**Math 4620/5620**

# Handout 2 Solution

**Problem 1. Linear regression.**

The difficulty here lies in the fact that he optimization problem as it is stated in the problem set is not linear: the absolute value or the maximum functions are not linear. So we need to reformulate these somehow using simple tricks that make the problems linear.

a) Note that our goal is to find values for $a_1$ and $a_0$ which minimize $\sum_{i=1}^{n} \left| y_i - (a_1 x_i + a_0) \right|$ .

Thus, $a_1$ and $a_0$ are variables, and $x_i$'s and $y_i$'s are given data. However, the above function is not linear. To make it linear, we need to introduce new variables. For i=1,…,n, let $z_i = \left| y_i - (a_1 x_i + a_0) \right|$. Then the new model is:

Minimize $\sum_{i=1}^{n} z_i$

subject to $z_i = \left| y_i - (a_1 x_i + a_0) \right|$, for each i=1,…,n

However, now we have non-linear functions in the constraints.

Suppose for each i=1,…n, we substitute $z_i = \left| y_i - (a_1 x_i + a_0) \right|$ by a pair of related constraints:

$z_i \geq y_i - (a_1 x_i + a_0)$ (1)

and $z_i \geq -y_i + (a_1 x_i + a_0)$ (2)

Note that (1) and (2) provide that $z_i \geq \left| y_i - (a_1 x_i + a_0) \right|$ . But since our model is trying to minimize $z_i$'s, in the optimal solution the value of each $z_i$ will be taken all the way down to $\left| y_i - (a_1 x_i + a_0) \right|$. Summarizing, the linear program is:

Minimize $\sum_{i=1}^{n} z_i$

subject to $z_i \geq y_i - (a_1 x_i + a_0)$, for each i=1,…,n (1)

$z_i \geq -y_i + (a_1 x_i + a_0)$, for each i=1,…,n (2)

b) We want to $\min_a \max_i \left| y_i - (a_1 x_i + a_0) \right|$. $a_1$ and $a_0$ are variables, and $x_i$'s and $y_i$'s are given data. But the maximum of absolute values is not a linear function. To make it linear, we need to introduce a new variable. Let $z = \max_i \left| y_i - (a_1 x_i + a_0) \right|$. Then the new model is:

Minimize z

subject to $z = \max_i \left| y_i - (a_1 x_i + a_0) \right|$

Now we have a non-linear function in the constraint. However, the following equivalent formulation takes care of that problem.

Minimize z

subject to $z \geq y_i - (a_1 x_i + a_0)$, for each i=1,…,n (1)

$z \geq -y_i + (a_1 x_i + a_0)$, for each i=1,…,n (2)

Note that (1) and (2) provide that $z \geq \max_i |y_i - (a_1 x_i + a_0)|$. But since our model is trying to minimize z, in the optimal solution the value of each z will be taken all the way down to $\max_i |y_i - (a_1 x_i + a_0)|$.

c) Just replace $(a_1 x_i + a_0)$ above with $(a_k x_i^k + a_{k-1} x_i^{k-1} + \ldots + a_1 x_i + a_0)$.