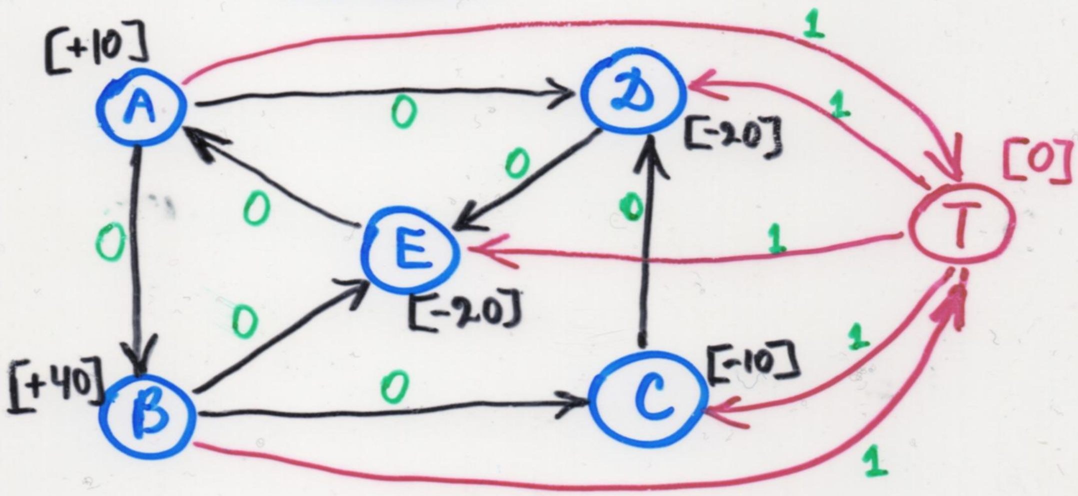
Phase 1 for Network Simplex

- How to get initial BFS?
  - Add an artificial node T.



  - $b_T = 0$ (no demand or supply for T)
  - Add artificial arcs from any supply node to T: $A \to T$, $B \to T$.
  - Add artificial arcs from T to any demand node: $T \to C$, $T \to D$, $T \to E$
  - The costs of original arcs are set to 0; The costs of artificial arcs are set to 1.

- The artificial problem defined this way is Min-Cost Flow problem with obvious initial BFS:
  - $X_{i,T} = b_i$ for any supply node i
  - $X_{T,j} = -b_j$ for any demand node j
  - $X_{i,j} = 0$ for all other arcs.

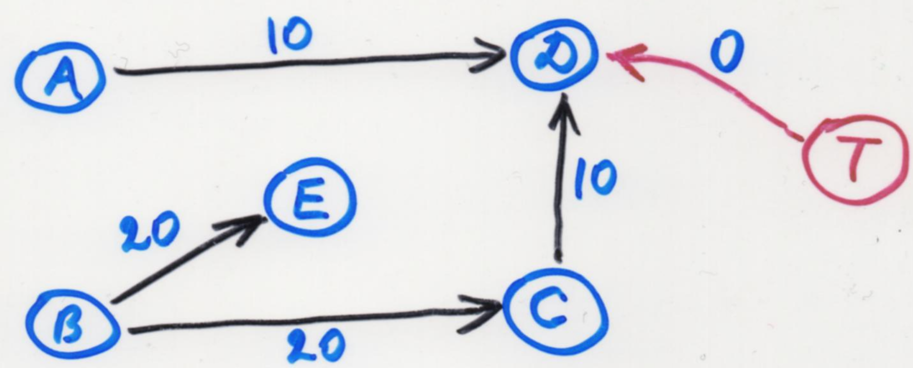
- Solve the artificial (Phase 1) problem by Network Simplex starting from that initial BFS.

- Two outcomes are possible:

  1) Suppose the original problem has a (basic) feasible sol-n.

  - Then that sol-n + an artificial arc with flow 0 is an optimal BFS for the artificial problem (with total cost 0).
  - Thus, solving the artificial problem will give an initial BFS for the original problem.

