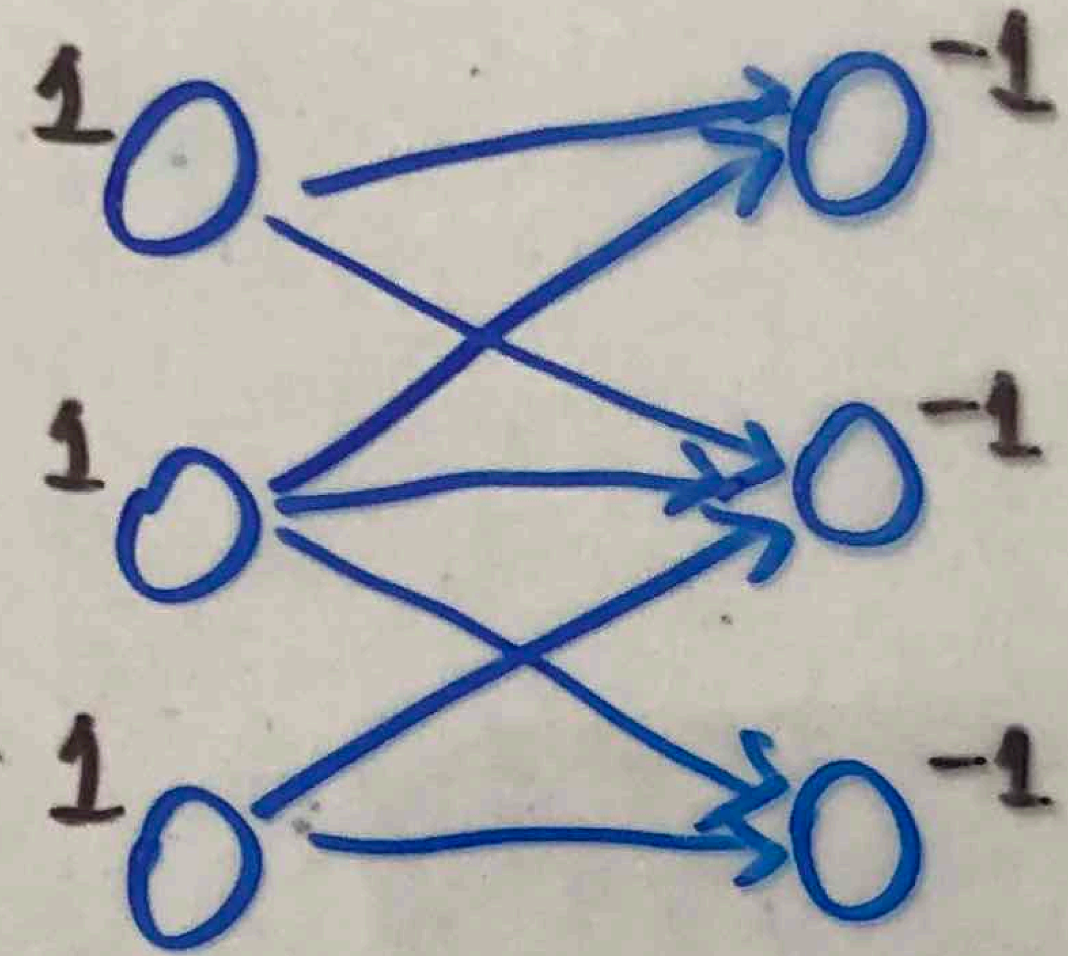


- $l_{ij} = 0$, $u_{ij} = +\infty$

Assignment Problem

A special case of transportation problem when

- $|S| = |T|$
- $b_j = 1$ for each supply node
- $b_j = -1$ for each demand node



Next time

- Other special cases
- How to solve network models