Ex.:



Optimal BFS for the artificial problem

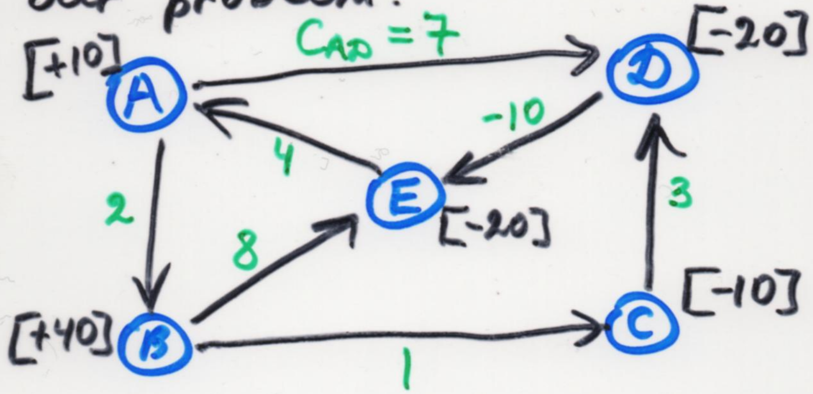
Dropping artificial node and arcs gives initial BFS for original problem.

2) Suppose the original problem is infeasible.

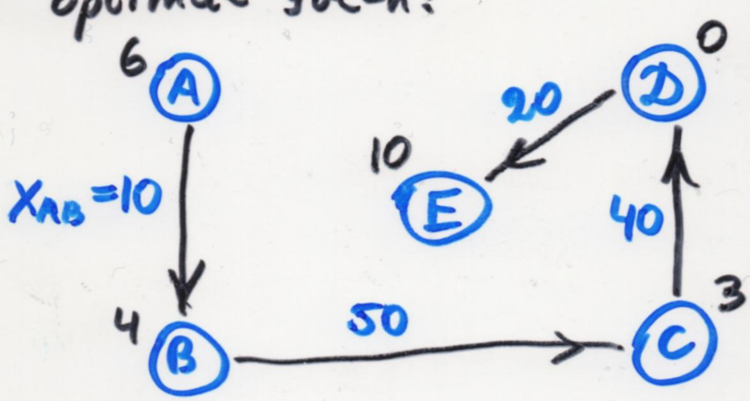
- Then the optimal sol-n of the artificial problem will contain artificial arcs with positive flow.
- The reverse is also true: If in the optimal sol-n of the artificial problem there are artificial arcs with positive flow then the original problem is infeasible.

Sensitivity Analysis for Network Simplex

Recall our problem:



and its optimal sol-n:



- What if C_{ij} is changed for some $i \to j$?

2 cases:

1) $i \to j$ is a nonbasic arc.

- Find allowable range to stay optimal for C_{ij}.
- In this case, node potentials π_i^*'s of the optimal sol-n are the same. Just need to ensure that

$$\pi_i^* - \pi_j^* \leq C_{ij}$$

Ex.: Let $i \to j = A \to D$. Then

$$\pi_A^* - \pi_D^* \leq C_{AD}$$

$$6 = 6 - 0 \leq C_{AD}$$

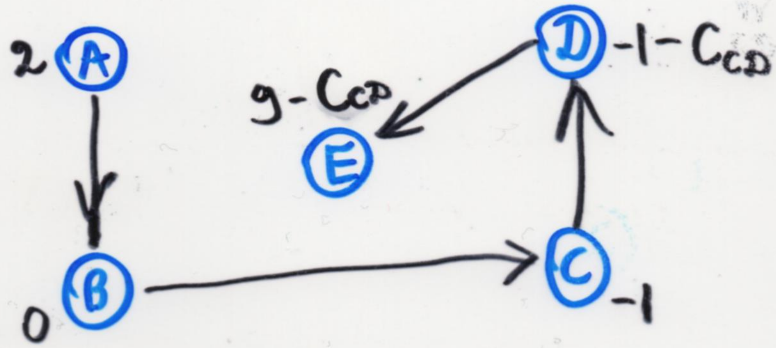
$C_{AD} \geq 6$ is allowable range to stay optimal for C_{AD}.

2) $i \to j$ is a basic arc.

- In this case, node potentials depend on the change of C_{ij}. Recompute those starting from a root.

Ex.: Let $i \to j = C \to D$.

- Node potentials:



- The dual stays feasible if

$A \to D$: $\pi_A - \pi_D \le C_{AD} \to 2-(-1-C_{CD}) \le 7$

$C_{CD} \le 4$

$E \to A$: $\pi_E - \pi_A \le C_{EA} \to 9 - C_{CD} - 2 \le 4$

$C_{CD} \ge 3$

$B \to E$: $\pi_B - \pi_E \le C_{BE} \to 0-(9-C_{CD}) \le 8$

$C_{CD} \le 17$

$\begin{cases} C_{CD} \le 4 \\ C_{CD} \ge 3 \\ C_{CD} \le 17 \end{cases} \to$ $3 \le C_{CD} \le 4$ is allowable range to stay optimal for C_{CD}.

- Can do similar analysis for changes of RHS.